Harmonic Measure

Tianyu Zhang

Table of Contents

1. Jordan Domains
    1.1 The Half Plane and the Disc
    1.2 Fatou’s Theorem and Maximal Functions
    1.3 Carathéodory’s Theorem
    1.4 Distortion and the Hyperbolic Metric
    1.5 The Hayman-Wu Theorem
    Summary of Chapter 1
2. Finitely Connected Domains
    2.1 The Schwarz Alternating Method
    2.2 Green Functions and Poisson Kernels
    2.3 Harmonic Conjugate
    2.4 Boundary Smoothness
    Summary of Chapter 2
3. Potential Theory
    3.1 Capacity and Green Function
    3.2 The Logarithmic Potential
    3.3 The Energy Method

1. Jordan Domains

To begin with, we construct harmonic measure and solve the Dirichlet problem in the upper half plane and the unit disc. We then prove the Fatou's theorem on non-tangential limits. Then we construct harmonic measure on domains bounded by Jordan curves, via the Riemann mapping theorem and the Carathéodory theorem on boundary correspondence. We review two topics from classical complex analysis, the hyperbolic metric and the elementary distortion theory for univalent functions. We conclude this chapter with the theorem of Hayman and Wu on length of level sets. Its proof is an elementary application of harmonic measure and the hyperbolic metric.

1.1 The Half Plane and the Disc

Denote $\mathbb{H} := \{z : \mathrm{Im}(z) > 0\}$ for the upper half plane and $\mathbb{R}$ for the real line. Suppose that $a < b$ are real numbers. Then the function

$$\theta := \theta(z) := \arg\left(\frac{z-b}{z-a}\right) = \mathrm{Im}\log\left(\frac{z-b}{z-a}\right)$$

is harmonic on $\mathbb{H}$, and

$$\theta = \begin{cases} \pi, & \text{on } (a,b) \\ 0, & \text{on } \mathbb{R}\backslash(a,b) \end{cases}$$

Viewed geometrically, $\theta(z) = \mathrm{Re}\big(\varphi(z)\big)$ where $\varphi(z)$ is any conformal mapping from $\mathbb{H}$ to the strip $\{0 < \mathrm{Re}(z) < \pi\}$ which maps (a,b) onto $\{\mathrm{Re}(z) = \pi\}$ and $\mathbb{R}\backslash(a,b)$ onto $\{\mathrm{Re}(z) = 0\}$.

Let $E \subset \mathbb{R}$ be a finite union of open intervals and write

$$E := \bigcup_{j=1}^{n} (a_j, b_j), \text{ with } b_{j-1} < a_j < b_j.$$

Set

$$\theta_j := \theta_j(z) := \arg\left(\frac{z-b_j}{z-a_j}\right)$$

and define the harmonic measure of E at $z \in \mathbb{H}$ to be

Definition: Harmonic Measure (for Set of Finite Union in Half Plane)

The harmonic measure of $E \subset \mathbb{R}$ at $z \in \mathbb{H}$ is defined to be

$$\omega(z, E, \mathbb{H}) := \sum_{j=1}^{n} \frac{\theta_j}{\pi}. \tag{1.1}$$

Remark 1.1: Some Elementary Properties for Harmonic Measure on Half Plane

(i) $0 < \omega(z, E, \mathbb{H}) < 1 \ \forall z \in \mathbb{H}$. (Positive)

(ii) $\omega(z, E, \mathbb{H}) \to 1$ as $z \to E$. (Boundary Limit on E)

(iii) $\omega(z, E, \mathbb{H}) \to 0$ as $z \to \mathbb{R}\backslash\overline{E}$. (Boundary Limit off E) ⋄

The function $\omega(z, E, \mathbb{H})$ is the **unique** harmonic function on $\mathbb{H}$ satisfies the above properties. The uniqueness of $\omega(z, E, \mathbb{H})$ is a consequence of the following lemma, known as the Lindelöf's maximum principle.

Lemma 1.1: Lindelöf's Maximum Principle

Suppose that the function $u(z)$ is harmonic and bounded above on a region Ω such that $\overline{\Omega} \neq \mathbb{C}$. Let F be a finite subset of $\partial\Omega$ and suppose that

$$\limsup_{z\to\zeta} u(z) \le 0 \ \forall \zeta \in \partial\Omega \backslash F. \tag{1.2}$$

Then $u(z) \le 0$ on Ω.

Proof:

Fix $z_0 \notin \overline{\Omega}$. Then the map $\dfrac{1}{z - z_0}$ transforms Ω into a bounded region, and thus we may assume that Ω is bounded. If (1.2) holds $\forall \zeta \in \partial\Omega$, then the lemma is just the ordinary maximum principle.

Denote $F := \{\zeta_1, \cdots, \zeta_N\}$, let $\varepsilon > 0$, and set

$$u_\varepsilon(z) := u(z) - \varepsilon \sum_{j=1}^{N} \log\left(\frac{\text{diam}(\Omega)}{|z - \zeta_j|}\right).$$

Then u_ε is harmonic on Ω and

$$\limsup_{z\to\zeta} u_\varepsilon(z) \le 0 \ \forall \zeta \in \partial\Omega.$$

Therefore, $u_\varepsilon \le 0 \ \forall \varepsilon$. Finally, since ε is arbitrary, sending $\varepsilon \downarrow 0$ gives

$$u(z) \le \lim_{\varepsilon\downarrow 0} \varepsilon \sum_{j=1}^{N} \log\left(\frac{\text{diam}(\Omega)}{|z - \zeta_j|}\right) = 0,$$

as desired.

□

Definition: Dirichlet Problem (over Half Plane)

Given a domain Ω and a function $f \in C(\partial\Omega)$, the Dirichlet problem for f on Ω is to find a function $u \in C(\overline{\Omega})$ such that

(i) $\Delta u = 0$ on Ω.

(ii) $u|_{\partial\Omega} = f$.

The following result treats the Dirichlet problem on the upper half plane $\mathbb{H}$.

Theorem 1.2: Existence and Uniqueness for Solution to Dirichlet Problem on $\mathbb{H}$

Suppose $f \in C(\mathbb{R} \cup \{\infty\})$. Then there exists a **unique** function

$$u := u_f \in C(\overline{\mathbb{H} \cup \{\infty\}})$$

such that u is harmonic on $\mathbb{H}$ and $u|_{\partial\mathbb{H}} = f$.

Proof:

Step I: Existence

We can assume that f is real-valued and $f(\infty) = 0$. For $\varepsilon > 0$, take disjoint open intervals

$$I_j := (t_j, t_{j+1}), j = 1, \cdots, n$$

and real constants c_j, so that the simple function

$$f_\varepsilon(t) := \sum_{j=1}^{n} c_j 1_{I_j},$$

where 1_{I_j} denotes the indicator function, satisfies

$$\|f_\varepsilon - f\|_{L^\infty(\mathbb{R})} < \varepsilon. \tag{1.3}$$

Set

$$u_\varepsilon(z) := \sum_{j=1}^{n} c_j \omega(z, I_j, \mathbb{H}).$$

If $t \in \mathbb{R} \backslash \bigcup_{j=1}^{n} \partial I_j$, then by **Remark 1.1** (ii) and (iii),

$$\lim_{\mathbb{H} \ni z \to t} u_\varepsilon(z) = f_\varepsilon(t).$$

Therefore by **(1.3)** and Lindelöf's maximum principle **Lemma 1.1**, the limit

$$u(z) := \lim_{\varepsilon \to 0} u_\varepsilon(z)$$

exists. Moreover, the limit is harmonic on $\mathbb{H}$ (by Harnack's theorem) and satisfies

$$\sup_{z \in \mathbb{H}} |u(z) - u_\varepsilon(z)| \leq 2\varepsilon.$$

We claim that for all $t \in \mathbb{R}$,

$$\lim\sup_{z \to t} |u_\varepsilon(z) - f(t)| \leq \varepsilon. \tag{1.4}$$

Claim: **(1.4)** holds $\forall t \in \mathbb{R} \cup \{\infty\}$

It is clear that **(1.4)** holds if $t \notin \bigcup_{j=1}^{n} \partial I_j$. To verify **(1.4)** at the endpoints $t_{j+1} \in \partial I_j \cap \partial I_{j+1}$, notice that

$$\sup_{z \in \mathbb{H}} \left| c_j \omega(z, I_j, \mathbb{H}) + c_{j+1} \omega(z, I_{j+1}, \mathbb{H}) - \left(\frac{c_j + c_{j+1}}{2}\right) \omega(z, I_j \cap I_{j+1}, \mathbb{H}) \right|$$
$$\leq \left| \frac{c_j - c_{j+1}}{2} \right|$$

where the blue terms equal to 1 by **Remark 1.1** (ii), the red term equals to 0 by **Remark 1.1** (iii), and the inequality holds by Lindelöf's maximum principle **Lemma 1.1**. Moreover,

$$\lim_{z \to t_{j+1}} \left(\frac{c_j + c_{j+1}}{2}\right) \omega(z, I_j \cup I_{j+1}, \mathbb{H}) = \frac{c_j + c_{j+1}}{2}$$

by **Remark 1.1** (ii). Now, let $t \in \mathbb{R}$, by **(1.4)** using in the first inequality,

$$\lim\sup_{z \to t} |u(z) - f(t)| \leq \sup_{z \in \mathbb{H}} |u(z) - u_\varepsilon(z)| + \lim\sup_{z \to t} |u_\varepsilon(z) - f(t)|$$
$$\leq 3\varepsilon$$

The same holds for $t = \infty$. Therefore, u extends to be continuous on $\overline{\mathbb{H}}$ and $u|_{\partial \mathbb{H}} = f$, proving the existence.

Step II: Uniqueness

The uniqueness of u follows from the maximum principle **Lemma 1.1**.

□

For $a < b$, elementary calculus gives

Remark 1.2: Harmonic Measure for Interval over $\mathbb{H}$

$$\omega\big(x+iy,(a,b),\mathbb{H}\big) = \frac{1}{\pi}\Big(\tan^{-1}\big(\frac{x-a}{y}\big) - \tan^{-1}\big(\frac{x-b}{y}\big)\Big)$$
$$= \int_a^b \frac{1}{\pi}\frac{y}{(t-x)^2+y^2}dt$$ ◇

Definition: Harmonic Measure (for Measurable Set on Half Plane)

If $E \subset \mathbb{R}$ is measurable, we define the harmonic measure of E at $z \in \mathbb{H}$ to be

$$\omega(z,E,\mathbb{H}) := \int_E \frac{1}{\pi}\frac{y}{(t-x)^2+y^2}dt. \tag{1.5}$$

Note that when E is a finite union of open intervals then this definition **(1.5)** agrees with the one in **(1.1)**.

Definition: Poisson Kernel over Half Plane

For $z = x+iy \in \mathbb{H}$, the density

$$P_z(t) := \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}$$

is said to be the Poisson kernel for $\mathbb{H}$.

Definition: Poisson Integral over Half Plane

If $f \in C\big(\mathbb{R}\cup\{\infty\}\big)$, the proof of **Theorem 1.2** shows that

$$u_f(z) = \int_{\mathbb{R}} f(t)P_z(t)dt,$$

and for this reason u_f is also called the Poisson integral of f.

Remark 1.3: Harmonic Measure as Transition Density and Harmonic Function

Note that the harmonic function $\omega(z,E,\Omega)$ is

(i) a harmonic function in its first variable z.

(ii) a Borel probability measure in its second variable E. ◇

Remark 1.4: Harmonic Measure Satisfies Harnack's Inequality

If $z_1, z_2 \in \mathbb{H}$, then

$$0 < C^{-1} \le \frac{\omega(z_1,E,\mathbb{H})}{\omega(z_2,E,\mathbb{H})} \le C < \infty,$$

where C depends on z_1 and z_2 but not on E. This inequality, known as the Harnack's inequality, is easily proved by comparing the kernels in **(1.5)**. ◇

Now let $\mathbb{D}$ be the unit disc $\{z : |z| < 1\}$ and let E be a finite union of open arcs on $\partial\mathbb{D}$. Then we define

Definition: Harmonic Measure (for Set of Finite Union over Unit Disc)

The harmonic measure of E at z in $\mathbb{D}$ is defined to be

$$\omega(z,E,\mathbb{D}) := \omega\big(\varphi(z),\varphi(E),\mathbb{H}\big), \tag{1.6}$$

where φ is the conformal mapping from $\mathbb{D}$ onto $\mathbb{H}$.

Remark 1.5: Definition in **(1.6)** Does Not Depend on the Conformal Mapping

This harmonic function satisfies properties analogous to **Remark 1.1** (i), (ii), and (iii). Thus by the Lindelöf's maximum principle **Lemma 1.1**, the definition **(1.6)** does not depend on the choice of φ. ◇

It follows by the change of variables $\varphi(z) := \dfrac{i(1+z)}{(1-z)}$ that

$$\omega(z, E, \mathbb{D}) = \int_E \frac{1-|z|^2}{|e^{i\theta}-z|^2}\frac{d\theta}{2\pi}.$$

An equivalent way to find this function is by a construction similar to **(1.1)**. Note that this is nothing but the Poisson integral formula, as the following theorem suggsts.

Theorem 1.3: Poisson Integral Formula over Unit Disc

Let $f(e^{i\theta})$ be an integrable function on $\partial\mathbb{D}$ and set

$$u(z) := u_f(z) = \int_0^{2\pi} f(e^{i\theta})\frac{1-|z|^2}{|e^{i\theta}-z|^2}\frac{d\theta}{2\pi}. \tag{1.7}$$

Then $u(z)$ is harmonic on D. If f is continuous at $e^{i\theta_0} \in \partial\mathbb{D}$, then

$$\lim_{\mathbb{D}\ni z\to e^{i\theta_0}} u(z) = f(e^{i\theta_0}). \tag{1.8}$$

The identity **(1.7)** tells us that the Poisson integral is harmonic over the unit disc $\mathbb{D}$ and the identity **(1.8)** gives us the boundary behavior. In fact, **(1.8)** also holds if the integrable function f is changed on a measure zero subset of $\partial\mathbb{D}\backslash\{e^{i\theta_0}\}$.

Definition: Poisson Integral (over Unit Disc)

The function $u := u_f$ is called the Poisson integral of f.

Definition: Poisson Kernel (over Unit Disc)

The kernel

$$P_z(\theta) := \frac{1}{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

is the Poisson kernel for the disc.

Definition: Solution to the Dirichlet Problem over Unit Disc

If $f \in C(\partial\mathbb{D})$ then

$$u(z) := \begin{cases} u_f(z), & z \in \mathbb{D} \\ f(z), & z \in \partial\mathbb{D} \end{cases}$$

is the solution of the Dirichlet problem for f on $\mathbb{D}$.

In the special case when $f(e^{i\theta})$ is continuous, **Theorem 1.3** follows from **Theorem 1.2** and a change of variables. Conversely, **Theorem 1.3** shows that **Theorem 1.2** can be extended to $f \in L^1\left(\dfrac{dt}{1+t^2}\right)$, again by change of variables.

Proof of **Theorem 1.3**:

Step I: Harmonicity of u.

We may suppose that f is real-valued. We have

$$\mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = 2\pi P_z(\theta)$$

by the definition of Poisson kernel over the unit disc. Thus, we see that u is the real part of the analytic function

$$\int_0^{2\pi} f(e^{i\theta})\frac{e^{i\theta}+z}{e^{i\theta}-z}\frac{d\theta}{2\pi}$$

and therefore that u is a harmonic function. One can also see that u is harmonic by differentiating the integral **(1.7)**.

Step II: Boundary condition when f is continuous at $e^{i\theta_0}$.

Suppose that f is continuous at $e^{i\theta_0}$ and let $\varepsilon > 0$. Then by the continuity

$$|f(e^{i\theta}) - f(e^{i\theta_0})| < \varepsilon$$

on any interval $I = (\theta_1, \theta_2)$ containing θ_0. Setting

$$u_\varepsilon(z) := \int_{[0,2\pi]\setminus I} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} f(e^{i\theta}) \frac{d\theta}{2\pi} + f(e^{i\theta_0})\omega(z, I, \mathbb{D}),$$

(note that the second identity in the right hand side comes from integration over I). Now we have

$$\begin{aligned} |u(z) - u_\varepsilon(z)| &= \left| \int_I \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \left(f(e^{i\theta}) - f(e^{i\theta_0})\right) \frac{d\theta}{2\pi} \right| \quad \text{(assumption and } u_\varepsilon\text{)} \\ &\leq \varepsilon\omega(z, I, \mathbb{D}) \quad \text{(Continuity and definition of } \omega\text{)} \\ &\leq \varepsilon \quad \text{(Conformal invaraince)} \end{aligned}$$

However, by the assumption of u_ε, one has

$$\lim_{z \to e^{i\theta_0}} u_\varepsilon(z) = f(e^{i\theta_0}).$$

Therefore, since f is continuous at $e^{i\theta_0}$, taking the difference in conjunctio with the continuity of harmonic function yields

$$\limsup_{z \to e^{i\theta_0}} |u(z) - f(e^{i\theta_0})| < \varepsilon.$$

Finally, since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ yields the desired identity **(1.8)**.

□

1.2 Fatou's Theorem and Maximal Functions

When $f \in L^1(\partial\mathbb{D})$ the limit **(1.8)** can fail to exist for every $\zeta \in \partial\mathbb{D}$. However, there is a substitute result known as Fatou's theorem, in which the approach $z \to \zeta$ is restricted to cones. This result allow us to extend **Theorem 1.3** to $f \in L^1(\partial\mathbb{D})$.

Definition: Cone (over Unit Disc)

For $\zeta \in \partial\mathbb{D}$ and $\alpha > 1$, we define the cone

$$\Gamma_\alpha(\zeta) := \left\{z : |z - \zeta| < \alpha(1 - |z|)\right\}.$$

Remark 1.6: Some Elementary Properties of Cones

(i) The cone $\Gamma_\alpha(\zeta)$ is **asymptotic** to a sector with vertex ζ and angle $2\sec^{-1}(\alpha)$ that is symmetric about the radius $[0,\zeta]$. (Asymptotic)

(ii) The cones $\Gamma_\alpha(\zeta)$ expanded as α increases. (Monotone in Angle) ⋄

Definition: Non-Tangential Limit (over Unit Disc)

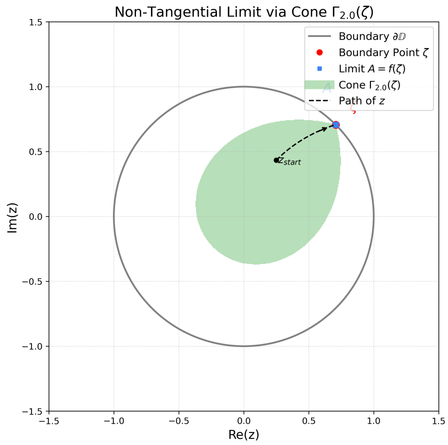
A function $u(z)$ on $\mathbb{D}$ has non-tangential limit A at $\zeta \in \partial\mathbb{D}$ if

$$\lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} u(z) = A \tag{1.9}$$

for every $\alpha > 1$.

Example 1.1: Example of Non-Tangential Limit over Unit Cone

Consider the function $u(z) := \exp\left\{\frac{z+1}{z-1}\right\}$. This function $u(z)$ is continuous on $\partial\mathbb{D}\backslash\{1\}$, and $|u(\zeta)| = 1$ on $\partial\mathbb{D}\backslash\{1\}$, but $u(z)$ has non-tangential limit 0 at $\zeta = 1$. ◇



(Figure 1.1: Non-Tangential Limit via Cone $\Gamma_2(\zeta)$.)

Definition: Non-Tangential Maximal Function (over Unit Disc)

With fixed $\alpha > 1$, the non-tangential maximal function of u at ζ is defined as

$$u^*_\alpha(\zeta) := \sup_{z \in \Gamma_\alpha(\zeta)} |u(z)|.$$

If u has a finite non-tangential limit at ζ, then

$$u^*_\alpha(\zeta) < \infty \ \forall \alpha > 1.$$

We will denote the Lebesgue measure for a set E as $\mathrm{Leb}(E)$ since we do not wish to abuse the use of λ. We will explicitly emphasize the dimension d by $\mathrm{Leb}_d(E)$ whenever necessary. Moreover, the almost everywhere property will be abbreviated as a.e. property.

Theorem 1.4: Fatou's Theorem

Let $f(e^{i\theta}) \in L^1(\partial\mathbb{D})$ and let $u(z)$ be the Poisson integral of f. Then

$$\lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} u(z) = f(\zeta) \text{ for Leb-a.e. } \zeta = e^{i\theta} \in \partial\mathbb{D} \text{ for every } \alpha > 1. \quad (1.10)$$

Moreover, for each $\alpha > 1$, one has

$$\mathrm{Leb}\Big(\big\{\zeta \in \partial\mathbb{D} : u^*_\alpha(\zeta) > \lambda\big\}\Big) \le \frac{3 + 6\alpha}{\lambda}\|f\|_{L^1(\partial\mathbb{D})}. \quad (1.11)$$

Fatou's theorem tells us that any positive harmonic function on the unit disc possesses a non-tangential limit at Leb-a.e. boundary points.

Definition: Solution to the Dirichlet Problem with Leb-a.e. Non-Tangential Limit

When $u(z)$ is the Poisson integral of $f \in L^1(\partial\mathbb{D})$ the function $u = u_f$ is also called the solution to the Dirichlet problem for f, even though u converges to f on $\partial\mathbb{D}$ only non-tangentially and only Leb-a.e..

Definition: Weak Type 1-1

An operator K is said to be of the weak type 1-1 over some finite sums of Dirac deltas if there exists a constant $C > 0$ such that for each $\lambda > 0$, the inequality

$$\mathrm{Leb}(\{|Kf| > \lambda\}) \le C\frac{n}{\lambda}$$

holds for every $f = \sum_{i=1}^{n} \delta_{x_i}$, where $x_1, \cdots, x_n$ are distinct points.

It is then obvious that **(1.11)** tells us that the operator $L^1(\partial\mathbb{D}) \ni f \to u_\alpha^*$ is weak-type 1-1. It follows from **(1.10)** that

$$u_\alpha^*(\zeta) < \infty \text{ Leb-a.e.},$$

but **(1.11)** is a sharper, quantitative result. Therefore, in the proof we shall use **(1.11)** to derive **(1.10)**.

Remark 1.7: Proof for Fatou's Theorem via Approximate Identity Argument

The proof of Fatou's theorem is a standard approximate identity argument from real analysis that derive a.e. convergence for all $f \in L^1(\partial\mathbb{D})$ from

(a) An estimate such as **(1.11)** for the maximal function.

(b) The a.e. convergence **(1.10)** for all functions in a dense subset of $L^1(\partial\mathbb{D})$ such as $C(\partial\mathbb{D})$. ◇

The approximate identity argument will be performed later, we here use another approach.

Proof of **Theorem 1.4**:

As promised, we first assume **(1.11)** and show that **(1.11)** implies **(1.10)**.

Step I: **(1.11)** implies **(1.10)**.

Fix α temporarily. We may assume that f is real-valued. Set

$$W_f(\zeta) := \limsup_{\Gamma_\alpha \ni z \to \zeta} |u_f(z) - f(\zeta)|$$

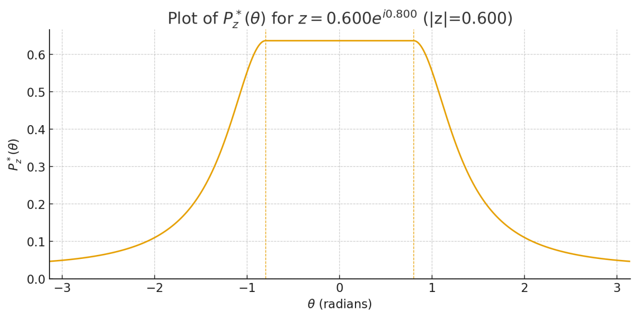
as the difference in **(1.10)**. Our goal is to show this value is arbitrarily small.

First, by the triangle inequality we have

$$W_f(\zeta) \le u_\alpha^*(\zeta) + |f(\zeta)|.$$

Now, using **Chebyshev's inequality**[1] gives

$$\left| \{\zeta : |f(\zeta)| > \lambda\} \right| \le \frac{\|f\|_{L^1(\partial\mathbb{D})}}{\lambda}.$$



(Figure 1.2: Plot for $P_z^*(\theta)$ for $z = 0.600e^{i0.800}(|z| = 0.600)$)

[1] **Theorem**: (Chebyshev's Inequality in L^p) If $f \in L^p(\mu)$ then $\forall \lambda > 0$, we have

$$\mu\big(\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\big) \le \frac{\int |f(x)|^p \, dx}{\lambda^p}.$$

Therefore,

$$\begin{aligned}&\text{Leb}\big(\{\zeta : W_f(\zeta) > \lambda\}\big)\\ &\leq \text{Leb}\big(\{\zeta : u_\alpha^*(\zeta) > \lambda/2\}\big) + \text{Leb}\big(\{\zeta : |f(\zeta)| > \lambda/2\}\big)\\ &\leq \frac{8+12\alpha}{\lambda}\|f\|_{L^1(\partial\mathbb{D})}\end{aligned} \tag{1.12}$$

where the first inequality holds by the sub-additivity for Lebesgue measure and the second inequality holds by assumption **(1.11)** and the above display.

Fix $\varepsilon > 0$ and let $g \in C(\partial\mathbb{D})$ be such that

$$\|f - g\|_{L^1(\partial\mathbb{D})} \leq \varepsilon^2.$$

Now by **(1.8)** in **Theorem 1.3**, $W_g(\zeta) = 0$, and hence

$$W_f(\zeta) = W_{f-g}(\zeta).$$

Applying **(1.12)** to $f - g$ yields

$$\text{Leb}\big(\{\zeta : W_f(\zeta) > \varepsilon\}\big) \leq \frac{(8+12\alpha)\varepsilon^2}{\varepsilon} = (8+12\alpha)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ yields **(1.10)** for any fixed α Leb-a.e.. Finally, by **Remark 1.6** (ii), the cones increases in the angle α, it follows that **(1.10)** holds $\forall \alpha > 1$, except for ζ in a set of Lebesgue measure zero, proving **(1.10)** as we unfixed α.

To prove **(1.11)**, we will dominate the non-tangential maximal function with a second, simpler maximal function. To this end we need a definition and a lemma.

Definition: Hardy-Littlewood Maximal Function

Let $f \in L^1(\partial\mathbb{D})$ and denote

$$Mf(\zeta) := \sup_{\zeta\in I} \frac{1}{\text{Leb}(I)} \int_I |f|\, d\theta$$

as the maximal average of $|f|$ over subarcs $I \subset \partial\mathbb{D}$ that contains ζ. The function Mf is called the Hardy-Littlewood function of f.

Remark 1.8: Hardy-Littlewood Max Is Simpler than Non-Tangential Max

The function Mf is simpler thatn u_α^* because it features characteristic functions of intervals instead of Poisson kernel. ⋄

Lemma 1.5: Hardy-Littlewood Max as Upper Bound for Non-Tangential Max

Let $u(z)$ be the Poisson integral of $f \in L^1(\partial\mathbb{D})$ and let $\alpha > 1$. Then

$$u_\alpha^*(\zeta) \leq (1+2\alpha)Mf(\zeta). \tag{1.13}$$

Proof:

Assume $\zeta = 1$. Fix z so that $\theta_0 := \arg(z)$ has $|\theta_0| \leq \pi$. Define

$$\begin{aligned}P_z^*(\theta) &:= \sup\big\{P_z(\varphi : |\theta| \leq |\varphi| \leq \pi\big\}\\ &= \begin{cases}\frac{1}{2\pi}\frac{1+|z|}{1-|z|}, & |\theta| \leq |\theta_0|\\ \max\big(P_z(\theta), P_z(-\theta)\big), & |\theta_0| < |\theta| \leq \pi\end{cases}\end{aligned}$$

Observe that the function P_z^* satisfies the following properties:

(i) $P_z^*(\theta)$ is an even function of $\theta \in [-\pi, \pi]$.

(ii) $P_z^*(\theta)$ is decreasing on $[0,\pi]$.

(iii) $P_z^*(\theta) \geq P_z(\theta)$.

The even function P_z^* is the smallest decreasing majorant of P_z on $[0,\pi]$. Without loss of generality, we may assume that $f(e^{i\theta}) \geq 0$, so that

$$\int f(e^{i\theta})P_z(\theta)d\theta \leq \int f(e^{i\theta})P_z^*(\theta)d\theta$$

by property (iii). The properties (i) and (ii) imply that

$$\int f(e^{i\theta})P_z^*(\theta)d\theta \leq \|P_z^*\|_{L^1(\partial\mathbb{D})}Mf(1) \tag{1.14}$$

because P_z^* is the increasing limit of a sequence of functions of the form

$$\sum_{j\geq 1} c_j\left(\frac{1}{2\theta_j}1_{(-\theta_j,\theta_j)}(\theta)\right)$$

with $c_j \geq 0$ and $\sum_{j\geq 1} c_j \leq \|P_z^*\|_{L^1(\partial\mathbb{D})}$.

Now we claim that when $z \in \Gamma_\alpha(1)$,

$$\|P_z^*\|_{L^1(\partial\mathbb{D})} \leq (1+2\alpha). \tag{1.15}$$

Since then **(1.13)** follows from property (iii), **(1.14)**, and **(1.15)**.

Claim: **(1.15)** holds $\forall z \in \Gamma_\alpha(1)$.

We shall consider two cases.

Case I: $-\dfrac{\pi}{2} \leq \theta_0 = \arg(z) \leq \dfrac{\pi}{2}$.

If so, then for $\beta = \arg\left(\dfrac{z-1}{z}\right)$,

$$\begin{aligned}
\frac{|\theta_0|}{1-|z|} &\leq \alpha\frac{|\theta_0|}{|1-z|} && \text{(Triangle inequality)}\\
&\leq \frac{\pi\alpha}{2}\frac{|\sin\theta_0|}{|1-z|} && (\text{Since } |\theta_0| \leq \pi/2)\\
&= \frac{\pi\alpha}{2}\frac{|\sin\beta|}{1} && (\text{definition of } \beta)\\
&\leq \frac{\pi\alpha}{2} && (\text{since } |\sin\beta| \leq 1)
\end{aligned}$$

Case II: $\dfrac{\pi}{2} \leq |\theta_0| \leq \pi$ and $z \in \Gamma_\alpha(1)$.

If so, then $|1-z| \geq 1$ and

$$\frac{|\theta_0|}{1-|z|} \leq \alpha\frac{|\theta_0|}{|1-z|} \leq \pi\alpha,$$

where the first inequality holds by triangle inequality and the second holds since $|1-z| \geq 1$.

Hence, in either cases, we all have

$$\|P_z^*\|_{L^1(\partial\mathbb{D})} = 2\int_{|\theta_0|}^{\pi} P_z(\theta)d\theta + \frac{2|\theta_0|}{2\pi}\frac{1+|z|}{1-|z|}$$
$$\leq (1+2\alpha)$$

where the first relation holds by the definition of P_z^* and the second relation holds by bounds in *Case I* and *Case II*. This proves the claim, thus **(1.15)** holds for each $z \in \Gamma_\alpha(1)$, and the desired **(1.13)** follows.

□

By **Lemma 1.5**, the inequality **(1.11)** will follow from the simpler inequality

$$\mathrm{Leb}\big(\{\zeta \in \partial\mathbb{D} : Mf(\zeta) > \lambda\}\big) \leq \frac{3\|f\|_{L^1(\partial\mathbb{D})}}{\lambda}, \tag{1.16}$$

which says that the operator $L^1(\partial\mathbb{D}) \ni f \to Mf$ is also weak type 1-1.

To prove **(1.16)**, we use a covering lemma.

Lemma 1.6: Measure Bound for Open Intervals via Disjointed Subintervals

Let μ be a positive Borel measure on $\partial\mathbb{D}$ and let $\{I_j\}$ be a finite sequence of open intervals in $\partial\mathbb{D}$. Then $\{I_j\}$ contains a pairwise disjoint subsequence $\{J_k\}$ such that

$$\sum_k \mu(J_k) \geq \frac{1}{3}\mu\Big(\bigcup_j I_j\Big). \tag{1.17}$$

Proof:

Because the family $\{I_j\}$ is finite, we may assume that no I_j is contained in the union of the others. Denoting

$$I_j := \big\{e^{i\theta} : \theta \in (a_j, b_j)\big\},$$

we may also assume that

$$0 \leq a_1 < a_2 < \cdots < a_n < 2\pi.$$

Then $b_{j+1} > b_j$, because otherwise $I_{j+1} \subset I_j$ and $b_{j-1} < a_{j+1}$, because otherwise $I_j \subset I_{j-1} \cup I_{j+1}$. If $n > 1$, then

$$b_n < b_1 + 2\pi \text{ and } b_{n-1} < a_1 + 2\pi.$$

Consequently, the family of even-numbered intervals I_j is pairwise disjoint. It left us to thow that the bound **(1.17)** holds.

Claim: **(1.17)** Holds

The family of odd-numbered intervals I_j is almost pairwise disjoint; only the first and the last intervals can intersect (since we have run a whole perior). We shall consider two cases, naturally, the even j and the odd j.

Case I: j even.

Now, if

$$\sum_{j \text{ even}} \mu(I_j) \leq \frac{1}{3}\mu\Big(\bigcup_j I_j\Big)$$

we take the even numbered intervals to be the subcolletion $\{J_k\}$.

Case II: j odd

Otherwise, we have

$$\sum_{j \text{ odd}} \mu(I_j) \geq \frac{2}{3}\mu\Big(\bigcup_j I_j\Big).$$

In this case, if

$$\mu(I_1) \leq \frac{1}{2}\sum_{j \text{ odd}} \mu\Big(\bigcup_j I_j\Big),$$

we take for $\{J_k\}$ the family of odd-numbered intervals, omitting the first interval I_1. Otherwise, if

$$\mu(I_1) > \frac{1}{2}\sum_{j \text{ odd}} \mu\Big(\bigcup_j I_j\Big),$$

we take $\{J_k\} = \{I_1\}$. Then in each case **(1.17)** hods for the subsequence $\{J_k\}$.

□

Lemma 1.7: Hardy-Littlewood Maximum Function Is Weak Type 1-1

The operator $f \to Mf$ is weak type $1-1$. That is, if $f \in L^1(\partial\mathbb{D})$ then

$$\text{Leb}\Big(\{\zeta \in \partial\mathbb{D} : Mf(\zeta) > \lambda\}\Big) \leq \frac{3\|f\|_{L^1(\partial\mathbb{D})}}{\lambda}. \tag{1.18}$$

Proof:

Let K be a compact subset of E_λ, where E_λ is defined to be

$$E_\lambda := \big\{\zeta \in \partial\mathbb{D} : Mf(\zeta) > \lambda\big\}.$$

For each $\zeta \in E_\lambda$, there is an open interval I such that $\zeta \in I$ and

$$\frac{1}{\text{Leb}(I)}\int_I |f|\,d\theta > \lambda.$$

So that

$$\text{Leb}(I) < \frac{1}{\lambda}\int_I |f|\,d\theta.$$

Since K is compact, we can cover K by finitely many such intervals, and we may assume, without loss of generality, that they are labeled via $\{I_j\}_{j=1}^n$, and by **Lemma 1.6**, there is a subcollection $\{J_k\}$ that is pairwise disjoint and **(1.17)** holds. Therefore, using sub-additivity in the first inequality and **(1.17)** in the second, one has

$$\begin{aligned}\text{Leb}(K) \leq \text{Leb}\Big(\bigcup_{j=1}^n I_j\Big) &\leq 3\sum_k \text{Leb}(J_k)\\ &\leq \frac{3}{\lambda}\sum_k \int_{J_k} |f|\,d\theta \quad \text{(by above display)}\\ &\leq \frac{3}{\lambda}\|f\|_{L^1(\partial\mathbb{D})} \quad (\text{Since } \cup_k J_k \subset \partial\mathbb{D}).\end{aligned}$$

Finally, since K is arbitrary, sending $\text{Leb}(K) \uparrow \text{Leb}(E_\lambda)$ yields **(1.18)**.

□

Now we can finish our proof for **(1.11)** and thus conclude the proof of Fatou's theorem **Theorem 1.4**.

Proof of **Theorem 1.4**: Continued

By **(1.13)** and **(1.18)**, the inequality **(1.11)** follows with constant

$$3 + 6\alpha,$$

proving Fatou's theorem.

□

Now, by Fatou's theorem **Theorem 1.4**, we can extend the result in **(1.8)** from f being continuous to $f \in L^1(\partial\mathbb{D})$. Consequently, the continuous boundary condition for the Dirichlet problem is generalized to boundedness. For the sakeness of simplicity, we shall use DP to denote the Dirichlet problem whenever necessary.

Corollary 1.4.1: Solution to DP over Unit Disc for Bounded Boundary Condition

If u is a bounded harmonic function on $\mathbb{D}$, then for every $\alpha > 1$ and for Leb-a.e. $\zeta = e^{i\theta} \in \partial\mathbb{D}$,

$$f(\zeta) = \lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} u(z)$$

exists, where $u(z)$ is the Poisson integral of f, and

$$\|f\|_{L^\infty(\partial\mathbb{D})} = \sup_{z\in\mathbb{D}} |u(z)|.$$

Proof:

We shall prove this result via **Banach-Alaoglu theorem**[2]. For the application available, we shall recover f in the unit ball.

Let $\{r_n\}_{n\geq 1} \subset \mathbb{R}$ be such that $r_n \to 1$ for n sufficiently large. Let

$$f_n(e^{i\theta}) := u(r_n e^{i\theta}).$$

By **Banach-Alaoglu Theorem**, the sequence $\{f_n\}_{n\geq 1}$ has a weak* limit in the dual space of $L^1(\partial\mathbb{D})$, namely, $f \in L^\infty(\partial\mathbb{D})$, such that

$$\begin{aligned} \|f\|_{L^\infty(\partial\mathbb{D})} &\leq \limsup_{n\to\infty} \|f_n\|_{L^\infty(\partial\mathbb{D})} \quad (L^\infty \text{ norm is l.s.c.}) \\ &\leq \sup_{z\in\mathbb{D}} |u(z)| \quad \text{(Lindelöf's Maximum Principle)} \end{aligned}$$

Since $u(r_n z)$ is the Poisson integral of f_n, and Poisson kernels are in L^1, u must be the Poisson integral of f. This gives

$$|u(z)| \leq \|f\|_{L^\infty(\partial\mathbb{D})}.$$

The desired result follows from **(1.10)** in Fatou's **Theorem 1.4**.

□

Remark 1.9: Harmonic Measure as Indicator Along Non-Tangential Limit over $\mathbb{D}$

In particular, this corollary implies that for any measurable set $E \subset \partial\mathbb{D}$, there exists a **unique** bounded harmonic function $u(z)$ on $\mathbb{D}$ such that $u(z)$ has non-tangential limit 1_E Leb-a.e.. It is the function

$$u(z) = \omega(z, E, \mathbb{D}).$$

◇

1.3 Carathéodory's Theorem

[2] **Theorem**: (Banach-Alaoglu Theorem) Let X be a separable Banach space. Then the closed unit ball $\mathscr{B}_1(0) := \{x \in X^* : \|x\| \leq 1\}$ of X^* is weak* sequentially compact.

Let Ω be a simply connected domain in the extended complex plane $\mathbb{C}^\infty$, where $\mathbb{C}^\infty = \mathbb{C} \cup \{\infty\}$.

Definition: Jordan Curve and Jordan Domain

A simply connected domain $\Omega \subset \mathbb{C}^\infty$ is said to be a Jordan domain if $\Gamma = \partial\Omega$ is a Jordan curve (continuous simply connected curve) in $\mathbb{C}^\infty$.

Theorem 1.8: Carathéodory's Theorem

Let φ be a conformal mapping from the unit disc $\mathbb{D}$ onto a Jordan domain Ω. Then φ has a continuous extension to $\overline{\mathbb{D}}$, namely $\widetilde{\varphi}$, and the extension is a one-to-one map from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$.

Note that, since φ maps $\mathbb{D}$ onto Ω, the continuous extension, denoted by $\widetilde{\varphi}$, must map $\partial\mathbb{D}$ onto $\Gamma := \partial\Omega$, and because $\widetilde{\varphi}$ is a one-to-one on $\partial\mathbb{D}$, $\varphi(e^{i\theta})$ parameterizes the Jordan curve Γ. Indeed, the Carathéodory's **Theroem 1.8** tells us that the bijectivity is preserved under the extension of $\widetilde{\varphi} : \overline{\mathbb{D}} \to \overline{\Omega}$ such that $\widetilde{\varphi}\,|_{\mathbb{D}} = \varphi$.

Before we prove Carathéodory's **Theroem 1.8**, we use it to solve the Dirichlet problem on a Jordan domain Ω. Recall that the Fatou's **Theorem 1.4** enables us to extend the boundary function from $C(\mathbb{D})$ to $L^1(\mathbb{D})$, now we are able, thanks to the result of Carathéodory's **Theroem 1.8**, to extend the domain from $\mathbb{D}$ to $\overline{\mathbb{D}}$, and to extend the range from Ω to $\overline{\Omega}$. This justifies the following definition.

Definition: Solution to DP over Jordan Domain for Bounded Boundary Function

Let f be a Borel function on Γ such that $f \circ \varphi$ is integrable on $\partial\mathbb{D}$. If $w = \varphi^{-1}(z)$, then

$$u(z) = u_f(z) = \int_0^{2\pi} f \circ \varphi(e^{i\theta}) \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi} \tag{1.19}$$

is harmonic on Ω, and by **Theroem 1.8** in conjunction with **Theorem 1.3**,

$$\lim_{\Omega \ni z \to \zeta} u(z) = f(\zeta) \tag{1.20}$$

whenever $\varphi^{-1}(\zeta) \in \partial\mathbb{D}$ is a point of continuity of $f \circ \varphi$. In particular, if f is continuous then **(1.20)** holds for every $\zeta \in \Gamma$ and $u(z) = u_f(z)$ solves the Dirichlet problem for f on Ω.

If f is a bounded Borel function on Γ, then $f \circ \varphi$ is Borel and the integral **(1.19)** is well-defined. Thus we derive the harmonic measure for this extension.

Definition: Harmonic Measure (over Jordan Domain)

For any Borel set $E \subset \Gamma$ we use **(1.19)** with $f := 1_E$ to define the harmonic measure of E relative to Ω by

$$\omega(z, E, \Omega) := \omega\left(w, \varphi^{-1}(E), \mathbb{D}\right) = \int_{\varphi^{-1}(E)} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} \frac{d\theta}{2\pi}. \tag{1.21}$$

Note that $\omega(z, E, \Omega)$ has the following properties.

(i) $E \mapsto \omega(z, E)$ is a Borel measure on $\partial\Omega$.

(ii) **(1.19)** can be rewritten as

$$u(z) = \int_{\partial\Omega} f(\zeta) d\omega(z, \zeta). \tag{1.22}$$

Note that, we have been implicitly stating that the harmonic measure is nothing but a measure transition kerenel. We shall mention this fact later, as one shall see the advantage of pointing this out explicitly.

Equations (1.21) and (1.22) do not depend on the choice of conformal mapping φ, as we have mentioned this fact in **Remark 1.5**. This is because, in the case of Jordan domain, that every conformal self map T of $\mathbb{D}$,

$$\omega\Big(T(w), T\big(\varphi^{-1}(E)\big), \mathbb{D}\Big) = \omega\big(w, \varphi^{-1}(E), \mathbb{D}\big).$$

When f is a bounded Borel function on $\partial\Omega$, (1.22) and Fatou's **Theorem 1.4** give

$$\sup_{z\in\Omega} |u(z)| = \|f\|_{L^\infty(\omega)}.$$

Moreover, **Corollary 1.4.1** shows that every bounded harmonic function on Ω can be expressed in the form (1.22), this is a direct result of the Riesz's representation theorem in the L^p space.

The principal goal of this book is to find geometric properties of the harmonic measure $\omega(z, E)$ more explicit than the definition (1.21). But (1.21) already points out the key issue:

Remark 1.10: Equivalent Question for Harmonic Measure over Jordan Domain

For a Jordan domain questions about harmonic measure are equivalent to questions about the boundary behavior of conformal mappings. ⋄

Proof of **Theroem 1.8**:

Without loss of generality, we may assume that Ω is bounded, otherwise we can apply Riemann mapping theorem. Fix $\zeta \in \partial\mathbb{D}$, we first show that φ has a continuous extension at ζ.

Step I: φ has a continuous extension at ζ.

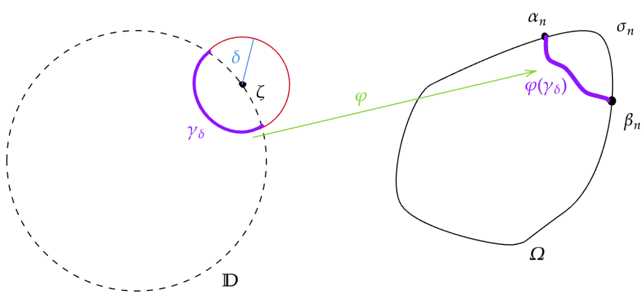
Let $0 < \delta < 1$, denote the open ball centered at ζ with radius δ as

$$B(\zeta, \delta) := \big\{z \in \Omega : |z - \zeta| < \delta\big\},$$

and set

$$\gamma_\delta := \mathbb{D} \cap \partial B(\zeta, \delta).$$

Then, since the conformal mappings preserve arcs, $\varphi(\gamma_\delta)$ is a Jordan arc with



(Figure 1.3: Proof of Continuity in the first step)

length

$$L(\delta) := \int_{\gamma_\delta} |\varphi'(z)|\, ds.$$

(To see this, let $z(t) := x(t) + iy(t)$, $ds = |dz| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$, thus we have $L(\gamma_\delta) = \int_{\gamma_\delta} |dz|$ for $t \in (a, b)$. Now, let $\Gamma(t) = \Gamma\big(z(t)\big)$, we have

$L\big(\varphi(\gamma_\delta)\big) = \int_a^b \left|\frac{d\Gamma}{dt}\right| dt$. By chain rule, $\frac{d\Gamma}{dt} = \frac{d}{dt}\varphi\big(z(t)\big) = \varphi'\big(z(t)\big)z'(t)$).

Now, one has

$$\begin{aligned} L^2(\delta) &\le \Big(\int_{\gamma_\delta} 1^2 ds\Big)\Big(\int_{\gamma_\delta} |\varphi'(z)|^2 ds\Big) \quad \text{(Cauchy-Schwartz inequality)} \\ &= L(\delta)\Big(\int_{\gamma_\delta} |\varphi'(z)|^2 ds\Big) \\ &\le \pi\delta \int_{\gamma_\delta} |\varphi'(z)|^2 ds \quad (L(\delta) \text{ is at most the circumference}) \end{aligned}$$

Therefore, for $\rho < 1$, dividing δ on both sides and integrating gives

$$\begin{aligned} \int_0^\rho \frac{L^2(\delta)}{\delta} d\delta &\le \pi \iint_{\mathbb{D}\cap B(\rho,\zeta)} |\varphi'(z)|^2 dxdy \quad \text{(Cauchy-Schwartz)} \\ &= \pi \cdot \text{Area}\Big(\varphi\big(\mathbb{D} \cap B(\zeta, \rho)\big)\Big) \\ &< \infty. \end{aligned} \tag{1.23}$$

Thus, there is a sequence $\delta_n \downarrow 0$ such that $L(\delta_n) \to 0$ by the fininteness. When $L(\delta_n) < \infty$, the curve $\varphi(\gamma_{\delta_n})$ has endpoints $\alpha_n, \beta_n \in \overline{\Omega}$, and both of these endpoints must lie on $\Gamma = \partial\Omega$. Indeed, if $\alpha_n \in \Omega$, then some point near α_n has two distinct pre-images in $\mathbb{D}$ because φ maps $\mathbb{D}$ onto Ω, and that is impossible as φ is one-to-one.

Furthermore, by the completeness of $\overline{\Omega}$ (closed and bounded thus by Heine-Borel theorem it is compact, and every compact subspace of a complete normed linear space is complete),

$$|\alpha_n - \beta_n| \le L(\delta_n) \to 0. \tag{1.24}$$

Let σ_n be that closed subarc of Γ having endpoints α_n and β_n and having small diameter. Then **(1.24)** implies that

$$\text{diam}(\sigma_n) \to 0,$$

because the Jordan curve Γ is homeomorphic to the circle. By the **Jordan curve theorem** (which states that $\mathbb{C}\backslash\Gamma$ is disconnected and consists of two

components, where Γ is a Jordan cuver), the curve

$$\sigma_n \cup \varphi(\gamma_{\delta_n})$$

devides the plane into two regions, and one of these regions, namely U_n, is bounded (again by **Jordan curve theorem**). Then $U_n \subset \Omega$, because $\mathbb{C}^\infty \backslash \overline{\Omega}$ is arcwise connected. Since

$$\operatorname{diam}(\partial U_n) = \operatorname{diam}\big(\sigma_n \cup \varphi(\gamma_{\delta_n})\big) \to 0,$$

we conclude that

$$\operatorname{diam}(U_n) \to 0. \tag{1.25}$$

Set

$$D_n := \mathbb{D} \cap \{z : |z - \zeta| < \delta_n\}.$$

We claim that

Claim: For n sufficiently large, $\varphi(D_n) = U_n$.

Suppose not, then by connectedness, $\varphi(\mathbb{D} \backslash \overline{D}_n) = U_n$ and

$$\operatorname{diam}(U_n) \geq \operatorname{diam}\Big(\varphi\big(B(0,1/2)\big)\Big) > 0.$$

Indeed, since D_n has diameter at most 1, and since D_n is centered at a point on $\partial\mathbb{D}$, it has $1/2$ inside $\mathbb{D}$ and $1/2$ outside $\mathbb{D}$, thus by the definition of U_n, U_n has diameter at least the diameter of $B(0,1/2)$. Then the result follows from the open mapping theorem. Finally, since this display holds, it would contradict our previous conslusion (1.25), thus for n sufficiently large, $\varphi(D_n)$ is necessarily U_n, proving the claim.

Therefore,

$$\operatorname{diam}\big(\varphi(D_n)\big) \to 0$$

and $\bigcap_n \overline{\varphi(D_n)}$ consists of a single point, because $\varphi(D_{n+1}) \subset \varphi(D_n)$. That means φ has a continuous extension to at $\zeta \in \partial\mathbb{D}$. Finally, as ζ is chosen arbitrary, we conclude that φ has a continuous extension to $\overline{\mathbb{D}}$.

Step II: φ is bijective, that is, it is one-to-one and onto.

Let $\widetilde{\varphi}$ denote the extension $\widetilde{\varphi} : \overline{\mathbb{D}} \to \overline{\Omega}$. Since $\widetilde{\varphi}(\mathbb{D}) = \Omega$, $\widetilde{\varphi}$ maps $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. To show that $\widetilde{\varphi}$ is one-to-one, suppose

$$\widetilde{\varphi}(\zeta_1) = \widetilde{\varphi}(\zeta_2) \text{ but } \zeta_1 \neq \zeta_2.$$

The argument used to show that $\alpha_n \in \Gamma$ also shows that $\widetilde{\varphi}(\partial\mathbb{D}) = \Gamma$, and so we can assume that $\zeta_j \in \partial\mathbb{D}$ for $j = 1,2$. The Jordan curve

$$\big\{\widetilde{\varphi}(r\zeta_1) : 0 \leq r \leq 1\big\} \cup \big\{\widetilde{\varphi}(r\zeta_2) : 0 \leq r \leq 1\big\}$$

bounds a domain $W \subset \Omega$, and then $\widetilde{\varphi}^{-1}(W)$ is one of the two components of

$$\mathbb{D}\backslash\Big(\{r\zeta_1 : 0 \leq r \leq 1\} \cup \{r\zeta_2 : 0 \leq r \leq 1\}\Big).$$

But since $\widetilde{\varphi}(\partial\mathbb{D}) \subset \Gamma$, one has

$$\widetilde{\varphi}\big(\partial\mathbb{D} \cap \partial\widetilde{\varphi}^{-1}(W)\big) \subset \partial W \cap \partial\Omega = \{\widetilde{\varphi}(\zeta_1)\}$$

and $\widetilde{\varphi}$ is constant on the arc $\partial\mathbb{D}$. It follows that $\widetilde{\varphi}$ is constant, either by Schwarz reflection principle (which states that an analytic function defined on some open set in the upper half of the complex plane can be extended across the real line) or by Jensen's formula (mean value equality), and this contradiction shows that

$$\widetilde{\varphi}(\zeta_1) \neq \widetilde{\varphi}(\zeta_2).$$

This proves the bijectivity of $\widetilde{\varphi}$, thus concluding the proof.

□

One can also prove that $\widetilde{\varphi}$ is one-to-one by repeating for $\widetilde{\varphi}^{-1}$ in the proof that φ is continuous. The Cauchy-Schwarz trick used to prove **(1.23)** is known as a length-area argument. The length-area method is the cornerstone of the theory of extremal length.

1.4 Distortion and the Hyperbolic Metric

Throughout, let $\mathbb{D}$ be the unit disc.

Definition: Hyperbolic Distance (over Unit Disc)

The hyperbolic distance from $z_1 \in \mathbb{D}$ to $z_2 \in \mathbb{D}$ is

$$\rho(z_1, z_2) = \rho_{\mathbb{D}}(z_1, z_2) := \inf \int_{z_1}^{z_2} \frac{|dz|}{1 - |z|^2}, \tag{1.26}$$

where the infimum is taken over all arcs in $\mathbb{D}$ connecting z_1 and z_2.

Remark 1.11: Hyperbolic Distance (over Unit Disc) Is Conformally Invariant

Let $\mathscr{M}$ denote the set of conformal self maps of $\mathbb{D}$:

$$T(z) := \lambda \frac{z - a}{1 - \bar{a}z}, \ a \in \mathbb{D} \text{ and } |\lambda| = 1.$$

When $zt \in \mathscr{M}$, we have

$$\frac{|T'(z)|}{1 - |T(z)|^2} = \frac{1}{1 - |z|^2},$$

and thus the hyperbolic distance is conformally invariant, namely,

$$\rho\big(T(z_1), T(z_2)\big) = \rho(z_1, z_2), \ T \in \mathscr{M}. \tag{1.27}$$

This conformal invariance is the main reason we are interested in the hyperbolic distance. ◇

Definition: Hyperbolic Metric

The hyperbolic metric is the infinitesimal form $\dfrac{|dz|}{1 - |z|^2}$ of the hyperbolic distance.

Remark 1.12: Hyperbolic Shortest Arc and Hyperbolic Length

Taking

$$T(z) := \frac{z - z_1}{1 - \bar{z}_1 z}$$

gives

$$\rho(z_1, z_2) = \rho\big(0, T(z_2)\big) = \int_0^{T(z_2)} \frac{|dz|}{1 - |z|^2}.$$

Therefore, the hyperbolically shortest arc from 0 to $T(z_2)$ is the radius $[0, T(z_2)]$, and its hyperbolic length is

$$\rho\big(0, T(z_2)\big) = \frac{1}{2} \log\left(\frac{1 + |T(z_2)|}{1 - |T(z_2)|}\right). \qquad ◇$$

In general,

$$\rho(z_1,z_2)=\frac{1}{2}\log\left(\frac{1+|\frac{z_2-z_1}{1-\bar z_1 z_2}|}{1-|\frac{z_2-z_1}{1-\bar z_1 z_2}|}\right), \tag{1.28}$$

and the hyperbolically shortest curve, or the geodesic, from z_1 to z_2 is a segment of diameter of $\mathbb{D}$ or an arc of a circle in $\mathbb{D}$ orthogonal to $\partial\mathbb{D}$.

By **(1.28)** we have

$$\left|\frac{z_2-z_1}{1-\bar z_1 z_2}\right|=\frac{e^{2\rho(z_1,z_2)}-1}{e^{2\rho(z_1,z_2)}+1}=\tanh\rho(z_1,z_2).$$

Denote

$$t:=t(d)=\tanh(d)=\frac{e^{2d}-1}{e^{2d}+1}.$$

Then the hyperbolic ball $B=\{z:\rho(z,a)<d\}$ is the Euclidean disc

$$\left\{z:\left|\frac{z-a}{1-\bar a z}\right|<t\right\},$$

and a conclusion shows that B has Euclidean radius

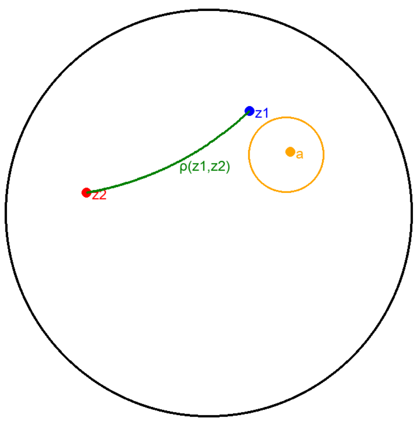
$$r(a,d)=\frac{t(1-|a|)^2}{1-t^2|a|^2} \tag{1.29}$$

and Euclidean distance to $\partial\mathbb{D}$

$$\mathrm{dist}(B,\partial\mathbb{D})=\left(\frac{1-t}{1+|a|t}\right)(1-|a|). \tag{1.30}$$

Therefore, if d is fixed, the Euclidean distance $\mathrm{dist}(B,\partial\mathbb{D})$ and the Euclidean diameter of B are both comparable to $\mathrm{dist}(a,\partial\mathbb{D})$.

However, if $a\neq 0$ the Euclidean center of B is not a. The following figure shows two hyperbolic balls with the same hyperbolic radius and two geodesics with the same hyperbolic length.



(Figure 1.4: Geodesics and Hyperbolic Balls)

Note that this figure confirms that the hyperbolic balls need not have their center being centered.

Now, assume that $\psi(z)$ is a univalent function in $\mathbb{D}$, that is, assume ψ is analytic and one-to-one on $\mathbb{D}$. After dilating, translating, and rotating the domain $\psi(\mathbb{D})$, ψ is normalized by $\psi(0) = 0$ and $\psi'(0) = 1$, so that

$$\psi(z) = z + a_2 z^2 + \cdots. \tag{1.31}$$

Important examples for univalent functions are the Koebe functions.

Definition: Koebe Function

The Koebe function is defined by

$$\psi(z) := \psi_\lambda(z) = \frac{z}{(1-\lambda z)^2}, \ |\lambda| = 1. \tag{1.32}$$

Note that

$$\psi_\lambda(z) = \sum_{n=1}^{\infty} n\lambda^{n-1} z^n$$

maps $\mathbb{D}$ to the complement of the radial slit $[-\bar{\lambda}/4, \infty]$.

Theorem 1.9: Koebe One-Quarter Theorem

Assume $\psi(z)$ is a univalent function on $\mathbb{D}$. If $\psi(z)$ has the form **(1.31)** then

$$|a_2| \leq 2 \tag{1.33}$$

and

$$\operatorname{dist}\big(0, \partial\psi(\mathbb{D})\big) \geq \frac{1}{4}. \tag{1.34}$$

Equality in **(1.33)** and **(1.34)** hold if and only if ψ is a Koebe function.

Note that this result tells us that the disc $\mathbb{D}$ under a univalent map always has radius at least $1/4$ and this bound is sharp.

Proof of **Theorem 1.9**:

We first prove that **(1.33)** implies **(1.34)**, then we prove **(1.33)**, and finally we deal with the equalities in both.

Step I: **(1.33)** $\Rightarrow$ **(1.34)**.

Suppose $w \notin \psi(\mathbb{D})$, we apply the noramlization argument. Let

$$g(z) := \frac{w\psi(z)}{w - \psi(z)},$$

note that g is a Möbius transformation. Moreover, one can check that

$$g(0) = 0 \text{ and } g'(0) = 1.$$

Therefore, g has the form **(1.31)**, namely,

$$g(z) = z + \left(a_2 + \frac{1}{w}\right) z^2 + \cdots,$$

so that by our assumption **(1.33)**,

$$\left| a_2 + \frac{1}{w} \right| \leq 2. \tag{1.35}$$

Now, **(1.35)** in conjunction with **(1.33)** yields

$$|w| \geq \frac{1}{4},$$

as desired.

Step II: **(1.33)** holds

To prove **(1.33)**, we define the odd function

$$f(z) := z\sqrt{\frac{\psi(z^2)}{z^2}} = z + \frac{a_2}{2}z^3 + \cdots.$$

By considering $h_1 : u = z^2$, $h_2 : \zeta = \psi(u)$, and $h_3 : f(z) = z\sqrt{\frac{\zeta}{u}}$, which are all univalent functions, thus their composition

$$f(z) := h_3 \circ h_2 \circ h_1(z)$$

is again univalent, and the $\mathbb{C}^\infty$-valued function

$$F(z) := \frac{1}{f(z)} = \frac{1}{z} - \frac{a_2}{2}z + \cdots = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \qquad (1.36)$$

is also univalent in $\mathbb{D}$. To complete the proof, we use the following lemma, which is called the Area lemma, for whose proof shall be established after we conclude the proof for **Theorem 1.9**.

Lemma 1.10: Area Theorem

If the univalent function $F(z)$ satisfies **(1.36)**, then

$$\sum_{n=1}^{\infty} n\,|\,b_n\,|^2 \le 1. \qquad (1.37)$$

Proof of **Theorem 1.9**: Continued

To establish **(1.33)** we apply **(1.37)** to $F := 1/f$. Since $b_1 = -a_2/2$ and $|\,b_1\,| \le 1$, we have $|\,a_2\,| \le 2$, proving **(1.33)**.

Step III: Equality in **(1.33)** and **(1.34)** holds $\Leftrightarrow$ ψ a is Koebe function

One can verify that equality in either **(1.33)** or **(1.34)** implies that ψ is a Koebe function. The converse holds by using **(1.32)**.

□

Proof of **Lemma 1.10**:

The lemma is called the "Area Theorem" because of its proof.

For $r < 1$, the Jordan curve

$$\Gamma_r := \{F(re^{i\theta}) : 0 \le \theta \le 2\pi\}$$

encloses an area $A(r)$, and by Green's theorem

$$A(r) = \frac{-i}{2}\int_{\Gamma_r} w\,d\overline{w} = \frac{-i}{2}\int_0^{2\pi} F(re^{i\theta})\frac{\partial \overline{F}}{\partial \theta}(re^{i\theta})d\theta.$$

Therefore by **(1.36)** and Fourier series expansion,

$$A(r) = \pi\left(\frac{1}{r^2} - \sum_{n=1}^{\infty} n\,|\,b_n\,|^2\, r^{2n}\right)$$

and

$$1 - \sum_{n=1}^{\infty} n\,|\,b_n\,|^2 = \lim_{r\to 1}\frac{A(r)}{\pi} \ge 0,$$

which yields **(1.37)**.

□

Theorem 1.11: Koebe's Estimate for Conformal Image

Let $\varphi(z)$ be a conformal mapping from the unit disc $\mathbb{D}$ onto a simply connected domain Ω. Then for all $z \in \mathbb{D}$,

$$\frac{1}{4}|\varphi'(z)|(1-|z|^2) \leq \operatorname{dist}\big(\varphi(z), \partial\Omega\big) \leq |\varphi'(z)|(1-z^2). \qquad (1.38)$$

Proof:

We shall prove the left hand side and the right hand side in **(1.38)** respectively.

Step I: LHS in **(1.38)**

Fix $z_0 \in \mathbb{D}$. Then the univalent function

$$\psi(z) = \frac{\varphi\left(\frac{z+z_0}{1+\overline{z_0}z}\right) - \varphi(z_0)}{\varphi'(z_0)(1-|z_0|^2)}$$

satisfies $\psi(0) = 0$ and $\psi'(0) = 1$. Hence if $w \notin \varphi(\mathbb{D})$, then by **(1.34)**,

$$\left|\frac{w-\varphi(z_0)}{\varphi'(z_0)(1-|z_0|^2)}\right| \geq \frac{1}{4}$$

and this gives the LHS in **(1.38)**.

Step II: RHS in **(1.38)**.

To prove the right hand side in **(1.38)**, fix $z \in \mathbb{D}$, take

$$f(w) := \varphi^{-1}\Big(\varphi(z) + \operatorname{dist}\big(\varphi(z), \partial\Omega\big)w\Big)$$

and apply the **Schwarz lemma**[3] at $w = 0$ to the function

$$g(w) := \frac{f(w) - z}{1 - \overline{z}f(w)}.$$

This yields the desired result.

□

We will often use the invariant form of **(1.38)**.

Corollary 1.11.1: Koebe's Estimate for Invariant Simply Connected Domain

Let ψ be a conformal mapping from a simply connected domain Ω_1 onto a simply connected domain Ω_2, and let $\psi(z_0) := w_0$. Then

$$\frac{|\psi'(z_0)|}{4} \leq \frac{\operatorname{dist}(w_0, \partial\Omega_2)}{\operatorname{dist}(z_0, \partial\Omega_1)} \leq 4|\psi'(z_0)|. \qquad (1.39)$$

Proof:

Applying **(1.38)** to

$$\varphi(z) := \psi\big(z_0 + \operatorname{dist}(z_0, \partial\Omega_1)z\big)$$

gives the left hand side of **(1.39)**. As for the right hand side in **(1.39)**, applying the same argument to ψ^{-1}.

□

Now we can extend our definition for hyperbolic distance over $\mathbb{D}$ to a simply connected domain $\Omega \subset \mathbb{C}$.

---

[3] **Theorem**: (Schwarz's Lemma) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then

(i) For all $z \in \mathbb{D}$, $|f(z)| \leq |z|$.

(ii) $|f'(0)| \leq 1$.

(iii) If either $f(z) = z$ for some non-zero $z \in \mathbb{D}$, or $|f'(0)| = 1$, then f is a rotation about 0.

Definition: Hyperbolic Distance (over Simply Connected Domain)

In a simply connected domain $\Omega \subset \mathbb{C}$, the hyperbolic distance is defined by moving back to $\mathbb{D}$ via a conformal map $\varphi : \mathbb{D} \to \Omega$. We write

$$\rho_\Omega(w_1, w_2) = \rho_\mathbb{D}(z_1, z_2)$$

where $\Omega \ni w_j := \varphi(z_j)$, $z_j \in \mathbb{D}, j = 1{,}2$.

Again, by **Remark 1.11**, $\rho_\Omega(w_1, w_2)$ does NOT depend on the choice of the conformal map φ.

Definition: Quasi-Hyperbolic Distance

The quasi-hyperbolic distance from $w_1 \in \Omega$ to $w_2 \in \Omega$ is defined by

$$Q_\Omega(w_1, w_2) := \inf \int_{w_1}^{w_2} \frac{|dw|}{\operatorname{dist}(w, \partial\Omega)},$$

in which the infimum is taken over all arcs in Ω joining w_1 and w_2.

Remark 1.13: Hyperbolic Distance Bound over Simply Connected Domain

Since **(1.38)** can be rewritten as

$$\frac{|dz|}{1 - |z|^2} \le \frac{|dw|}{\operatorname{dist}(w, \partial\Omega)} \le \frac{4|dz|}{1 - |z|^2},$$

where $w := \varphi(z)$, we have

$$\rho_\Omega(w_1, w_2) \le Q_\Omega(w_1, w_2) \le 4\rho_\Omega(w_1, w_2). \tag{1.40}$$

Consequently, the geometric statement following **(1.29)** and **(1.30)** about hyperbolic distances near $\partial\mathbb{D}$ remains approximately true in every simply connected domain with non-trivial boundary. ◇

Definition: Whitney Square

Let Ω be any proper open subset of $\mathbb{C}$. Then there exist closed squares $\{S_j\}_{j\ge1}$ having pairwise disjoint interiors and sides parallel to the axes, such that

(i) S_j has side length $\ell(S_j) = 2^{-n_j}$ for each $j \ge 1$.

(ii) $\Omega = \bigcup_{j\ge1} S_j$.

(iii) $\operatorname{diam}(S_j) \le \operatorname{dist}(S_j, \partial\Omega) < 4\operatorname{diam}(S_j)$.

The squares $\{S_j\}_{j\ge1}$ are called Whitney squares.

Here is one way to construct Whitney squares in the case $\operatorname{diam}(\Omega) < \infty$, the construction for the case $\operatorname{diam}(\Omega) = \infty$ is also possible, but here we leave it as an exercise. Let

$$2^{-N+1} \le \operatorname{diam}(\Omega) < 2^{-N+2}$$

and partition the plane into squares having sides parallel to the axes and side length 2^{-N}. We call these 2^{-N}-squares. The construction is done by induction.

We start with the base case. Include in the family $\{S_j\}_{j\ge1}$ any 2^{-N}-square $S \subset \Omega$ satisfying (iii), and divide each of the remaining 2^{-N}-squares into four squares of side length 2^{-N-1}.

Next, the induction step, include $\{S_j\}_{j\ge1}$ any of these new 2^{-N-1}-squares contained in Ω satisfying (iii), and continue.

Remark 1.14: Whitney Squares As Substitute for Hyperbolic Balls

Whitney squares can be viewed as replacement of hyperbolic balls since there are universal constants $r_1 < r_2$ such that each S_j contains a hyperbolic ball of radius r_1 and is contained in a hyperbolic ball of radius r_2. ◇

Remark 1.15: Whitney Squares Are ALMOST Conformal Invariant

Assume that Ω is simply connected, let $\varphi : \mathbb{D} \to \Omega$ be a conformal mapping, let $\{S_j\}_{j\geq 1}$ be the Whitney squares for Ω and let $\{T_k\}_{k\geq 1}$ be the Whitney squares for $\mathbb{D}$. Then by **(1.40)** there is a constant M, NOT depending on the choice of φ such that for each $k \geq 1$,

(a) $\varphi(T_k)$ is contained in at most M Whitney squares S_j.

(b) $\varphi^{-1}(S_k)$ is contained in at most M Whitney squares T_j.

In particular, for each $d > 0$, there is an $M(d)$ such that every hyperbolic ball $\{z \in \Omega : \rho_\Omega(z, a) < d\}$ in Ω is covered by $M(d)$ Whitney squares. ◇

Theorem 1.12: Growth, Distortion, and Angular Distortion for Univalent Maps

Let $\psi(z)$ be a univalent function satisfying $\psi(0) = 0$ and $\psi'(0) = 1$. Then

(i) $$\frac{|z|}{(1+|z|)^2} \leq |\psi(z)| \leq \frac{|z|}{(1-|z|)^2}. \qquad \text{(Growth Theorem)}$$

(ii) $$\frac{1-|z|}{(1+|z|)^3} \leq |\psi'(z)| \leq \frac{1+|z|}{(1+|z|)^3}. \qquad \text{(Distortion Theorem)}$$

(iii) $$\frac{1-|z|}{|z|(1+|z|)} \leq \frac{|\psi'(z)|}{|\psi(z)|} \leq \frac{1+|z|}{|z|(1-|z|)}. \qquad \text{(Angular Distortion)}$$

Note that, shapes in $\mathbb{D}$ are distorted under a univalent map according to ψ'. For instance, fast changes in the size of $|\psi'(z)|$ cause by nearby curves of the same length to be mapped to curves of very different length, or fast changes in $\arg\big(\psi'(z)\big)$ make straight line segments to be mapped to curves with sharp bends.

Proof of **Theorem 1.12**:

The critical inequality is (ii) and we shall prove (ii) first, then use (ii) to prove (i) and (iii) respectively.

Step I: Distortion theorem

Fix $z_0 \in \mathbb{D}$ and take

$$f(z) := \frac{\psi\left(\frac{z+z_0}{1+\bar{z}_0 z}\right) - \psi(z_0)}{\psi'(z_0)(1-|z_0|)^2}. \tag{1.41}$$

Then f is univalent on $\mathbb{D}$, $f(0) = 0$, and $f'(0) = 1$. Now,

$$\begin{aligned} |f''(0)| &= \left|\psi''(z_0) + \frac{1-|z_0|^2}{\psi'(z_0)} - 2\bar{z}_0\right| \quad \text{(Definition of } f\text{)} \\ &\leq 4 \quad \text{(Koebe's One-Quarter Theorem)} \end{aligned}$$

Substituting $z = re^{i\theta}$ into the above display yields

$$\left|\frac{e^{i\theta}\psi''(z_0)}{\psi'(z_0)} - \frac{2|z_0|}{1-|z_0|^2}\right| \leq \frac{4}{1-|z_0|^2}.$$

Applying the general formula of Radial derivative identity

$$\mathrm{Re}\left(\frac{zg'(z)}{|z|}\right) = \frac{\partial \mathrm{Re}(g)}{\partial r}$$

to $g := \log \psi'$ yields

$$\frac{2r-4}{1-r^2} \leq \frac{\partial}{\partial r} \log|\psi'(z_0)| \leq \frac{2r+4}{1-r^2}. \tag{1.42}$$

Integrating **(1.42)** along the radius [0,z] yields both inequalities in (ii).

Step II: Growth Theorem

To prove the upper bound in (i), integrate the upper bound in (ii) along [0,z].

To prove the lower bound, we can assume that $|\psi(z)| \leq \frac{1}{4}$, because

$$\frac{|z|}{(1+|z|)^2} \leq \frac{1}{4}.$$

Then by **(1.34)**, there exists an arc $\gamma \in \mathbb{D}$ with

$$\psi(\gamma) = [0,\psi(z)].$$

Integrating $|\psi'(z)||dz|$ along γ yields the desired lower bound.

Step III: Angular Distortion

Finally, applying (i) at $-z_0$ to the function f defined in **(1.41)** yields inequalities in (iii), concluding the proof.

□

Remark 1.16: Equalities in **Theorem 1.12** Holds ⇔ ψ Is Koebe

Once again, the equalities in **Theorem 1.12**, as well as in **(1.42)**, hold if and only if ψ is a Koebe function, following directly from **Theorem 1.9**. ◇

1.5 The Hayman-Wu Theorem

We give a very elementary proof, based o an idea of the late K.Øyma (1992), of the theorem of Hayman and Wu. Hayman-Wu theorem will be a recurrent topic throughout. This result states that the preimage of a line or circle L under a conformal mapping from the unit disc $\mathbb{D}$ to a simply connected domain Ω has total length bounded by an absolute constant. The best known constant is in $[\pi^2, 4\pi)$.

Theroem 1.13: Hayman-Wu Theorem

Let φ be a conformal mapping from $\mathbb{D}$ to a simply connected domain Ω and let L be any line. Then

$$\mathrm{length}\left(\varphi^{-1}(L \cap \Omega)\right) \leq 4\pi. \tag{1.43}$$

For the proof, we adapted the one developed by Øyma and modified by Rohde. It will be convenient to replace the hyperbolic metric $\rho(z_1, z_2)$ by the pseudohyperbolic metric.

Definition: Pseudohyperbolic Metric (over Unit Disc)

The pseudohyperbolic metric defined over $\mathbb{D}$ is given by

$$\delta_{\mathbb{D}}(z_1, z_2) := \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = \tanh \rho(z_1, z_2).$$

Definition: Pseudohyperbolic Metric (over Simply Connected Domain)

The pseudohyperbolic metric defined over a simplified connected domain Ω is given by

$$\delta_\Omega(w_1,w_2) := \delta_{\mathbb{D}}\big(\varphi^{-1}(w_1),\varphi^{-1}(w_2)\big).$$

Proof of Theorem 1.13:

We can assume that φ is analytic and one-to-one in a neighborhood of $\overline{\mathbb{D}}$ and that $L = \mathbb{R}$.

Step I: Construction of Jordan Domain

Let L_k denote the components of $\Omega \cap L$ and let Ω_k be that component of

$$\Omega \cap \{\bar{z} : z \in \Omega\}$$

such that $L_k \subset \Omega_k$. Then Ω_k is a Jordan domain symmetric about $\mathbb{R}$.

Step II: Construction of conformal mapping

By symmetry there is a conformal mapping $\psi_k : \Omega_k \to -i\mathbb{H}$ such that

$$\psi_k(L_k) = \mathbb{R}^+ \text{ and } \psi_k \text{ extends continuously to } \overline{\Omega_k}.$$

For $\zeta \in \partial\varphi^{-1}(\Omega_k) \cap \partial\mathbb{D}$, set

$$\alpha := \varphi(\zeta),\ x := |\psi_k(\alpha)|,\ \beta := \psi_k^{-1}(x), \text{ and } z := \varphi^{-1}(\beta).$$

Then the composition

$$\Phi := \varphi^{-1}\circ\psi_k^{-1}\big(|\psi_k\circ\varphi|\big)$$

(note that Φ is a composition of conformal mappings hence it is conformal) is a smooth map of

$$\varphi^{-1}\Big(\bigcup_{k\geq 1}\partial\Omega_k \cap \partial\Omega\backslash P\Big) \subset \partial\mathbb{D}$$

onto

$$\varphi^{-1}\Big(\bigcup_{k\geq 1} L_k\Big)\backslash\widetilde{P},$$

where P and $\widetilde{P}$ are finite sets. Now, to prove Hayman-Wu theorem, it suffices to prove the following claim.

Claim: $|\nabla\Phi| \leq 2$.

To prove this claim, suppose that $I = (\zeta, \widetilde{\zeta})$ is an open interval contained in $\varphi^{-1}(\partial\Omega_k) \cap \partial\mathbb{D}$. Set

$$\widetilde{\alpha} := \varphi(\widetilde{\zeta}),\ \widetilde{x} := |\psi_k(\widetilde{\alpha})|,\ \widetilde{\beta} := \psi_k^{-1}(\widetilde{x}), \text{ and } \widetilde{z} := \varphi^{-1}(\widetilde{\beta}).$$

Then by **Schwarz-Pick's therom**[4] using in the second relation, one has

$$\begin{aligned}
\delta_{\mathbb{D}}\big(\Phi(\zeta),\Phi(\widetilde{\zeta})\big) &= \delta_\Omega(\beta,\widetilde{\beta}) \quad \text{(definition of } \delta_\Omega\text{)}\\
&\leq \delta_{\Omega_k}(\beta,\widetilde{\beta}) \quad \text{(Schwartz-Pick's Theorem)}\\
&= \delta_{-\mathbb{H}}(x,\widetilde{x}) \quad (\psi_k \text{ is conformal and definition of } \beta,\widetilde{\beta})\\
&= \left|\frac{x-\widetilde{x}}{x+\widetilde{x}}\right| \quad \text{(definition of } \delta_{-i\mathbb{H}}\text{)}
\end{aligned}$$

Therefore, one has

$$\omega\Big(x,\psi_k\big(\varphi(I)\big),-i\mathbb{H}\Big) = \omega\big(z,I,\varphi^{-1}(\Omega_k)\big) \leq \omega\big(\Phi(\zeta),I,\mathbb{D}\big),$$

[4] **Theorem**: (Schwarz-Pick's Theorem) Suppose that $\psi : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then either ψ is a hyperbolic contraction, or ψ is a hyperbolic isometry.

where the first relation holds by the conformal invariance in **Remark 1.5** and the second holds by Lindelöf's maximum principle **Lemma 1.1**.

Finally, sending $\widetilde{\zeta} \to \zeta$ yields

$$\frac{|\nabla \Phi|}{1 - |\Phi(\zeta)|^2} \leq \pi \frac{1}{2\pi} \frac{1 - |\Phi(\zeta)|^2}{|\zeta - \Phi(\zeta)|^2}$$

by the growth theorem in **Theorem 1.12**. This concludes the proof.

□

Because it does not depend on the conformal mapping, the claim we proved in Hayman-Wu **Theorem 1.13** is actually stronger than **(1.43)**.

In chapter 7 we shall see that for some $1 < p < 2$, independent of φ,

$$\int_{L \cap \Omega} |(\varphi^{-1})'(w)|^p |dw| < C_p \tag{1.44}$$

but the largest permissible p is unkown. A slit disc shows that **(1.44)** fails at $p = 2$, and a counterexample for some $p < 2$, due to Baerstein, will be given in the eighth chapter. In chapter 10 we shall determine the class of curves L for which the Hayman-Wu thereom **(1.43)** holds.

Summary of Chapter 1

Solving the Dirichlet problem on a domain Ω is equivalent to constructing a harmonic measure on its boundary $\partial\Omega$. Our aim is to let the domain Ω and the boundary condition be as general as possible, we first construct the harmonic measures in nice domains. Before the construction of harmonic measure in any of the domains, we need to demonstrate what properties are desired: Some elementary properties (**Remark 1.1**), conformal invariance (**Remark 1.5**), and the Harnack's inequality (**Remark 1.4**).

In the first section, we start with construction of the **Harmonic Measure (for Set of Finite Union in Half Plane)**. The uniqueness is guaranteed by Lindelöf's maximum principle **Lemma 1.1** (and so are the later versions). We formulated the Dirichlet problem on upper half plane and proved the desired solutions via the harmonic measure — Existence and Uniqueness for Solution to Dirichlet Problem on $\mathbb{H}$ in **Theorem 1.2**. Then we extend our definition of harmonic measure from finite union in $\mathbb{H}$ to **Harmonic Measure (for Measurable Set on Half Plane)**; we defined **Poisson Kernel (over Half Plane)** and **Poisson Integral (over Half Plane)**. The conformal invariance of harmonic measure (**Remark 1.5**) enables us to define **Harmonic Measure (for Set of Finite Union over Unit Disc)**, which in turn formulates Poisson integral formula in **Theorem 1.3**. The corresponding Poisson kernel and Poisson integral on $\mathbb{D}$ are formulated, as well as the Dirichlet problem on $\mathbb{D}$.

We now have harmonic measures over $\mathbb{H}$ and $\mathbb{D}$, which are related through a conformal mapping. For us to extend the definition of harmonic measure to a general domain, we need to make sure that the conformal mapping always does the job correctly. This leads us to consider one of the most extreme case - **Non-Tangential Limit (over Unit Disc)**. To control the non-tangential limit, it suffices to control the

Non-Tangential Maximal Function (over Unit Disc), which happens to be bounded above by **Hardy-Littlewood Maximal Function** via **Lemma 1.5**, as for the Hardy-Littlewood maximal function, it is also weak type 1-1 by **Lemma 1.7**. Together, we prove Fatou's **Theorem 1.4**, which tells us that any positive harmonic function on $\mathbb{D}$ possesses a non-tangential limit at almost all boundary points. Finally, we formulate and solve the Dirichlet problem over unit disc with bounded boundary data in **Corollary 1.4.1**. So far we have relaxed the boundary condition but we have not generalized the domain, as a trade off we see that the harmonic measure over $\mathbb{D}$ is the indicator function along the non-tangential limit (**Remark 1.9**).

In the second section we relaxed the conditions in Dirichlet problem but we did not extend the underlying domains, we now do it in the third section by proving the Carathéodory's **Theorem 1.8** that extends a conformal mapping φ from $\mathbb{D}$ onto a Jordan domain Ω, to the conformal mapping $\widetilde{\varphi}$ from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$, such that $\widetilde{\varphi}\,|_{\mathbb{D}} = \varphi$. We formulate the solution to Dirichlet problem over Jordan domain with bounded boundary data, therefore a harmonic measure over Jordan domain. This construction tells us that the questions about harmonic measure on Jordan domains are equivalent to the questions about the boundary behavior of conformal mappings (**Remark 1.10**).

Extension of harmonic measure from $\mathbb{H}$ to $\mathbb{D}$ and finally to simply connected domain Ω such that the extension solves the Dirichlet problem over Ω and relaxes the boundary data from continuity to boundedness. All the constructions are dependent on the behavior of conformal mappings, then it is natural to consider how good the conformal mappings are, especially how good is its image, as we do not wish change in the boundary data. To answer this, in the fourth section, we define the conformal invariant hyperbolic distance and hyperbolic metric. We consider the univalent function ψ on $\mathbb{D}$ which is analytic, one-to-one, $\psi(0) = 0$, and $\psi'(0) = 1$. A particular example for this function is Koebe's function. We proved Koebe's One Quarter **Theorem 1.9**, which tells us that the disc under univalent maps always has radius at least $1/4$ and this bound is sharp. With the help of this result, we are able to estimate the image of $\mathbb{D}$ under conformal mappings (**Theorem 1.11**), as well as for image of simply connected domains (**Corollary 1.11.1**). Finally, we defined the **Whitney Square**, which is an almost conformal invariant substitute for hyperbolic balls (**Remark 1.15** and **Remark 1.14** respectively). We proved the Growth rate, Distortion, and Angular Distortion for univalent mappings in **Theorem 1.12**. All the inequalities in our estimates are equalities provided the univalent function is Koebe.

As an application, we prove the Hayman-Wu **Theorem 1.13**, which states that the preimage of a line or circle L under a conformal mapping from $\mathbb{D}$ to a simply connected domain Ω has total length bounded by an absolute constant. Moreover, the best known value is somewhere in $[\pi^2, 4\pi)$.

2. Finitely Connected Domains

In this chapter we solve the Dirichlet problem on a domain bounded by a finite number of Jordan curves. For a simply connected Jordan domain the problem was solved in the first chapter via Carathéodory's **Theorem 1.8**. For a multiple connected domain the problem will be reduced to the simply connected case using the Schwarz alternating method.

Solving the Dirichlet problem on a domain Ω is equivalent to constructing a harmonic measure on $\partial\Omega$. In the second section we describe harmonic measures in terms of the normal derivative of Green's fnction in the case when $\partial\Omega$ consists of analytic curves. In the fourth section we study the relation between the smoothness of $\partial\Omega$ and the smoothness of the Poisson kernel (the Radon-Nikodym derivative of harmonic measure agains arc length). This relation hinges on two classical estimates for conjugate functions which we shall prove in the third section.

2.1 The Schwarz Alternating Method

We start with two definitions and a result.

Definition: Finitely Connected Jordan Domain

Let Ω be a plane domain such that $\partial\Omega$ is a finite union of pairwise disjoint Jordan curves

$$\partial\Omega := \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_p.$$

We say Ω is a finintely connected Jordan domain.

Definition: Piecewise Continuous Function

A bounded funtion f on $\partial\Omega$ is said to be piecewise continuous if there is a finite set $E \subset \partial\Omega$ such that

(i) f is continuous on $\partial\Omega \backslash E$.

(ii) f has left and right limits at each point of E.

In this section we solve the Dirichlet problem for piecewise continuous boundary functions on a finitely connected Jordan domain. For the sakeness of simplicity, we shall denote F.C.J.D. for finitely Connected Jordan domain whenever necessary.

Remark 2.1: In Proving Solution to DP We Can Assume Bounded Domain

There is a technique which we did not use in the first chapter but will be used quite often later. This technique is that, in proving solution to Dirichlet problem over simply connected set Ω, we can always assume Ω is bounded. Indeed, the Riemann Mapping Theorem establishes a conformal equivalence between any proper simply connected domain and bounded unit disc $\mathbb{D}$. ◇

Theorem 2.1: Solution to DP on F.C.J.D. with Bounded Piecewise Continuous Data

Let Ω be a finitely connected Jordan domain and let f be a bounded piecewise continuous function on $\partial\Omega$. Then there exists a unique function $u(z) = u_f(z)$, bounded and harmonic on Ω such that

$$\lim_{z\to\zeta} u(z) = f(\zeta) \tag{2.1}$$

at every point of continuity ζ of f. Moreover,

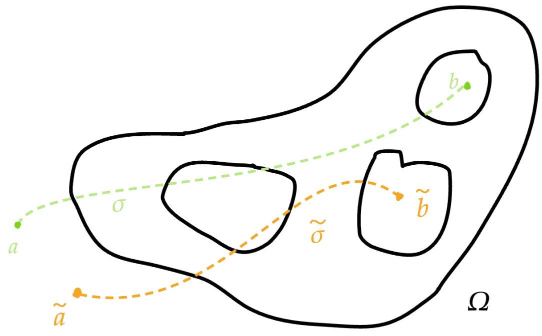
$$\sup_{\Omega} |u| \le \sup_{\partial\Omega} |f|. \tag{2.2}$$

Proof:

By **Remark 2.1**, we may assume, without loss of generality, that Ω is bounded. The uniqueness of u_f is an immediate consequence from Lindelöf's maximum principle **Lemma 1.1**.

The existence of u_f in case $p = 1$, that is, when Ω is a Jordan domain, was formulated in the definition of **Harmonic Measure (over Jordan Domain)**. Therefore, we may, without loss of generality, assume that $p > 1$.

Step I: Construct Dirichlet problems inductively on the Jordan domain



(Figure 2.1: Separate Finitely Connected Jordan Domain via Jordan Arcs)

Take a Jordan arc σ with endpoints $a, b \in \overline{\Omega}$ such that $\Omega_1 := \Omega \backslash \sigma$ is simply connected and such that $\sigma \cap \partial\Omega$ is a finite set.

Then

$$\varphi(z) := \sqrt{\frac{z-a}{z-b}}$$

has a single-valued analytic branch defined on Ω_1, and we can solve the Dirichlet problem on Ω_1 by translating it to the Jordan region $\varphi(\Omega_1)$.

Take a second Jordan arc $\widetilde{\sigma}$ such that

$\widetilde{\Omega}_1 := \Omega \backslash \widetilde{\sigma}$ is simply connected, $\widetilde{\sigma} \cap \partial\Omega$ is a finite set, and $\sigma \cap \widetilde{\sigma} = \varnothing$.

We can also solve the Dirichlet problem on $\widetilde{\Omega}_1$.

Step II: Construct Harmonic Function by Harnack's Principle

Continuing our induction on the Jordan domain in the first step, we shall construct solutions to each subdomain iteratively. This step shall give us a sequence of bounded positive harmonic functions, and then Harnack's theorem tells us that this sequence converges to a bounded harmonic function.

Let $E \subset \partial\Omega$ be a finite set and define

$$F := E \cup (\sigma \cap \partial\Omega) \cup (\widetilde{\sigma} \cap \partial\Omega).$$

Suppose, without loss of generality, that $f \in C(\partial\Omega \backslash E)$ is positive and bounded.

To start, let u_1 be the solution to the Dirichlet problem on Ω_1 with boundary value

$$u_1(\zeta) := \begin{cases} f(\zeta), & \zeta \in \partial\Omega \\ \max_{\partial\Omega} f, & \zeta \in \sigma \end{cases}$$

Then u_1

(a) is harmonic on Ω_1 and continuous on $\overline{\Omega} \backslash F$.

(b) matches its boundary data on $\partial\Omega_1 \backslash F$.

Next, let $\widetilde{u}_1$ be the solution to Dirichlet problem on $\widetilde{\Omega}_1$ with boundary data $u_1(\zeta), \zeta \in \partial\widetilde{\Omega}_1$. Then $\widetilde{u}_1$

(a') is harmonic on $\widetilde{\Omega}_1$ and continuous on $\overline{\Omega}\backslash F$.

(b') matches its boundary data on $\partial\Omega_1\backslash F$.

In particular, by (a) and (a'),

$$u_1 = \widetilde{u}_1 \text{ on } \partial\Omega\backslash F.$$

By Lindelöf's maximum principle **Lemma 1.1**, we have

$$\widetilde{u}_1 \le \max_{\partial\Omega} f =: u_1 \text{ on } \sigma \cap \Omega$$

and therefore

$$\widetilde{u}_1 \le u_1 \text{ on } \overline{\Omega}\backslash F.$$

Now, let u_2 be the solution to the Dirichlet problem on Ω_1 with boundary data $\widetilde{u}_1(\zeta), \zeta \in \partial\Omega_1$. On $\sigma \cap \Omega$ we have

$$u_2 = \widetilde{u}_1 \le u_1,$$

while on $\partial\Omega\backslash F$ we have

$$u_2 = f = u_1.$$

Therefore by Lindelöf's maximum principle **Lemma 1.1**,

$$u_2 \le u_1 \text{ on } \Omega_1.$$

Consequently

$$u_2 \le \widetilde{u}_1 \text{ on } \overline{\Omega}\backslash F.$$

Continuing this way we obtain a decreasing sequence

$$u_1 \ge \widetilde{u}_1 \ge u_2 \ge \widetilde{u}_2 \ge u_3 \ge \cdots$$

of positive functions, which are harmonic on Ω_1 and $\widetilde{\Omega}_1$ respectively. Now by Harnack's principle, the limit

$$u(z) = \lim_{n\to\infty} u_n(z) = \lim_{n\to\infty} \widetilde{u}_n(z)$$

is a bounded harmonic function on Ω such that

$$u \le u_1 \le \max_{\partial\Omega} f.$$

Step III: Our harmonic function solves DP with given boundary condition
To complete the proof, it suffices to prove

$$\lim_{z\to\zeta} u(z) = f(\zeta)$$

whenever $\zeta \in \partial\Omega$ is a point of continuity of f.
We may assume that $\zeta \notin \sigma \cup E$. Take a neighborhood V of ζ such that

$W := V \cap \Omega$ is a Jordan domain and such that

$$W \cap (E \cup \sigma) = \varnothing.$$

Let φ be a conformal map from $\mathbb{D}$ onto W. By Carathéodory's **Theorem 1.8** $\varphi(\partial\mathbb{D}) = \partial W$, and for $w := \varphi(z) \in W$,

$$u_n(w) = \int_{\varphi^{-1}(\partial\Omega)} \frac{1-|z|^2}{|e^{i\theta}-z|^2} f\circ\varphi(e^{i\theta})\frac{d\theta}{2\pi} + \int_{\partial\mathbb{D}\backslash\varphi^{-1}(\partial\Omega)} \frac{|1-|z|^2}{e^{i\theta}-z|^2} u_n\circ\varphi(e^{i\theta})\frac{d\theta}{2\pi}.$$

Because $\varphi^{-1}(\partial\Omega)$ is a neighborhood of $\varphi^{-1}(\zeta)$ in $\partial\mathbb{D}$, the first integral approaches $f(\zeta)$ as $w \to \zeta$.

Finally, because $|u_n| \le \sup_{\partial\Omega} |f|$, the second integral tends to 0, uniformly in n as $w \to \zeta$. Therefore (2.1) holds and thus the proof is complete.

□

Combining steps 1-3 in the proof for **Theorem 2.1**, this construction for the proof of solution to Dirichlet problem with boundary data is called Schwarz alternating method. This method worth a remark to demonstrate how it works.

Remark 2.2: Schwarz Alternating Method

When working with finding solution to Dirichlet problem over a finitely connected Jordan domain, the Schwarz alternating method is the collection of the following steps:

(i) Use Jordan arcs to partition the finitely connected Jordan domain into disjoint subdomains, this relaxes the original problem to the Dirichlet problem over a Jordan domain.

(ii) Solve Dirichlet problem over each subdomains, then use Lindelöf's maximal principle **Lemma 1.1** to make sure the positive bounded harmonic functions converge locally uniformly to zero (that is, they are decreasing). Then by Harnack's theorem they converge to a bounded harmonic function.

(iii) Show that the harmonic function we obtained satisfies the boundary condition. ⋄

As we have proved the existence and the uniqueness for the solution to the Dirichlet problem over finitely connected Jordan domains with bounded piecewise continuous boundary data, the harmonic measure in this case is also characterized.

Definition: Harmonic Measure (over Finitely Connected Jordan Domain)

If Ω is a finitely connected domain. **Theorem 2.1** shows that the map

$$f \mapsto u_f(z)$$

is a bounded linear functional on $C(\partial\Omega)$.The harmonic measure of a relatively **open subset** $U \subset \partial\Omega$ is therefore defined by

$$\omega(z, U) := \omega(z, U, \Omega) := \sup\{u_f(z) : f \in C(\partial\Omega), 0 \le f \le 1_U\},$$

and of an **arbitrary subset** $E \subset \partial\Omega$ is

$$\omega(z, E) := \omega(z, E, \Omega) := \inf\{\omega(z, U) : U \text{ open in } \partial\Omega, U \supseteq E\}.$$

Remark 2.3: Harmonic Measure over Finitely Connected Jordan Domain Is Borel

The above definition, which mimics the usual proof of Riesz representation theorem, shows that $\omega(z, E)$ is a Borel measure on $\partial\Omega$ such that

$$u_f(z) = \int_{\partial\Omega} f(\zeta) d\omega(z, \zeta) \qquad (2.3)$$

for f continuous. ⋄

Note that when Ω is connected, this definition of harmonic measure agrees with the definition in **Harmonic Measure (over Jordan Domain)**.

Remark 2.4: Harmonic Measure Satisfies Harnack's Inequality

For every $z_1, z_2 \in \Omega$ there exists, by virtue of Harnack's inequality, a constant $c := c(z_1, z_2)$ such that

$$\frac{1}{c}\omega(z_1, E) \le \omega(z_2, E) \le c\omega(z_1, E) \tag{2.4}$$

and the constants $c(z_1, z_2)$ remain uniformly bounded if z_1 and z_2 remain in a compact subset of Ω. ◇

It is natural to ask, as **Remark 2.2** suggests, that if the boundary condition can be weakened to bounded and Borel? The answer is positve. We shall demonstrate this into the following example.

Example 2.1: Boundary Data over F.C.J.D. Can Be Weakened to Bounded Borel

If $f \in L^1(\partial\Omega, d\omega)$, and in particular if f is bounded and Borel, there is a sequence $\{f_n\}_{n\ge 1} \subset C(\partial\Omega)$ such that for some fixed $z_0 \in \Omega$,

$$\int |f_n(\zeta) - f(\zeta)|\, d\omega(z_0, \zeta) \to 0.$$

Denote

$$u_n(z) := \int f_n(\zeta) d\omega(z, \zeta)$$

and

$$u(z) := u_f(z) = \int f(\zeta) d\omega(z, \zeta) \tag{2.5}$$

Then by **(2.4)**,

$$u_n(z) \to u(z) \text{ for all } z \in \Omega.$$

Again by **(2.4)**, we also see that the harmonic functions $\{u_n(z)\}_{n\ge 1}$ are uniformly bounded on compact subsets of Ω. Then by Harnack's principlet the limit function $u(z)$ is harmonic on Ω. ◇

Note that in **Example 2.1**, we did not use Schwarz alternating method directly. We shall give a reason why after the new version for definition of solution to Dirichlet problem.

Definition: Solution to DP over F.C.J.D. with Bounded Borel Boundary Data

The harmonic function u defined in **(2.5)** is called the solution to the Dirichlet problem for f on Ω. If f is bounded, then we also have

$$\sup_{z\in\Omega} |u_f(z)| \le \|f\|_{L^\infty(\Omega, d\omega)}.$$

Moreover, if f is bounded and continuous at $\zeta \in \partial\Omega$, then

$$\lim_{z\to\zeta} u(z) = f(\zeta).$$

The reason we cannot apply Schwarz alternating method in this situation is that the condition for applying Lindelöf's maximal principle **Lemma 1.1** is not satisfied. That is to say, to apply Schwarz alternating method we need to guarantee the condition for Lindelöf's maximal principle, as well as the condition for Harnack's theorem.

Remark 2.5: Condition for Applying Schwarz Alternating Method

Note that the Schwarz alternating method cannot be applied directly to a bounded Borel function because the conditions of Lindelöf's maximal principle **Lemma 1.1** holds only for piecewise continuous functions. ◇

The next section will give a much more explicit description of the measure $\omega(z, E)$ when $\partial\Omega$ has some additional smoothness.

2.2 Green Functions and Poisson Kernels

Again let Ω be a finitely connected Jordan domain, and assume that Ω is bounded. For fixed $\omega \in \Omega$, let $h(z,\omega)$ be the solution to the Dirichlet problem for the boundary value

$$f(\zeta) = \log|\zeta - \omega| \in C(\partial\Omega), \zeta \in \partial\Omega,$$

and define

Definition: Green Function with Pole (over Bounded Domain)

The Green function with pole ω is defined by

$$g(z,\omega) := \log\frac{1}{|z-\omega|} + h(z,\omega). \tag{2.6}$$

Remark 2.6: Some Elementary Properties of Green Function with Pole

The Green function with pole ω has the following properties:

(i) $g(z,\omega)$ is continuous in $z \in \overline{\Omega}\backslash\{\omega\}$.

(ii) $g(z,\omega) > 0$ on Ω.

(iii) $g(\zeta,\omega) = 0$ on $\partial\Omega$.

(iv) $z \mapsto g(z,\omega)$ is harmonic on $\Omega\backslash\{\omega\}$.

(v) $z \mapsto g(z,\omega) - \log\dfrac{1}{|\omega - z|}$ is harmonic at ω. ⋄

The properties are easily derived from **Theorem 2.1** and the definition **(2.6)**. By the Lindelöf's maximum principle **Lemma 1.1**, (iii), (iv), and (v) determine $g(z,\omega)$ uniquely.

Definition: Green Function with Pole (over Unbounded Domain)

When Ω is unbounded, we fix $a \notin \overline{\Omega}$, then we use Poisson kernel or the inversion argument to define the Green function.

(i) For $\omega \neq \infty$, we let $h(z,\omega)$ solve the Dirichlet problem on Ω for

$$f(\zeta) := \log\left|\frac{\zeta-\omega}{\zeta-a}\right|,$$

and define

$$g(z,\omega) := \log\left|\frac{z-a}{z-\omega}\right| + h(z,\omega).$$

(ii) For $\omega = \infty$, we instead use inversion

$$f(\zeta) := \log\left|\frac{1}{\zeta-a}\right|$$

to define $h(z,\infty)$ and set

$$g(z,\infty) := \log|z-a| + h(z,\infty).$$

These definitions are independent of the choice of a, and with them the properties in **Remark 2.6** still holds and (iii), (iv), and (v) still determine $g(z,\omega)$ uniquely.

Now it is natural to consider the Green function under the conformal mapping.

Definition: Green Function with Pole (under Conformal Mapping)

Suppose φ is a conformal mapping from one finitely connected Jordan domain Ω onto another finitely connected Jordan domain $\widetilde{\Omega}$. Then

$$\varphi(z) \to \partial\widetilde{\Omega} \ \forall z \to \partial\Omega$$

since $\varphi : \Omega \to \widetilde{\Omega}$ is a homeomorphism. It follows that the Green function with pole ω under conformal mapping is

$$g_{\widetilde{\Omega}}\big(\varphi(z), \varphi(\omega)\big) = g_{\Omega}(z, \omega), \tag{2.7}$$

because Green function is uniquely determined by (iii), (iv), and (v) in **Remark 2.6**.

Remark 2.7: Green Function with Pole over F.C.J.D. Is Conformal Invariant

If Ω is the unit disc $\mathbb{D}$ then

$$g(z, \omega) = \log \left| \frac{1 - z\overline{\omega}}{z - \omega} \right|. \tag{2.8}$$

Consequently Green function for any simply connected Jordan domain Ω can be expressed in terms of the conformal mapping $\psi : \Omega \to \mathbb{D}$. ⋄

Theorem 2.2: Green Function as Log of Conformal Mapping over F.C.J.D.

Let Ω be a simply connected domain bounded by a Jordan curve, let $\omega \in \Omega$ and let $\psi : \Omega \to \mathbb{D}$ be a conformal mapping with $\psi(\omega) = 0$. Then

$$g(z, \omega) = -\log|\psi(z)|.$$

Proof:

Direct calcuation gives

$$\begin{aligned} g_{\Omega}(z, \omega) &= g_{\mathbb{D}}\big(\psi(z), \psi(\omega)\big) \\ &= g_{\mathbb{D}}\big(\psi(z), 0\big) \quad \text{(by assumption } \psi(\omega) = 0) \\ &= \log \left| \frac{1 - \psi(z) \cdot 0}{\psi(z) - 0} \right| \\ &= -\log|\psi(z)| \end{aligned}$$

where the first equality holds by the definition in (2.7) and the third equality by (2.8) in **Remark 2.7**.

□

Definition: Analytic Arc

An analytic arc is the image $\psi\big((-1,1)\big)$ of the open interval under a one-to-one and analytic map ψ defined on a neighborhood of $(-1,1)$.

Definition: Jordan Analytic Curve

A Jordan analytic curve is a Jordan curve that is a finite union of (open) analytic arcs.

The following lemma states that every finitely connected Jordan domain has a representation whose boundary contains pairwise disjoint analytic Jordan curves (thus the Schwarz alternating method **Remark 2.2** may be applied) and there exists an homeomorphic extension to the boundary. Note that this result does not tell us that the Schwarz alternating method is closed under finitely many conformal mappings, as **Remark 2.5** already told us that for this method to work, the boundary data must be at least piecewise continuous.

Lemma 2.3: F.C.J.D. Has Partition and Homoemorphism Extension on Boundary

Let Ω be a finitely connected Jordan domain. Then there exists a finitely connected Jordan domain Ω^* such that

(i) $\partial\Omega^*$ consists of finitely many pairwise disjoint analytic Jordan curves.

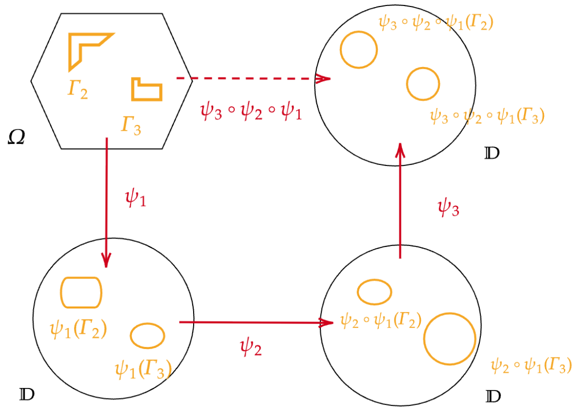
(ii) There exists a conformal map from Ω onto Ω^* which extends to be a homeomorphism from $\overline{\Omega}$ onto $\overline{\Omega}^*$.

Proof:

Denote, for $p \geq 1$,

$$\partial\Omega := \Gamma_1 \cup \cdots \cup \Gamma_p$$

where each Γ_p is a Jordan curve, $1 \leq j \leq p$. We proceed the proof with induction.



(Figure 2.2: Inductive construction for the proof)

Base Step:

Let Ω_1 be the component of $\mathbb{C}^\infty \backslash \Gamma_1$ containing Ω, and let ψ_1 be a conformal map from Ω_1 onto $\mathbb{D}$.

Induction Step:

Let Ω_2 be the component of $\mathbb{C}^\infty \backslash \psi_1(\Gamma_2)$ containing $\psi_1(\Omega)$, and let ψ_2 be a conformal map from Ω_2 onto $\mathbb{D}$. Repeating this process for each bounded curve, we obtain a conformal map ψ_p from Ω to a region Ω^* such that $\partial\Omega^*$ consists of finitely many pairwise disjoint analytic Jordan curves. Applying Carathéodory's **Theorem 1.8** to each ψ_k, we see that ψ_p extends to a homeomorphism from $\overline{\Omega}$ onto $\overline{\Omega}^*$.

□

Theorem 2.4: Green Function with Pole is Symmetric over F.C.J.D.

Let Ω be a finitely connected Jordan domain and let $z_1, z_2 \in \Omega$. Then

$$g(z_1, z_2) = g(z_2, z_1). \tag{2.9}$$

Proof:

By **Lemma 2.3** (i), we may assume that $\partial\Omega$ consists of analytic Jordan curves. When $\partial\Omega$ consists of analytic curves, an argument of Schwarz reflection principle, which we shall use many times and in proving the **Lemma 2.5**, shows that there is a neighborhood V of $\partial\Omega$ to which $z \mapsto g(z, \omega)$ has a harmonic extension. Hence $g(z, \omega)$ is analytic on some neighborhood V of $\partial\Omega$ and we can use Green's theorem in the form

$$\iint_{\mathcal{U}} (u\Delta v - v\Delta u)dxdy = \int_{\partial \mathcal{U}} \left(u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}}\right)ds,$$

where $\vec{n}$ is the unit vector pointing out from the domain $\mathcal{U}$. Fix distinct z_1, z_2 in Ω, we apply Green's theorem on the domain

$$\Omega_\varepsilon := \Omega \backslash \big(\{|z - z_1| \le \varepsilon\} \cup \{|z - z_2| \le \varepsilon\}\big),$$

when ε is small, with

$$u(z) := g(z, z_1) \text{ and } v(z) := g(z, z_2).$$

Now by **Remark 2.6** (iii), $u = v = 0$ on $\partial\Omega$, thus

$$\int_{\partial\Omega} \left(u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}}\right)ds = 0.$$

Now, by **Remark 2.6** (iv), u and v are harmonic on Ω_ε, the area integral in Green's theorem vanishes. We conclude that

$$\begin{aligned}
&\varepsilon\int_0^{2\pi} g(z_1 + \varepsilon e^{i\theta}, z_1)\frac{\partial}{\partial r}g(z_1 + \varepsilon e^{i\theta}, z_2)\frac{d\theta}{2\pi} \\
&\qquad -\varepsilon\int_0^{2\pi} g(z_1 + \varepsilon e^{i\theta}, z_2)\frac{\partial}{\partial r}g(z_1 + \varepsilon e^{i\theta}, z_1)\frac{d\theta}{2\pi} \\
&= \varepsilon\int_0^{2\pi} g(z_2 + \varepsilon e^{i\theta}, z_2)\frac{\partial}{\partial r}g(z_2 + \varepsilon e^{i\theta}, z_1)\frac{d\theta}{2\pi} \\
&\qquad -\varepsilon\int_0^{2\pi} g(z_2 + \varepsilon e^{i\theta}, z_1)\frac{\partial}{\partial r}g(z_2 + \varepsilon e^{i\theta}, z_2)\frac{d\theta}{2\pi}
\end{aligned} \tag{2.10}$$

For ε sufficiently small, using the indices we used in **Lemma 2.3** (i),

$$g(z_j + \varepsilon e^{i\theta}, z_j) \le 2\log\left(\frac{1}{\varepsilon}\right) \ \forall k \ne j,$$

we see that $g(z, z_j)$ has bounded derivatives near z_k (since bounded analytic Green function has locally bounded derivatives). This means that the first and the third integrals in **(2.10)** tends to 0 as $\varepsilon \to 0$. Now by **(2.6)**,

$$-\frac{\partial}{\partial r}g(z_j + \varepsilon e^{i\theta}, z_j) = \frac{\partial}{\partial \varepsilon}\log\varepsilon + O(1),$$

so that, as $\varepsilon \downarrow 0$, the second integral in **(2.10)** tends to $g(z_1, z_2)$ and the fourth integral in **(2.10)** tends to $g(z_2, z_1)$. Finally, by **Remark 2.6** (ii), the Green function is positive on Ω, it follows that the equality holds and hence the symmery is proved.

□

The Schwarz reflection principle argument in the proof is of frequent use, so we write it into the following remark.

Remark 2.8: Schwarz Reflection Principle Extends Harmonic Locally on Boundary

The Schwarz reflection principle does not only apply to the reflection over the real axis, it also applies to a neighborhood of the boundary $\partial\Omega$, that is, there is a neighborhood V of $\partial\Omega$ to which $z \mapsto g(z, \omega)$ has a harmonic extension. ◇

Lemma 2.5: Sufficiency for Harmonic Extension to Analytic Curve over F.C.J.D.

Suppose Ω is a finitely connected Jordan domain and suppose $\gamma \subset \partial\Omega$ is an analytic arc. Let $u(z)$ be a harmonic function in Ω.

(a) If $\lim_{z\to\zeta} u(z) = 0 \ \forall \zeta \in \gamma$, then there exists an open set $W \supset \gamma \cup \Omega$ such that u extends to be harmonic on W.

(b) If in addition that $u(z) > 0$ on Ω, then

$$\frac{\partial}{\partial \vec{n}} u(\zeta) < 0 \ \forall \zeta \in \gamma. \tag{2.11}$$

Proof:

We first show that there exists a harmonic extension, then we prove assertion (a) and assertion (b).

Step I: Existence of harmonic extension over neighborhood of ζ.

Let $\gamma \in \Gamma$, where Γ is a Jordan curve bounding Ω. Because γ is an analytic arc, by **Remark 2.8**, there exists a neighborhood V of ζ and a conformal mapping $\psi : V \to \mathbb{D}$ such that for $\zeta \in \gamma$ fixed,

(i) $\psi(\zeta) = 0$.

(ii) $\psi(V \cap \Omega) := \mathbb{D}^- := \mathbb{D} \cap \{\mathrm{Im}(\omega) < 0\}$.

(iii) $\psi(\gamma \cap V) = (-1,1)$.

Set

$$v(\omega) := \begin{cases} u \circ \psi^{-1}(\omega), & \omega \in \mathbb{D}^- := \mathbb{D} \cap \{\mathrm{Im}(\omega < 0\} \\ -u \circ \psi^{-1}(\overline{\omega}), & \omega \in \mathbb{D}^+ := \mathbb{D} \cap \{\mathrm{Im}(\omega) > 0\} \\ 0, & \omega \in (-1,1) \end{cases}$$

Then by **Remark 2.6** (i), v is continuous in $\mathbb{D}$; by **Remark 2.6** (iv), v has mean value property over sufficiently small circles centered at any $\omega \in \mathbb{D}$. Hence v is harmonic in $\mathbb{D}$ and

$$\widetilde{u} := v \circ \psi$$

defines a harmonic extension of u to V.

Step II: Assertion (a)

Suppose $\widetilde{u}_1$ and $\widetilde{u}_2$ are extensions of u to a neighborhood V_1 and V_2 such that $V_1 \cap V_2 \cap \gamma$ is connected, then $\widetilde{u}_1 = \widetilde{u}_2$ in the component of $V_1 \cap V_2$ that contains $V_1 \cap V_2 \cap \gamma$. It follows that u has a harmonic extension to some open set $W \supset \gamma \cup \Omega$.

Step III: Assertion (b)

If in addition $u > 0$ on Ω, then clearly $\frac{\partial}{\partial \vec{v}} u \leq 0$ on γ since otherwise (a) does not hold. The inequality **(2.11)** then holds if and only if $\frac{\partial}{\partial y} v(0) < 0$. Thus it suffices to prove the following claim.

Claim: $\frac{\partial}{\partial y} v(0) < 0$.

We prove by contradiction. On $\mathbb{D}$, by Schwarz lemma, there exists an analytic function

$$h := \overline{v} - iv \text{ with } \mathrm{Im}(h) = -v \text{ and } h(0) = 0.$$

The Taylor expansion of h at 0 is

$$h(\omega) = a_n\omega^n + O(|\omega|^{n+1}), a_n \neq 0.$$

But if $n \geq 2$, then

$$h(\mathbb{D}^-) \cap \mathbb{D}^+ \neq \varnothing,$$

which is a contradiction. It follows that $a_1 := h'(0) = -\dfrac{\partial}{\partial y}v(0) \neq 0$. But $\widetilde{u} := v \circ \psi$ and $\dfrac{\partial}{\partial \vec{v}}u \leq 0$ on γ, since ψ is conformal, $\dfrac{\partial}{\partial y}v(0)$ cannot be positive.

□

When $\partial\Omega$ consists of analytic curves, Green function provides a formula for harmonic measure that generalizes the Poisson integral formula for $\mathbb{D}$.

Theorem 2.6: Harmonic Measure as Generalization of Poisson Integral Formula

Assume $\partial\Omega$ consists of finitely many pairwise disjoint analytic Jordan curves and let $z \in \Omega$. Then

(a) Green function $g(\zeta, z)$ extends to be harmonic (and hence real analytic) on a neighborhood of $\partial\Omega$ and

$$\frac{-\partial}{\partial \vec{n_\zeta}}g(\zeta, z) > 0 \text{ on } \partial\Omega \tag{2.12}$$

where $\vec{n_\zeta}$ is the unit outer normal vector at $\zeta \in \partial\Omega$.

(b) If in addition $u \in C(\overline{\Omega})$ is harmonic on Ω then

$$u(z) = \int_{\partial\Omega} -\frac{\partial}{\partial \vec{n_\zeta}}g(\zeta, z)u(\zeta)\frac{ds(\zeta)}{2\pi}. \tag{2.13}$$

In particular, the second assertion gives the generalization for the Poisson kernel over finitely connected Jordan domain. Note that in classical potential theory our definition for harmonic measure is defined to be the common value between Perron function and the generalized Poisson integral. So far we did not introduce the Perron function and our definition is different from the classical one, and now the reason is obvious: since our definition is more flexible, and we are always ready for another generalization.

Definition: Poisson Kernel (over Finitely Connected Jordan Domain)

In (2.13), the term defined by

$$P_z(\zeta) := -\frac{1}{2\pi}\frac{\partial}{\partial \vec{n_\zeta}}g(\zeta, z), \zeta \in \partial\Omega,$$

is called the Poisson kernel over finitely connected Jordan domain Ω.

Proof of **Theorem 2.6**:

Fix $z \in \Omega$. By **Lemma 2.5**, $g(\zeta, z)$ extends to be harmonic (and real analytic) on some neighborhood of $\partial\Omega$ and then **(2.12)** is an immediate consequence of **(2.11)**. To prove **(2.13)**, we first assume that u is analytic on a neighborhood of $\partial\Omega$.

Step I: **(2.13)** holds when u is analytic on a neighborhood of $\partial\Omega$.

We apply Green's theorem on

$$\Omega_\varepsilon := \Omega \backslash \{\omega : |\omega - z| < \varepsilon\}$$
with ε sufficiently small and $v(\omega) := g(\omega, z)$. By **Remark 2.6** (iv) we have
$$\Delta_\omega g(\omega, z) = \Delta u = 0 \text{ on } \Omega_\varepsilon$$
and by **Remark 2.6** (iii) we have
$$g = 0 \text{ on } \partial\Omega.$$
Therefore, Green's theorem yields
$$\int_{\partial\Omega} \frac{-\partial}{\partial \vec{n_\zeta}} g(\zeta, z) u(\zeta) \frac{ds(\zeta)}{2\pi} = \varepsilon \int_0^{2\pi} g(z + \varepsilon e^{i\theta}, z) \frac{\partial}{\partial r} u(z + \varepsilon e^{i\theta}) \frac{d\theta}{2\pi}$$
$$- \varepsilon \int_0^{2\pi} u(z + \varepsilon e^{i\theta}) \frac{\partial}{\partial r} g(z + \varepsilon e^{i\theta}, z) \frac{d\theta}{2\pi}.$$
Since for $\varepsilon > 0$ sufficiently small, we have
$$g(z + \varepsilon e^{i\theta}, z) \leq 2 \log\left(\frac{1}{\varepsilon}\right)$$
by **Remark 2.6** (v). Moreover, since u is analytic by assumption, we have
$$\lim_{\varepsilon \to 0} \varepsilon \int_0^{2\pi} g(z + \varepsilon e^{i\theta}, z) \frac{\partial}{\partial r} u(z + \varepsilon e^{i\theta}) \frac{d\theta}{2\pi} = 0.$$
As we have seen, **(2.6)** yields
$$\frac{\partial}{\partial r} g(z + \varepsilon e^{i\theta}, z) = \frac{-\partial}{\partial \varepsilon} \log \varepsilon + O(1),$$
while
$$u(z + \varepsilon e^{i\theta}) = u(z) + O(\varepsilon)$$
since u is analytic on a neighborhood of $\partial\Omega$. It follows that
$$- \varepsilon \int_0^{2\pi} u(z + \varepsilon e^{i\theta}) \frac{\partial}{\partial r} g(z + \varepsilon e^{i\theta}, z) \frac{d\theta}{2\pi} = u(z) + O(\varepsilon),$$
and this gives **(2.13)** when u is analytic on a neighborhood of $\partial\Omega$.
Step II: **(2.13)** holds in general case
To prove **(2.13)** in general case, for $\delta > 0$ sufficiently large we define
$$\Omega^\delta := \{\omega \in \Omega : g(\omega, z) > \delta\}.$$
By uniqueness of Green function up to a harmonic correction, Ω^δ has Green function
$$g_\delta(\omega, z) = g(\omega, z) - \delta.$$
Therefore,
$$\frac{\partial}{\partial \vec{n_\zeta}} g_\delta(\zeta, z) = \frac{\partial}{\partial \vec{n_\zeta}} g(\zeta, z) \text{ on } \partial\Omega^\delta.$$
In a neighborhood N of a point $\zeta_0 \in \partial\Omega$, the function
$$\varphi := g + i\overline{g} \qquad (\overline{g} \text{ denotes the complex conjugate})$$
is a conformal map and
$$\frac{1}{2\pi} \int_{N \cap \partial\Omega^\delta} \frac{-\partial}{\partial \vec{n_\zeta}} g(\zeta, z) u(\zeta) ds = \int_{\{Re(z) = \delta\} \cap \varphi(N)} u \circ \varphi^{-1} ds$$
by the definition of conformal mapping φ and change of variables; note that the right hand side convergs, as $\delta \downarrow 0$, to

$$\int_{\{Re(z)=0\}\cap\varphi(N)} u\circ\varphi^{-1}ds = \frac{1}{2\pi}\int_{\partial\Omega}\frac{-\partial}{\partial\overrightarrow{n_\zeta}}g(\zeta,z)u(\zeta)ds.$$

Therefore, as $\delta\downarrow 0$, one has

$$\frac{1}{2\pi}\int_{\partial\Omega^\delta}\frac{-\partial}{\partial\overrightarrow{n_\zeta}}g(\zeta,z)u(\zeta)ds \to \frac{1}{2\pi}\int_{\partial\Omega}\frac{-\partial}{\partial\overrightarrow{n_\zeta}}g(\zeta,z)u(\zeta)ds.$$

But **(2.13)** holds for u on Ω^δ because u is analytic on a neighborhood of $\partial\Omega^\delta$, as we have shown in the first step. Thus **(2.13)** holds on Ω by the above conve-rgence.

□

People used to believe that harmonic measures can be singular with respect to Hausdorff measure on "very wild" bou-ndaries, but Garnett, Marshall, et. al proved that the absolute continuity holds on analytic bounds in 1980s.

Corollary 2.6.1: Absolute Continuity and Analyticity of Harmonic Measure on FCJD

If $\partial\Omega$ consists of finitely many pairwise disjoint analytic Jordan curves and if $z\in\Omega$, then

$$d\omega(z,\zeta) = \frac{-\partial}{\partial\overrightarrow{n_\zeta}}g(z,\zeta)\frac{ds(\zeta)}{2\pi}. \tag{2.14}$$

In other words, harmonic measure for $z\in\Omega$ is absolutely continuous with respect to the arc length on $\partial\Omega$. The density (Radon-Nikodym derivative)

$$\frac{d\omega}{ds} = \frac{-1}{2\pi}\frac{\partial}{\partial\overrightarrow{n_\zeta}}g(z,\zeta) = P_z(\zeta)$$

is real analytic on $\partial\Omega$, and

$$c_1 < \frac{d\omega}{ds} < c_2 \tag{2.15}$$

for positive constants c_1 and c_2.

Proof:

Using **(2.2)** in **Theorem 2.1**, **(2.13)** in **Theorem 2.6**, and the fact that Borel measures are determined by their actions on continuous functions gives the equality **(2.14)**. The inequalities in **(2.15)** are an immediate consequence of **(2.12)** in **Theorem 2.6**.

□

Remark 2.9: Comparing Harmonic Measure to Geometric Measure

One objective of this book is to compare harmonic measure for general domains to more geometrical measures such as arc length, and **Corollary 2.6.1** is the first result of this kind. ◇

Theorem 2.7: Solution to DP over F.C.J.D. with Bounded Borel Boudnary Data

Assume $\partial\Omega$ consists of finitely many pairwise disjoint analytic Jordan curves and for $\zeta\in\partial\Omega$ and $\alpha>1$ define

$$\Gamma_\alpha(\zeta) := \left\{z\in\Omega : |z-\zeta| < \alpha\text{dist}(z,\partial\Omega)\right\}.$$

If $u(z)$ is a bounded harmonic function on Ω, then

(i) For ds-almost every $\zeta\in\partial\Omega$ the limit

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} u(z) = f(\zeta) \tag{2.16}$$

exists.

(ii) u can be written as the Poisson integral formula

$$u(z) = \int_{\partial\Omega} P_z(\zeta) f(\zeta) ds(\zeta), \tag{2.17}$$

(iii) The following isometry holds

$$\sup_{\Omega} |u(z)| = \|f\|_{L^\infty}. \tag{2.18}$$

Conversely, if f is a bounded Borel function on $\partial\Omega$, then **(2.17)** defines a bounded harmonic function $u(z)$ on Ω such that **(2.18)** holds ds-almost everywhere. Moreover, if f is continuous at $\zeta_0 \in \partial\Omega$ then

$$\lim_{z\to\zeta_0} u(z) = f(\zeta_0). \tag{2.19}$$

Remark 2.10: Isometry Between Space of Bounded Harmonic Functions and L^∞

By **(2.18)**, we see **(2.16)** and **(2.17)** establish an isometry between the space of bounded harmonic function on Ω and $L^\infty(\partial\Omega, ds)$ when $\partial\Omega$ consists of analytic curves. ◇

Proof of **Theorem 2.7**:

We shall use a simple localization argument to prove the first assertion in the first step, then we prove assertion (ii) and (iii) in the second step, we finally deal with the converse direction in the third step.

Step I: Assertion (i)

A simple localization argument gives the existence of the non-tangential limit f. If I is an open arc on $\partial\Omega$, there exists a neighborhood $V \supset I$ such that

$$V \cap \partial\Omega = I \text{ and } V \cap \Omega \text{ is simply connected.}$$

Moreover, there exists a conformal mapping ψ defined on V such that

$$\psi(V \cap \Omega) = \mathbb{D} \text{ and } \psi(I) \text{ is an arc on } \partial\mathbb{D}.$$

It follows that ψ maps conical approach regions at $\zeta \in V \cap \partial\Omega$ into cones at $\psi(\zeta)$:

$$\psi\big(V \cap \Gamma_\alpha(\zeta) \cap \mathscr{B}_\delta(\zeta)\big) \subset \Gamma_{\beta(\alpha)}\big(\psi(\zeta)\big),$$

where $\mathscr{B}_\delta(\zeta) := \big\{z : |z-\zeta| < \delta := \delta(\zeta)\big\}$. Then, if u is a bounded harmonic function on Ω, we can apply Fatou's **Theorem 1.4** to $u \circ \psi^{-1}$ to obtain **(2.16)** ds-almost everywhere on $V \cap \partial\Omega$.

Step II: Assertion (ii) and (iii)

The proof of **(2.17)** is exactly the same as the proof of **(2.12)** except that the (Lebesgue's) dominated convergence theorem (LDCT) is applied in **(2.13)**. By **(2.16)** we have

$$\|f\|_\infty \le \sup_{\Omega} |u(z)|. \quad \text{(LDCT)}$$

Since

$$P_z \ge 0 \text{ and } \int_{\partial\Omega} P_z ds = 1$$

by the definition of Poisson kernel, one has, via LDCT once more,

$$\sup_{\Omega} |u(z)| \le \|f\|_\infty.$$

Hence **(2.16)** and **(2.17)** together implies **(2.18)**, proving both assertions.
Step III: Converse
To prove the converse, let $f \in L^{\infty}(\partial\Omega, ds)$. Then the discussion following **(2.5)** shows that **(2.17)** defines a bounded harmonic function $u(z)$ on Ω and

$$\sup_{\Omega} |u(z)| \leq \|f\|_{\infty}.$$

Therefore, by **(2.16)**, u has **almost-everywhere**[5] a non-tangential limit, which we will temporarily call it F, and u is the Poisson integral of F by **(2.17)**.
It suffices to prove the following claim.
Claim: $F = f$ **almost-everywhere**
Let V be a neighborhood of an open arc I such that

$$I = V \cap \partial\Omega \text{ and } V \cap \Omega \text{ is simply connected.}$$

For $h \in L^{\infty}(\partial\Omega, ds)$, define

$$v_h(z) := \int_{\partial\Omega} P_z(\zeta)h(\zeta)ds(\zeta) - \int_I P_z(\zeta, V)h(\zeta)ds(\zeta),$$

where $P_z(\zeta, V)$ is the Poisson integral for $z \in U \cap \Omega$. If $h \in C(\partial\Omega)$ then by **(2.3)**, **Theorem 2.1**, **(2.12)**, and **(1.20)**, one has

$$\lim_{z \to \zeta} v_h(z) = 0 \ \forall \zeta \in I,$$

and hence by **Lemma 2.5** (i), v_h extends to be harmonic in a neighborhood W of I which does not depend on h. Thus, by **Exercise 1.5** (e) or **(2.25)** below, if J is a compact subset of I and if $\varepsilon > 0$, then there exists a neighborhood N of J depending only on $\|h\|_{\infty}$ and ε, such that

$$|v_h| < \varepsilon \text{ in } N.$$

Now take $\{h_n\}_{n \geq 1} \subset C(\partial\Omega)$ so that

$$\lim_{n \to \infty} h_n = f \text{ in } L^1 \text{ and } \|h_n\|_{\infty} \leq \|f\|_{\infty}.$$

For each $z \in V \cap N$, $v_{h_n}(z) \to v_f(z)$, and so

$$|v_f(z)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$v_f(z) \xrightarrow[z \to \zeta \in J]{} 0.$$

Now by **Theorem 1.3**,

$$F(\zeta) - f(\zeta) = \lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} v_f(z) = 0 \textbf{ almost-everywhere} \text{ on } J.$$

Consequently, $F = f$ **almost-everywhere** and **(2.16)**, as well as **(2.18)** holds for all $f \in L^{\infty}(\partial\Omega, ds)$. Finally, if f is continuous at $\zeta_0 \in I$, then

$$\int_I P_z(\zeta, V)f(\zeta)ds(\zeta)$$

is continuous at ζ_0 by **(1.20)**. Thus, **(2.19)** follows from the continuity of v_f.

□

[5] By saying so we mean u has a non-tangential limit μ-almost everywhere for whatever measure μ we do not care.

Remark 2.11: Alternative Proof for **Theorem 2.7** and Conformal Estimate

The converse can also be proved in another way. Using the real analyticity of $g(\omega, z)$, one can refine the proof of **Lemma 1.5** and show that

$$\sup_{\Gamma_\alpha(\zeta)} |u(z)| \le C(\alpha, \Omega) M_s f(\zeta), \tag{2.20}$$

where u is the Poisson integral **(2.17)** and where the maximal function $M_s f(\zeta)$ is the supremum of the averages of f over arcs $\gamma \subset \partial\Omega$ with $\zeta \in \gamma$:

$$M_s f(\zeta) := \sup_{\zeta \in \gamma} \frac{1}{\ell(\gamma)} \int_\gamma |f| \, ds.$$

A variation on the covering lemma shows that M_s is weak-type 1-1, and an approximation, as in the proof of **Theorem 1.4**, then yields **(2.16)** for the Poisson integral of f. This is the argument that MUST be used in the Euclidean space $\mathbb{R}^d$, $d \ge 3$.

With the same care, the conformal mapping proof of **(2.16)** in the text can also be parleyed into a proof of the maximal estimate **(2.20)**. ⋄

Remark 2.12: Equivalent Definition for Harmonic Measure on F.C.J.D.

Let Ω be any finitely connected Jordan domain and let φ be a conformal map, given in **Lemma 2.3**, of Ω onto a domain Ω^*, where $\partial\Omega^*$ consists of analytic Jordan curves. Since $\varphi : \overline{\Omega} \to \overline{\Omega}^*$, harmonic measure can be transplanted from Ω^* to Ω via φ, just as it was in **Section 1.3** for simply connected Jordan domains. This gives an alternative but equivalent definition of harmonic measure for Ω. ⋄

In **Section 2.4** we shall consider two questions. Let Ω be a finitely connected Jordan domain.

Question I: If $\partial\Omega$ has some degree of differentiability and if $f \in C(\partial\Omega)$ also has some degree of differentiability along $\partial\Omega$, how smooth is the solution $u_f(z)$ as z approaches $\partial\Omega$?

Question II: What smoothness condition on $\partial\Omega$, weaker than real-analyticity, will ensure that

$$\frac{\partial}{\partial \overrightarrow{n_\zeta}} g(z, \zeta) \text{ exists on } \partial\Omega \text{ and } \textbf{(2.14)} \text{ and } \textbf{(2.15)} \text{ still hold?}$$

The two questions are equivalent. Their answers will depend on Kellogg's theorem about the boundary behavior of conformal mappings. The proof of Kellogg's theorem in turn depends on the estimates for conjugate functions in the next section.

2.3 Harmonic Conjugate

Let $f \in L^1(\partial\mathbb{D})$ be real. For convenience we write $f(\theta)$ for $f(e^{i\theta})$. We shall benefit from the fact that the Poisson integral of $f \in L^1(\partial\mathbb{D})$ is always harmonic and real.

Definition: Harmonic Conjugate (Conjugate Function)

If $u(z)$ is the Poisson integral of f on $\mathbb{D}$, then $u(z)$ is harmonic and real and there exists a unique harmonic function $\widetilde{u}(z)$ such that

$$\widetilde{u}(0) = 0 \text{ and } F = u + i\widetilde{u} \text{ is analytic on } \mathbb{D}.$$

The function $\widetilde{u}$ is called the harmonic conjugate or conjugate funciton of u.

One may recall the definition of harmonic minorant (respectively, hamonic majora-nt). Heuristically, both the harmonic minorant and the harmonic conjugate are serving as a harmonic "correction", while the minorant serves as the continuity correction and the conjugate serves as a differentiability correction.

Proposition 2.8: Non-Tangential Limit for Conjugate Function Exists A.E.

The non-tangential limit

$$\tilde{f}(\theta) := \lim_{\Gamma_\alpha(e^{i\theta})\ni z\to e^{i\theta}} \widetilde{u}(z) \tag{2.21}$$

exists a.e..

Proof:

This has an easy proof from Fatou's **Theorem 1.4**: We may assume $f \geq 0$, so that

$$G(z) := \exp\left\{-u(z) + i\widetilde{u}(z)\right\}$$

is bounded and analytic on $\mathbb{D}$. By **Corollary 1.4.1**, G has non-tangential limit $G(e^{i\theta})$ almost everywhere. Since

$$|G(e^{i\theta})| = e^{-f(\theta)} \text{ and } f \in L^1,$$

$|G(e^{i\theta})| > 0$ a.e.. At such $e^{i\theta}$, G is continuous and non-zero on the cone

$$k := \overline{\Gamma_\alpha(e^{i\theta})}.$$

Consequently,

$$\log G = -(u + i\widetilde{u})$$

has a continuous extension to

$$K \cap \left\{z : \left|G(z) - G(e^{i\theta})\right| < \frac{1}{2}\left|G(e^{i\theta})\right|\right\}$$

and the limit **(2.21)** exists at $e^{i\theta}$.

□

There is a close connection between harmonic conjugate and conformal mappings.

Remark 2.13: Connection Between Harmonic Conjugate and Conformal Map

If u is harmonic and if $|u| < \dfrac{\pi}{2}$, then

$$\varphi(z) := \int_0^z \exp\left\{i(u + i\widetilde{u})(\zeta)\right\} d\zeta$$

is a conformal map from $\mathbb{D}$ to a finitely connected domain and

$$u = \arg \varphi'.$$

Indeed, if $a \neq b \in \mathbb{D}$, then

$$\varphi(b) - \varphi(a) = (b - a)\int_0^1 \varphi'\big(a + t(b - a)\big)dt \neq 0$$

because $\operatorname{Re}(\varphi') > 0$. ⋄

When f is bounded, or even continuous, it can happen that $\tilde{f}$ is not bounded.

Example 2.2: Bounded Continuous Function with Unbounded Harmonic Conjugate

Let $u + i\widetilde{u}$ be the conformal map of $\mathbb{D}$ onto the region

$$\left\{0 < x < \frac{1}{1 + |y|}\right\}.$$

Then u is continuous on $\overline{\mathbb{D}}$ by Carathéodory's **Theorem 1.8**, but $\widetilde{u}$ is not bounded. $\diamond$

The next two assertions get around the obstruction that $\tilde{f}$ may be unbounded even when f is continuous.

Theorem 2.9: Zygmund's Exponential Integrability for Harmonic Conjugate

Let $f \in L^\infty(\partial\mathbb{D})$ be real with $\|f\|_\infty \leq 1$.

(a) For $0 < \lambda < \dfrac{\pi}{2}$ there is a constant C_λ, depending only on λ, such that

$$\frac{1}{2\pi}\int \exp\{\lambda |\tilde{f}(\theta)|\}d\theta \leq C_\lambda.$$

(b) If $f \in C(\partial\mathbb{D})$, then for all $\lambda < \infty$,

$$\sup_{0<r<1} \frac{1}{2\pi}\int \exp\{\lambda |\widetilde{u}(re^{i\theta})|\}d\theta < \infty.$$

Proof:

Step I: Assertion (a)

Let $u(z)$ be the Poisson integral of f in $\mathbb{D}$ and consider the analytic function

$$g(z) := \widetilde{u}(z) - iu(z).$$

For $r < 1$, $g(z)$ satisfies

$$e^{\lambda g(0)} = \frac{1}{2\pi}\int_0^{2\pi} e^{\lambda g(re^{i\theta})}d\theta$$

because $\widetilde{u}(0) = 0$. Therefore, by Euler's identity,

$$\cos \lambda u(0) = \frac{1}{2\pi}\int_0^{2\pi} \exp\{\lambda\widetilde{u}(re^{i\theta})\}\cos \lambda u(re^{i\theta})d\theta.$$

But if $0 < \lambda < \dfrac{\pi}{2}$, then

$$0 < \cos\lambda < \cos\lambda u \leq 1$$

since $|u| \leq 1$. It follows that

$$\frac{1}{2\pi}\int_0^{2\pi} \exp\{\lambda\widetilde{u}(re^{i\theta})\}d\theta \leq \sec\lambda.$$

Then by **Proposition 2.8** in conjunction with Fatou's lemma, one has

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\lambda\tilde{f}(\theta)}d\theta \leq \sec\lambda.$$

By repeating this argument with $-f(\theta)$, we then obtain assertion (a) with constant $C_\lambda := 2\sec\lambda$.

Step II: Assertion (b)

To prove (b), fix $\lambda < \infty$ and take a trigonometric polynomial

$$p(\theta) = \sum_{n=0}^{N}\left(a_n \cos n\theta + b_n \sin n\theta\right)$$

such that $\|f - p\|_\infty < \dfrac{\pi}{2\lambda}$. Then the conjugate

$$\widetilde{p}(re^{i\theta}) = \sum_{n=0}^{N} r^n\left(a_n \sin n\theta - b_n \cos n\theta\right)$$

is bounded, while (a) gives

$$B_\lambda := \sup_{0<r<1} \frac{1}{2\pi} \int \exp\left\{ \lambda \left| \widetilde{(u-p)}(re^{i\theta}) \right| \right\} d\theta < \infty.$$

Therefore, since

$$|\tilde{u}| \le |\tilde{p}| + \left| \widetilde{(u-p)} \right|,$$

we have

$$\sup_{0<r<1} \frac{1}{2\pi} \int \exp\{ \lambda |\tilde{u}(re^{i\theta})| \} d\theta \le B_\lambda e^{\lambda \|\tilde{p}\|_\infty} < \infty,$$

which yields assertion (b).

$\square$

Definition: Alpha-Hölder Class and Alpha-Hölder Continuous Function

Let $0 < \alpha < 1$. The α-Hölder class C^α is

$$C^\alpha := \left\{ f \in L^\infty(\partial\mathbb{D}) : \sup_{t>0} \frac{\| f(\theta + t) - f(\theta) \|_\infty}{t^\alpha} < \infty \right\}.$$

Every $f \in C^\alpha$ agrees almost everywhere with a function continuous on $\partial\mathbb{D}$.

A function $f \in C^\alpha$ is called an α-Hölder continuous function.

Definition: Alpha-Hölder Norm

The class C^α is given the α-Hölder norm

$$\|f\|_{C^\alpha} := \|f\|_\infty + \sup_{t>0} \frac{\|f(\theta+t) - f(\theta)\|_\infty}{t^\alpha}. \tag{2.22}$$

In a moment we shall prove Privalov's theorem that $\tilde{f} \in C^\alpha$ whenever $f \in C^\alpha$. In the Poisson integral formula

$$u(z) = \mathrm{Re} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t) dt,$$

the kernel

$$\frac{e^{it} + z}{e^{it} - z} \text{ is } \begin{cases} \text{analytic,} & \text{in } z \in \mathbb{D} \\ \text{real,} & \text{at } z = 0 \end{cases}$$

Then the uniqueness of $\tilde{u}$ shows that

$$u(z) + i\tilde{u}(z) =: F(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t) dt. \tag{2.23}$$

Definition: Herglotz Integral of Alpha-Hölder Continuous Class

The analytic function $F(z)$ defined in (2.23) is called the Herglotz integral of $f \in C^\alpha$.

We shall denote

$$\nabla u := \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$$

when u is differentiable on an open plane set. If u is harmonic and bounded on $\mathbb{D}$ then u is the Poisson integral of some $f \in L^\infty(\partial\mathbb{D})$ and by (2.23) the Cauchy-Riemann equations

$$|\nabla u(z)| = |F'(z)| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{it} f(t)}{(e^{it} - z)^2} dt \right|. \tag{2.24}$$

Therefore,

$$\begin{aligned} |\nabla u(z)| &\leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|e^{it} - z|^2} dt \right) \|f\|_\infty \quad \text{(Hölder's Inequality)} \\ &\leq 2(1 - |z|)^{-1} \|f\|_\infty \end{aligned} \tag{2.25}$$

We will often make use of the following consequence of (2.25):

Remark 2.14: Locally Bounded Harmonic Function Has Local Bounded PDE

If $u(z)$ is harmonic and $|u(z)| \leq M$ on $\mathscr{B}_R(z)$ for some $M > 0$ and $\mathscr{B}_R(z)$ denotes the open ball centered at z with radius $R > 0$. Then

$$\sup_{\mathscr{B}_{R/2}(z)} |\nabla u| \leq \frac{4M}{R}. \tag{2.26}$$

◇

To prove (2.26) simply apply (2.25) to $U(\omega) := u(z + R\omega)$. The next theorem shows that $f \in C^\alpha$ if and only if the estimate (2.25) can be upgraded to

$$|\nabla u| = O\big((1 - |z|)^{\alpha - 1}\big).$$

Theorem 2.10: Criterion of Alpha-Hölder Continuous Class with Norm Bound

Let $0 < \alpha < 1$, let $f \in L^\infty(\partial\mathbb{D})$ be real, and let $u(z)$ be the Poisson integral of f. Then the following conditions are equivalent:

(a) $f \in C^\alpha$.

(b) $\tilde{f} \in C^\alpha$.

(c) $|\nabla u(z)| = O\big((1 - |z|)^{\alpha - 1}\big)$.

(d) $u \in C^\alpha(\overline{\mathbb{D}})$, that is, $\forall z_1, z_2 \in \mathbb{D}$,

$$|u(z_1) - u(z_2)| = O\big(|z_1 - z_2|^\alpha\big).$$

Moreover, there exists a constant C_1, independent of α, such that

(b’) $$\|\tilde{f}\|_{C^\alpha} \leq \frac{C_1}{\alpha(1 - \alpha)} \|f\|_{C^\alpha}. \tag{2.27}$$

(c’) $$|\nabla u(z)| \leq \frac{C_1}{1 - \alpha} (1 - |z|)^{\alpha - 1} \|f\|_{C^\alpha}, \tag{2.28}$$

and

(d’) $$\sup_{z_1 \neq z_2} \frac{|u(z_1) - u(z_2)|}{|z_1 - z_2|^\alpha} \leq \frac{C_1}{\alpha} \sup_{|z|<1} \left\{ (1 - |z|)^{1-\alpha} |\nabla u(z)| \right\}. \tag{2.29}$$

The equivalence (a) ⇔ (b) was first proved by Privalov who worked directly with the imaginary part of the integral (2.23); (a) ⇔ (c) was proved by Hardy and Littlewood in 1931.

Proof of **Theorem 2.10**:

Clearly (d) ⇒ (a) since if (d) holds then u is uniformly continuous over $\mathbb{D}$.
We first show (a) ⇒ (c) and establish (2.28), then we show (c) ⇒ (d) and establish (2.29). Finally, we show (a) ⇔ (b) and inequality (2.27) will follow because $|\nabla u| = |\nabla \tilde{u}|$ by an application of Cauchy-Riemann equation.

Step I: (a) ⇒ (c) and construction of (2.28).

Assume (a) holds, that is, assume u is the Poisson integral of $f \in C^\alpha$. We prove (2.28). By (2.25) we may assume $|z| \geq \frac{1}{2}$. Let $e^{it_0} := \frac{z}{|z|}$. Since

$$\int \frac{e^{it}}{(e^{it}-z)^2}dt = 0,$$

(2.24) yields

$$|\nabla u(z)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{it}\big(f(t)-f(t_0)\big)}{(e^{it}-z)^2}dt\right|,$$

so by (2.25) again

$$\begin{aligned}|\nabla u(z)| &\leq \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{|f(t)-f(t_0)|}{|e^{it}-z|^2}dt\\ &= \frac{1}{\pi}\int_{|t-t_0|<|1-z|}\frac{|f(t)-f(t_0)|}{|e^{it}-z|^2}dt + \frac{1}{\pi}\int_{1-|z|\leq|t-t_0|\leq\pi}\frac{|f(t)-f(t_0)|}{|e^{it}-z|^2}dt.\end{aligned}$$

The inequality

$$1-|z| \leq |e^{it}-z| \text{ holds for all } t,$$

it follows that

$$\begin{aligned}\frac{1}{\pi}\int_{|t-t_0|<1-|z|}\frac{|f(t)-f(t_0)|}{|e^{it}-z|^2}dt &\leq \frac{2\|f\|_\alpha}{(1-|z|)^2}\int_0^{1-|z|}\frac{t^\alpha}{\pi}dt\\ &= \frac{2\|f\|_\alpha}{\pi(1+\alpha)}(1-|z|)^{\alpha-1}.\end{aligned}$$

When $1-|z| \leq |t-t_0| \leq \pi$, the inequality

$$|t-t_0|^2 \leq c\cdot|e^{it}-z|^2 \text{ for all } c \text{ independent of } z,$$

thus the second term

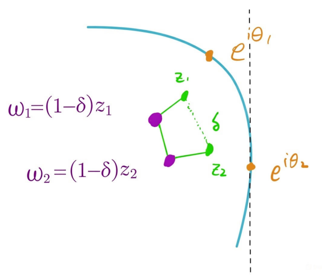
$$\begin{aligned}\frac{1}{\pi}\int_{1-|z|<|t-t_0|<\pi}\frac{|f(t)-f(t_0)|}{|e^{it}-z|^2}dt &\leq \frac{2c\|f\|_\alpha}{\pi}\int_{1-|z|}^{\pi}t^{\alpha-2}dt\\ &\leq \frac{2c\|f\|_\alpha}{\pi(1-\alpha)}(1-|z|)^{\alpha-1}.\end{aligned}$$

Combining these bounds together, we obtain (2.28) with

$$C_1 := \sup_{0<\alpha<1}\frac{2}{\pi}\left(c+\frac{1-\alpha}{1+\alpha}\right) = \frac{2(c+1)}{\pi}.$$

Step II: (c) ⇒ (d) and (2.29)

If (c) holds, we may assume $(1-|z|)^{1-\alpha}|\nabla u(z)| \leq 1$.



(Figure 2.3: Choice of z_j, ω_j, and parallelogram)

Let $z_j = r_j e^{i\theta_j} \in \mathbb{D}$, as illustrated below

We may assume $|z_j| \geq \dfrac{1}{2}$ and $\delta = |z_1 - z_2| \leq \dfrac{1}{2}$. Set $\omega_j := (1-\delta)z_j$. Then

$|u(z_1) - u(z_2)| \leq |u(z_1) - u(\omega_1)| + |u(z_2) - u(\omega(2)| + |u(\omega_1) - u(\omega_2)|$.

However,

$$\begin{aligned}|u(z_j) - u(\omega_j)| &= \left|\int_{(1-\delta)r_j}^{r_j} \frac{\partial u}{\partial t}(te^{i\theta_j})dt\right| \quad \text{(definition of } \omega_j \text{ and } \delta)\\ &\leq \int_{1-\delta}^{1} (1-t)^{\alpha-1}dt \quad \text{(assumption (c))}\\ &\leq \frac{\delta^\alpha}{\alpha},\end{aligned}$$

while

$$\begin{aligned}|u(\omega_1) - u(\omega_2)| &\leq |\omega_1 - \omega_2| \max_{j=1,2}(1-|\omega_j|)^{\alpha-1} \quad \text{(assumption (c))}\\ &\leq \delta^\alpha \quad \text{(definition of } \omega_j \text{ and } \delta)\end{aligned}$$

Therefore (d) and (2.29) hold.

Step III: (a) ⇔ (b) and (2.27)

Finally, suppose (a) holds. Then (c) holds and

$$|\nabla \widetilde{u}| = O\big((1-|z|)^{\alpha-1}\big).$$

Therefore (d) and (2.29) hold for $\widetilde{u}$, and $\widetilde{u}$ extends continuously to $\partial\mathbb{D}$ where $\widetilde{u}$ has boundary value $\tilde{f}$ by Poisson integral properties (see **Theorem 1.6** (ii) Ransford). It follows that $\tilde{f} \in C^\alpha$ because (d) ⇒ (a). Finally, since

$$\tilde{\tilde{f}} = -f + u(0),$$

it follows that (a) ⇔ (b) and hence (2.27) holds.

□

It is useful to introduce the following notion.

Definition: k times Continuously Differentiable

Let k be a non-negative integer, let $0 \leq \alpha < 1$ and let $f \in C(\partial\mathbb{D})$. We say $f \in C^{k+\alpha}$ if f is k times continuously differentiable on $\partial\mathbb{D}$ and $\left(\dfrac{d}{d\theta}\right)^k f \in C^\alpha$ provided $\alpha > 0$.

If $F(z)$ is analytic on $\mathbb{D}$ we say $F \in C^{k+\alpha}(\overline{\mathbb{D}})$ if F and its first k derivatives F', F'', ⋯, $F^{(k)}$ extend continuously to $\overline{\mathbb{D}}$ and if there is C such that

$$|F^{(k)}(z_1) - F^{(k)}(z_2)| \leq C|z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in \overline{\mathbb{D}}$.

Corollary 2.10.1: Criterion for Alpha-Hölder Class Extension to Boundary of Unit Disc

Assume k is a non-negative integer and assume $0 < \alpha < 1$. Let $f \in C(\partial\mathbb{D})$ be real and let $F(z) = u(z) + i\widetilde{u}(z)$ be the Herglotz integral of f. Then

$$F \in C^{k+\alpha}(\overline{\mathbb{D}}) \Leftrightarrow f \in C^{k+\alpha}(\partial\mathbb{D}).$$

Proof:

Because $f \in \text{Re}(f)$, it is clear that $f \in C^{k+\alpha}(\partial\mathbb{D})$ if $F \in C^{k+\alpha}(\overline{\mathbb{D}})$. Assume

$f \in C^{k+\alpha}(\partial\mathbb{D})$. If $f(\theta)$ has Fourier series

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

then F has Taylor series

$$F(z) = a_0 + 2\sum_{n=1}^{\infty} a_n z^n.$$

Therefore $\dfrac{df}{d\theta}$, which has Fourier series

$$\frac{df}{d\theta} \sim \sum_{n=-\infty}^{\infty} ina_n e^{in\theta},$$

has Herglotz integral $izF'(z)$ and the corollary follows from a special case $k = 0$, which is **Theorem 2.10**.

□

Remark 2.15: **Theorem 2.10** Fails when $\alpha = 1$, **Corollary 2.10.1** Fails when $\alpha = 0$

Theorem 2.10 fails when $\alpha = 1$. The harmonic conjugate of a continuously differentiable function on $\partial\mathbb{D}$ need not have a continuous derivative, and the conjugate of a Lipschitz function, that is, a function satisfying

$$|f(\theta + t) - f(\theta)| \leq M|t|,$$

need not to be a Lipschitz function.

For the same reason, **Corollary 2.10.1** fails when $\alpha = 0$ and k is a non-negative integer. ◇

However, conjugation does preserve the Zygmund class.

Definition: Zygmund Class

The Zygmund class, denoted as Z^*, is the class of continuous functions f on $\partial\mathbb{D}$ such that

$$\sup_{t>0} \frac{\|f(\theta + t) + f(\theta - t) - 2f(\theta)\|_\infty}{t} < \infty.$$

Definition: Zygmund Norm and Zygmund Function

The Zygmund class has norm

$$\|f\|_{Z^*} := \|f\|_\infty + \sup_{t>0} \frac{\|f(\theta + t) + t(\theta - t) - 2f(\theta)\|_\infty}{t}.$$

When $f \in Z^*$, we say f is a Zygmund function.

Define

$$|\nabla_2 u(z)| = |\nabla u_x| = \left(|u_{xx}(z)|^2 + u_{yx}(z)|^2\right)^{1/2}$$

$$= |F''(z)| = \left|\frac{2}{\pi}\int_{-\pi}^{\pi} \frac{e^{it}f(t)}{(e^{it} - z)^2} dt\right|$$

where $F = u + i\widetilde{u}$ is the Herglotz integral of $f \in L^1(\partial\mathbb{D})$. If $f \in C^\alpha$, then **(2.26)** and **(2.28)**, applied to u_x in the disc $\mathscr{B}_{\frac{1-|z|}{2}}(z)$, gives us

$$|\nabla_2 u(z)| = O\left((1 - |z|)^{\alpha - 2}\right). \qquad (2.30)$$

Conversely, if **(2.30)** holds, then integrating F'' along radii shows that

$$|\nabla u(z)| = O\big((1-|z|)^{\alpha-1}\big) \text{ and } f \in C^\alpha.$$

Thus (2.30) provides yet another characterization of C^α functions, in terms of second derivatives. Zygmund functions have a similar characterization.

Theorem 2.11: Criterion for Zygmund Boundary Data with Zygmund Norm Bound

Let $f \in L^\infty(\partial\mathbb{D})$ be real and let $u(z)$ be the Poisson integral of f. Then the followings are equivalent:

(a) $f \in Z^*$.

(b) $\tilde{f} \in Z^*$.

(c) $|\nabla_2 u(z)| = O\big((1-|z|)^{-1}\big)$.

There is a constant C such that

$$\|\tilde{f}\|_{Z^*} \le C\|f\|_{Z^*} \tag{2.31}$$

and

$$\frac{\|f\|_{Z^*}}{C} \le \|f\|_\infty + \sup_{z\in\mathbb{D}}\left\{(1-|z|)|\nabla_2 u(z)|\right\} \le C\|f\|_{Z^*}. \tag{2.32}$$

Notice that by (2.30) and (2.32), $Z^* \subset C^\alpha$ for $\alpha < 1$. In particular, if $f \in Z^*$, then f and $\tilde{f}$ are continuous. On the other hand, if f is Lipschitz, then clearly $f \in Z^*$. To make our contents self-contained, we define the Lipschitz function formally and point out that a function lives in C^α is not necessarily Lipschitz.

Definition: Lipschitz Function, Class of Lipschitz Function

A function f such that $|f(x)-f(y)| \le C|x-y|$ for all x and y where C is a constant independent of x and y is called a Lipschitz function. In particular, we denote $C^{0,1}$ as the space of all Lipschitz functions.

Definition: Norm on Class of Lipschitz Function

The norm of $f \in C^{0,1}$ is defined to be

$$\|f\|_{C^{0,1}} := \|f\|_\infty + \sup_{t>0}\frac{\|f(\theta+t)-f(t)\|_\infty}{t}.$$

Remark 2.16: Zygmund Class, Lipschitz Class, and Alpha-Hölder Class

Provided $\alpha < 1$, one has

Continuous Differentiability $\Rightarrow$ Lipschitz $\Rightarrow$ Zygmund

$\Rightarrow$ α-Hölder $\Rightarrow$ Continuity

That is to say,

$$C^1 \subset C^{0,1} \subset Z^* \subset C^\alpha \subset C \text{ for } \alpha < 1,$$

where C^1 is the space of continuously differentiable functions and C is the class of continuous functions. ◇

Proof of **Theorem 2.11**:

The logic is the same as in the proof of **Theorem 2.10**. First assume (a) holds and we establish (c) and the right hand side of (2.32). Then we assume (c) and prove (a) and establish the left hand side of (2.32). Finally we prove (a) ⇔ (b) and establish (2.31).

Step I: (a) ⇒ (c) and right hand side of (2.32)

Assume (a) holds. Fix $z \in \mathbb{D}$, we may assume that $z = |z| = \mathrm{Re}(z)$ and $|z| > \dfrac{1}{2}$. Because

$$\int \frac{e^{i\theta}}{(e^{i\theta}-z)^2}d\theta = 0$$

and $f(-\theta)$ has Herglotz integral $\overline{F(\overline{z})}$, we have

$$\frac{\partial^2}{\partial x^2}u(z) = \mathrm{Re}F''(z) = \frac{1}{\pi}\int \frac{e^{it}\big(f(t)+f(-t)-2f(0)\big)}{(e^{it}-|z|)^3}dt$$

and using **(2.25)** in the last relation yields

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x^2}u(z)\right| &= |\mathrm{Re}F''(z)| \quad (\text{assumption on } |z|)\\
&\le \frac{1}{\pi}\int_{|t|\le 1-|z|}\frac{\|f\|_{Z^*}|t|}{(1-|z|)^3}dt + \frac{c}{\pi}\int_{1-|z|<|t|\le\pi}\frac{\|f\|_{Z^*}}{|t|^2}dt\\
&\le C(1-|z|)^{-1}\|f\|_{Z^*}
\end{aligned}$$

where the middle relation holds by integration by parts and c comes from the bound in α-Hölder continuity.

Unfortunately, this trick does not help us with $|\mathrm{Im}F''(z)| = \left|\frac{\partial^2}{\partial x\partial y}u\right|$. Instead we apply **(2.26)** to $u := \frac{\partial^2}{\partial x^2}u$ on $\mathscr{B}_{\frac{1-|z|}{2}}(z)$, which yields

$$\left|\frac{\partial^3}{\partial y\partial x^2}u\right| \le \frac{C\|f\|_{Z^*}}{(1-|z|)^2}.$$

An integration in conjunction with the above two display then gives

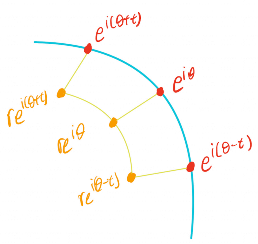
$$\left|\frac{\partial^2}{\partial y\partial x}u(z)\right| \le |F''(0)| + \int_0^{|z|}\frac{C\|f\|_{Z^*}}{(1-s)^2}ds \le C'\frac{\|f\|_{Z^*}}{1-|z|}.$$

Thus (c) is proved, and the right hand side of **(2.32)** holds.

Step II: (c) $\Rightarrow$ (a), and the left hand side of **(2.32)** holds

Now assume (c) holds. Then by **(2.30)**, $f \in C^\alpha$ and f and $\tilde{f}$ are both continuous by **Theorem 2.10**. Fix θ and t with $0 < t \le \pi$ and set $r := 1 - \frac{t}{\pi}$. Then

$$\begin{aligned}
f(\theta+t)+f(\theta-t)-2f(\theta) = f(\theta+t) &- u(re^{i(\theta+t)}) + f(\theta-t)\\
&- u(re^{i(\theta-t)}) + 2u(re^{i\theta}) - 2f(\theta)\\
&+ u(re^{i(\theta+t)}) - u(rei\theta) + u(re^{i(\theta-t)}) - u(re^{i\theta})
\end{aligned} \tag{2.33}$$



(Figure 2.4: Choice of points in calculation)

Because

$$\left|u(re^{i(\theta+t)})+u(re^{i(\theta-t)})-2u(re^{i\theta})\right|\leq|t|^2\sup_{|\omega|=r}\left|\frac{\partial^2}{\partial\theta^2}u(\omega)\right|,$$

the last two terms on the right hand side of (2.33) are $O(|t|)$ by assumption (c). Using assumption (c) once again we have

$$\lim_{s\to 1}(1-s)\frac{\partial}{\partial s}u(se^{i\alpha})=0,$$

an integration by parts shows that

$$f(\alpha)-u(re^{i\alpha})=\int_r^1(1-s)\frac{\partial^2}{\partial s^2}u(se^{i\theta})ds+(1-r)\frac{\partial}{\partial r}u(re^{i\alpha}).$$

Therefore when (c) holds, the sum of the first two lines of the right hand side of (2.33) is

$$\begin{aligned}&\frac{t}{\pi}\left(u_r(re^{i(\theta+t)})+u_r(re^{i(\theta-t)})-2u_r(re^{i\theta})\right)+O(t)\\ \leq&\frac{t}{\pi}\left|u_r(re^{i(\theta+t)})-u_r(re^{i\theta})\right|+\frac{t}{\pi}\left|u_r(re^{i(\theta-t)})-u_r(re^{i\theta})\right|+O(t)\\ \leq&\frac{2t^2}{\pi}\sup_{|\omega|=r}\left|\frac{\partial^2}{\partial r\partial\theta}u(\omega)\right|+O(t)\leq C''t.\end{aligned}$$

Thus (a) is proved and the left hand side of (2.32) is established following the constants in the previous argument.

Step III: (a) ⇒ (b) and (2.31)

Since $|\nabla_2\tilde{u}|=|\nabla_2 u|$ by Cauchy-Riemann equation, it follows that (a) ⇔ (b) and (2.31) follows.

□

2.4 Boundary Smoothness

Let Ω be a Jordan domain with boundary Γ and let φ be a conformal mapping from $\mathbb{D}$ onto Ω, so that φ extends to a homeomorphism from $\partial\mathbb{D}$ to the Jordan curve $\Gamma=\partial\Omega$. In this section we examine the connection between the smoothness of Γ and the differentiability of φ on $\partial\mathbb{D}$. When Γ has some degree of smoothness, we also study the relation between the differentiability of $f\in C(\Gamma)$ and the differentiability of its solution u_f to the Dirichlet problem at points of Γ.

The results do not depend on the choice of the mapping $\varphi:\mathbb{D}\to\Omega$ because any other such map has the form $\varphi\circ T$, with $T\in\mathscr{M}$ (the set of all conformal self maps of $\mathbb{D}$). We first show that the smoothness of φ in a neighborhood of $\varphi^{-1}(\zeta)$ depends only on the smoothness of Γ in a neighborhood of ζ.

Theorem 2.12: Analytic Continuation of Riemann Maps Across Shared Arcs in Nested Jordan Domains

Let Ω_1 and Ω_2 be Jordan domains such that $\Omega_1\subset\Omega_2$ and let $\gamma\in\partial\Omega_1\cap\partial\Omega_2$ be an open subarc. Let φ_j be a conformal map of $\mathbb{D}$ onto Ω_j. Then $\psi:=\varphi_2^{-1}\circ\varphi_1$ has an analytic continuation across $\varphi_1^{-1}(\gamma)$, and $\psi'\neq 0$ on $\varphi_1^{-1}(\gamma)$.

Proof:

The analytic function $\psi := \varphi_2^{-1} \circ \varphi_1$ from $\mathbb{D}$ into $\mathbb{D}$ has a continuous and unimodular ($|\psi(z)| = 1$) extension to the arc $\varphi_1^{-1}(\gamma)$. By Schwartz reflection ψ has an analytic and one-to-one extension to a neighborhood of $\varphi_1^{-1}(\gamma)$ in $\mathbb{C}$ and hence $\psi' \neq 0$ on $\varphi_1^{-1}(\gamma)$.

$\square$

Let Γ be an arc parameterized as $\{\zeta(t) : a < t < b\}$.

Definition: Tangent of Arc

We say Γ has a tangent arc at $\zeta_0 := \zeta(t_0)$ if

$$\lim_{t \downarrow t_0} \frac{\zeta(t) - \zeta_0}{|\zeta(t) - \zeta_0|} = e^{i\tau} \tag{2.34}$$

and

$$\lim_{t \uparrow t_0} \frac{\zeta(t) - \zeta_0}{|\zeta(t) - \zeta_0|} = -\, e^{i\tau} \tag{2.35}$$

where $0 \le \tau \le 2\pi$.

Definition: Unit Tangent Vector of Arc

If both (2.34) and (2.35) are valid, then Γ has a unit tangent vector $e^{i\tau}$ at ζ_0. Note that once Γ admits a tangent vector it admits a unit tangent vector by normalizing.

Except for reversals of orientation, the existence of a tangent at ζ_0 and its value $e^{i\tau}$ do not depend on the choice of the parameterization $t \mapsto \zeta(t)$.

Definition: Continuous Tangent of Arc

We say that Γ has a continuous tangent if Γ has a tangent at each $\zeta \in \Gamma$ and if $e^{i\tau(\zeta)}$ is continuous on Γ (in ζ).

Theorem 2.13: Criterion for Tangent and Continuous Tangent on Jordan Boundary

The curve Γ has a tangent at $\zeta = \varphi(e^{i\theta})$ if and only if the limit

$$\lim_{\mathbb{D} \ni z \to e^{i\theta}} \arg\left(\frac{\varphi(z) - \zeta}{z - e^{i\theta}}\right) \tag{2.36}$$

exists and is finite. In that case,

$$\lim_{\mathbb{D} \ni z \to e^{i\theta}} \arg\left(\frac{\varphi(z) - \zeta}{z - e^{i\theta}}\right) = \tau(\zeta) - \theta - \frac{\pi}{2} \mod 2\pi. \tag{2.37}$$

The curve Γ has a continuous tangent if and only if $\arg \varphi'(z)$ has a continuous extension to $\overline{\mathbb{D}}$. Moreover, if Γ has a continuous tangent, then $\varphi \in C^\alpha(\partial \mathbb{D})$ for all $\alpha < 1$ and Γ rectifiable (a rectifiable curve is a curve of finite length).

There exist Jordan domains Ω and conformal maps $\varphi : \mathbb{D} \to \Omega$ such that $\partial\Omega$ has a continuous tangent but $\varphi \notin C^1(\partial \mathbb{D})$. An example can be built from the connection in the previous section between conjugate functions and conformal maps.

Example 2.3: Conformal Map with Continuous Tangent But Not Continuously Differentiable

If $u(e^{i\theta})$ is continuous and $|u| < \frac{\pi}{2}$, then $u(e^{i\theta}) = \arg \varphi'(e^{i\theta})$, where φ is a conformal mapping onto a Jordan domain, but

$$\widetilde{u}(e^{i\theta}) = -\log|\varphi'(e^{i\theta})|$$

may not be bounded above or below, but $\widetilde{u} \notin C^1(\partial \mathbb{D})$. $\diamond$

Proof of Theorem 2.13:

Set

$$v(z) := \arg\left(\frac{\varphi(z)-\zeta}{z-e^{i\theta}}\right).$$

Since $\dfrac{\varphi(z)-\zeta}{z-e^{i\theta}}$ is holomorphic and $v(z)$ is the imaginary part of a holomorphic function, it follows that $v(z)$ is harmonic on $\mathbb{D}$. Moreover, v is continuous on $\overline{\mathbb{D}}\backslash\{e^{i\theta}\}$.

Step I: Criterion for Tangent

If Γ has a tangent ar $\zeta = \varphi(e^{i\theta})$ or if the limit **(2.36)** exists at $e^{i\theta}$, then $v(z)$ is bounded on $\mathbb{D}$. Thus, in either case, v is the Poisson integral of $v(e^{i\theta})$ by an application of **Corollary 1.4.1**. Therefore, by Carathéodory's **Theorem 1.8**, v has a continuous extension to $e^{i\theta}$ if and only if $v|_{\partial\mathbb{D}}$ has a continuous extension to $e^{i\theta}$, and if and only if Γ has a tangent at ζ by the definition **(2.34)** and **(2.35)** for tangent. Furthermore, **(2.37)** holds when v is continuous at $e^{i\theta}$ (the modulo term 2π comes from aperiodic).

Step II: Criterion for Continuous Tangent

We shall prove the sufficiency first, then the necessity, and finally the assertion in the moreover part.

Step II.1: Sufficiency

Now suppose Γ has a continuous tangent. Then $\tau\circ\varphi$ is continuous on $\partial\mathbb{D}$ since this is the composition between two continuous functions. If $h \neq 0$, then

$$A_h(z) := \arg\left(\frac{\varphi(ze^{ih})-\varphi(z)}{z(e^{ih}-1)}\right)$$

is continuous on $\overline{\mathbb{D}}$ and harmonic on $\mathbb{D}$ (use the same argument as we did for v in the first step). For $|z| < 1$, a direct computation yields

$$\lim_{h\to 0} A_h(z) = \arg\varphi'(z).$$

For $|z| = 1$, there exist k such that $0 < |k| < |h|$ such that

$$\arg\left(\frac{\varphi(ze^{ih})-\varphi(z)}{h}\right) = \arg\big(\varphi(ze^{ih}-\varphi(z)\big) \quad \text{(Mean Value Theorem)}$$

$$= (\tau\circ\varphi)(ze^{ik}) \quad (\text{Definition of } \tau\circ\varphi)$$

Consequently, by using the above two displays, one obtains

$$\lim_{h\to 0} A_h(z) = (\tau\circ\varphi)(z) - \arg(z) - \frac{\pi}{2},$$

with the convergence uniformly on $\partial\mathbb{D}$ by Dini's theorem (see Ransford), and hence $\arg(\varphi')$ is the Poisson integral of the continuous function

$$(\tau\circ\varphi)(e^{i\theta}) - \theta - \frac{\pi}{2}$$

by **(1.7)** in **Theorem 1.3**.

Step II.2: Necessity

Now suppose $\arg\varphi'(z)$ has a continuous extension to $\overline{\mathbb{D}}$. For $r < 1$, the curve

$$\Gamma_r := \left\{\varphi(re^{i\theta}) : 0 \le \theta \le 2\pi\right\}$$

has tangent $e^{i(\tau\circ\varphi)(re^{i\theta})}$ satisfying

$$e^{i(\tau\circ\varphi)(re^{i\theta})} = ie^{i\theta}e^{i\arg\varphi'(re^{i\theta})},$$

and $e^{i(\tau\circ\varphi)(z)}$ has a continuous extension to $\overline{\mathbb{D}}\backslash\{0\}$ by an application of Carathéodory's **Theorem 1.8**. Then for $h > 0$, we have

$$\arg\big(\varphi(e^{i(\theta+h)}) - \varphi(e^{i\theta})\big) = \lim_{r\to 1}\arg\big(\varphi(re^{i(\theta+h)}) - \varphi(re^{i\theta})\big) \quad \text{(Continuity)}$$

$$= \lim_{r\to 1}\tau\circ\varphi(re^{i(\theta+k_r)}) \quad \text{(Mean Value Theorem)}$$

where $0 < k_r < h$. Therefore,

$$\lim_{h\downarrow 0}\arg\big(\varphi(e^{i(\theta+h)}) - \varphi(e^{i\theta})\big) = \lim_{z\to e^{i\theta}}\tau\circ\varphi(z),$$

and a similar argumetn applies for $h < 0$:

$$\lim_{h\uparrow 0}\arg\big(\varphi(e^{i(\theta+h)}) - \varphi(e^{i\theta})\big) = \lim_{z\to e^{i\theta}}\tau\circ\varphi(z) + \pi.$$

Thus Γ has a continuous tangent by definition.

Step II.3: Moreover Part of Criterion for Continuous Tangent

If Γ has a continuous tangent, then by **Theorem 2.9** (b),

$$\sup_{r<1}\int\left|\varphi'(re^{i\theta})\right|^{\lambda}d\theta = \sup_{r<1}\int e^{-\lambda(\widetilde{\arg\varphi'})(re^{i\theta})}d\theta =: B_\lambda < \infty$$

for all $\lambda < \infty$, where B_λ depends on φ and λ. Take $\lambda := \dfrac{1}{1-\alpha}$, where $0 < \alpha < 1$. Let $a < b < a + \pi$. Then for any $r < 1$,

$$\int_a^b \left|\varphi'(re^{i\theta})\right|d\theta \le |b-a|^{\alpha}\Big(\int_a^b\left|\varphi'(re^{i\theta})\right|^{\lambda}d\theta\Big)^{1-\alpha} \quad \text{(Hölder's Inequality)}$$

$$\le |b-a|^{\alpha}B_\lambda^{1-\alpha} \quad \text{(Definition of } B_\lambda)$$

Therefore if $a = \theta_0 < \theta_1 < \cdots < \theta_n = b$, one has

$$\sum_{j=1}^{n}\left|\varphi(re^{i\theta_j}) - \varphi(re^{i\theta_{j-1}})\right| \le \int_a^b\left|\varphi'(re^{i\theta})\right|d\theta \le |b-a|^{\alpha}B_\lambda^{1-\alpha}. \quad (2.38)$$

Sending $r \to 1$ yields the rectifiability of Γ and $\varphi \in C^{\alpha}$ for all $\alpha < 1$.

□

Let k be a non-negative integer and let $0 \le \alpha < 1$.

Definition: Alpha-Hölder Class for Arc

We say that the curve Γ is of the class $C^{k+\alpha}$ if

(i) Γ is rectifiable.

(ii) In the arc parameterization

$$\Gamma = \{\gamma(s) : 0 \le s \le \ell(\Gamma) := \text{length}(\Gamma)\},$$

the function γ is k times continuously differentiable and

$$\frac{d^k\gamma}{ds^k} \in C^{\alpha} \text{ for } \alpha > 0.$$

Now we can state and prove the main result of this section.

Theorem 2.14: Kellogg's Theorem

Let $k \ge 1$ and $0 < \alpha < 1$. Then the following conditions are equivalent:

(a) Γ is of class $C^{k+\alpha}$.

(b) $\arg\varphi' \in C^{k-1+\alpha}(\partial\mathbb{D})$.

(c) $\varphi \in C^{k+\alpha}(\overline{\mathbb{D}})$ and $\varphi' \neq 0$ on $\overline{\mathbb{D}}$.

Remark 2.17: $\alpha = 0$ in **Theorem 2.14** Ruins Equivalence

Note that if $\alpha = 0$ and $k \geq 1$ then (a) $\not\Rightarrow$ (c) but (a) $\Leftrightarrow$ (b) still. ⋄

We need an elementary lemma.

Lemma 2.15: Hölder Continuity Equivalence for Analytic Functions and Its Inverse

Let k be a positive integer, and let $0 \leq \alpha < 1$. Let $f \in C^1([0,1])$ satisfying $f' > 0$ and $g \equiv f^{-1}$. Then

$$g \in C^{k+\alpha} \Leftrightarrow f \in C^{k+\alpha}.$$

Proof:

The case $\alpha = 0$ and $k = 1$ is clear since

$$g'(y) = \frac{1}{f' \circ g(y)} > 0.$$

Case I: $\alpha = 0$ and $k \geq 2$

If $\alpha = 0$ and $k \geq 2$, the proof is by induction: if $f \in C^k$ and $g \in C^{k-1}$, then $f' \circ g \in C^{k-1}$ and because $f' > 0$,

$$g' = \frac{1}{f' \circ g} \in C^{k-1}.$$

Hence $g \in C^k$.

Case II: $\alpha > 0$ and $k = 1$

Now suppose that $\alpha > 0$ and $k = 1$. If $f' \in C^\alpha$ then

$$\begin{aligned}
|g'(y_1) - g'(y_2)| &= \left| \frac{1}{f' \circ g(y_1)} - \frac{1}{f' \circ g(y_2)} \right| \quad \text{(Definition of } g'\text{)} \\
&\leq \frac{\left| f' \circ g(y_2) - f' \circ g(y_1) \right|}{\min |f'|^2} \quad \text{(Minimizing Denominator)} \\
&\leq C |g(y_2) - g(y_1)|^\alpha \quad (\alpha\text{-Hölder}) \\
&\leq C' |y_2 - y_1|^\alpha \quad (\alpha\text{-Hölder})
\end{aligned}$$

and so $g' \in C^\alpha$.

Case III: $\alpha > 0$ and $k \geq 2$.

Finally, assume $\alpha > 0$ and $k \geq 2$. If $f \in C^{k+\alpha}$ then $g \in C^k$ and $g^{(k)}$ can be written as a sum of products of the functions

$$g^{(1)}, \cdots, g^{(k)}, f^{(2)} \circ g, \cdots, f^{(k)} \circ g$$

by applying chain rule. All these functions are C^1, except perhaps $f^{(k)} \circ g$, but $f^{(k)} \circ g \in C^\alpha$ and thus $g^{(k)} \in C^\alpha$. The converse holds by swapping the role of f and g in the above argument.

□

Proof of **Theorem 2.14**:

We first prove (b) $\Leftrightarrow$ (c), then we prove (c) $\Rightarrow$ (a), finally we prove (a) $\Rightarrow$ (b).

Step I: (b) $\Leftrightarrow$ (c)

We first prove the sufficiency.

Step I.1: (b) $\Rightarrow$ (c)

If $\arg \varphi' \in C^{k-1+\alpha}(\partial\mathbb{D})$, $0 < \alpha < 1$, then by **Corollary 2.10.1**, one has

$$\log|\varphi'| \in C^{k-1+\alpha}.$$

Taking exponential results in

$$\varphi' \in C^{k-1+\alpha}(\overline{\mathbb{D}})$$

and $\varphi' \neq 0$.

Step I.2: (c) ⇒ (b).

Conversely, if $\varphi' \in C^{k-1+\alpha}(\overline{\mathbb{D}})$ and $\varphi' \neq 0$, then

$$e^{i\arg(\varphi')} = \frac{\varphi'}{|\varphi'|} \in C^{k-1+\alpha}$$

and $\arg\varphi' \in C^{k-1+\alpha}(\partial\mathbb{D})$.

Step II: (c) ⇒ (a)

Assume (c) holds. If

$$s(\theta) := \int_0^\theta \left|\varphi'(e^{it})\right| dt,$$

then $s' > 0$ and $s'(\theta) = |\varphi'(e^{i\theta})| \in C^{k-1+\alpha}$. Thus, a direct application of **Lemma 2.15** yields

$$\theta'(s) \in C^{k-1+\alpha}.$$

By **(2.37)** in **Theorem 2.13**, one has

$$\arg\frac{d\gamma}{ds} = \arg\varphi'(e^{i\theta(s)}) + \frac{\pi}{2} + \theta(s). \tag{2.39}$$

Since $\arg\varphi' \in C^{k-1+\alpha}$ and $\theta' \in C^{k-1+\alpha}$, we conclude from **Corollary 2.10.1** that

$$\frac{d\gamma}{ds} \in C^{k-1+\alpha} \text{ and } \Gamma \text{ is of the class } C^{k+\alpha},$$

as desired.

Step III: (a) ⇒ (b).

Now assume (a) holds, i.e., assume Γ is of class $C^{k+\alpha}$. Then by an application of **Theorem 2.13**, we obtain $\arg\varphi' \in C$; and by **(2.39)**, we have

$$\frac{d(\gamma\circ s)(\theta)}{ds} \in C.$$

Step III.1: (a) ⇒ (b) when $k = 1$.

If $k = 1$, so that $\frac{d\gamma}{ds} \in C^\alpha$, then

$$\left|\frac{d\gamma}{ds}s(\theta_1) - \frac{d\gamma}{ds}s(\theta_2)\right| \leq C|s(\theta_1) - s(\theta_2)|^\alpha \tag{2.40}$$

so that by **(2.38)**,

$$\left|\frac{d\gamma}{ds}s(\theta_1) - \frac{d\gamma}{ds}s(\theta_2)\right| \leq C|\theta_1 - \theta_2|^{(1-\varepsilon)\alpha}$$

for any $\varepsilon > 0$. Thus, by **Corollary 2.10.1**,

$$\arg\varphi' \in C^{(1-\varepsilon)\alpha} \text{ and } \varphi' \in C^{(1-\varepsilon)\alpha}.$$

Thus, by the definition of s, we have

$$s'(\theta) = |\varphi'(e^{i\theta})| \in C^{(1-\varepsilon)\alpha} \text{ and } |s(\theta_1) - s(\theta_2)| \leq K|\theta_1 - \theta_2|.$$

But then by **(2.39)** and **(2.40)**, $\arg\varphi' \in C^\alpha$. Moreover, by the equivalent relations (a) and (b) in **Theorem 2.10**, we obtain

$$s'(\theta) = |\varphi'(e^{i\theta})| \in C^{\alpha} \text{ and } s \in C^{1+\alpha}.$$

This concludes the case when $k = 1$ according to the definition of s.

Step III.2: (a) $\Rightarrow$ (b) when $k \geq 2$

Finally, when $k \geq 2$ we use induction. If $\dfrac{d\gamma}{ds} \in C^{k-1+\alpha}$ and if $s(\theta) \in C^{k-1+\alpha}$, then by (2.39), $\arg \varphi' \in C^{k-1+\alpha}$, so that

$$s' = |\varphi'| = \in C^{k-1+\alpha}.$$

Using **Theorem 2.10** once more, it follows that $s \in C^{k+\alpha}$, as desired

□

Let ℓ be a non-negative integer and let $0 \leq \beta \leq 1$. If Γ is of class $C^{k+\alpha}$ and if $f \in C(\Gamma)$, we say $f \in C^{\ell+\beta}(\Gamma)$ if $(f \circ \gamma)(s) \in C^{\ell+\beta}$, when viewed as a function of arc length on Γ.

Corollary 2.14.1: Change of Hölder Class Coefficients under Conformal Map on Arc

Suppose Γ is of class $C^{k+\alpha}$, where $k + \alpha > 1$, and suppose that $f \in C^{\ell+\beta}(\Gamma)$. Set

$$n + \sigma := \min(k + \alpha, \ell + \beta) \text{ where } 0 < \sigma < 1 \text{ and } n \in \mathbb{Z}^+ \cup \{0\}.$$

Let φ be a conformal map of $\mathbb{D}$ onto Ω, let G be the Herglotz integral of $f \circ \varphi$, and let $F := G \circ \varphi^{-1}$. Then

$$F \in C^{n+\sigma}(\overline{\Omega}).$$

Proof:

Using **Corollary 2.10.1**, **Theorem 2.14**, and **Lemma 2.15**.

□

The same result holds for finitely connected Jordan domains whose boundary curves are of class $C^{k+\alpha}$, except that harmonic conjugate and Herglotz integrals cannot be defined in multiply connected domains.

Corollary 2.14.2: Change of Hölder Coefficients under Conformal Maps on F.C.J.D.

Let $\partial\Omega$ be a fininte union of pairwise disjoint $C^{k+\alpha}$ Jordan curves, where $k + \alpha > 1$, and let $f \in C(\partial\Omega)$ be a $C^{\ell+\beta}$ function of arc length on each component of $\partial\Omega$. Set

$$n + \sigma := \min(k + \alpha, \ell + \beta) \text{ where } 0 < \sigma < 1 \text{ and } n \in \mathbb{Z}^+ \cup \{0\}.$$

Then $u(z) := u_f(z)$ and its first n partial derivatives extend continuously to $\overline{\Omega}$ and

$$|D^n u(z_1) - D^n u(z_2)| \leq K |z_1 - z_2|^{\sigma}$$

for all $z_1, z_2 \in \overline{\Omega}$, where D^n denotes any n-th partial derivative.

Proof:

By **Theorem 2.12** it suffices to work in some neighborhood of $\zeta \in \partial\Omega$. Let J be the component of $\partial\Omega$ such that $\zeta \in J$. Let Ω_1 be that component of $\mathbb{C}^{\infty} \backslash J$ such that $\Omega \subset \Omega_1$, and let u_1 be the solution to the Dirichlet problem on Ω_1 with boundary value f.

Near J, u_1 has the required smoothness by **Corollary 2.14.1**. If φ_1 is a conformal map of $\mathbb{D}$ onto Ω, then

$$v := (u - u_1) \circ \varphi_1 \text{ is harmonic on } A := \{r < |z| < 1\}$$

for some $r < 1$, v is continuous on $\overline{A}$, and $v = 0$ on $\{|z| = 1\}$. Now according to **Remark 2.8**, the reflection of v extends to be harmonic across $\partial\mathbb{D}$. Hence $v \in \mathbb{C}^\infty$ and

$$u = u_1 + v \circ \varphi_1^{-1}$$

has as much smoothness as u_1 and φ_1 both have.

□

Corollary 2.14.2 answers Question 1 from the end of **Section 2.2**. The next corollary answers Question 2.

Corollary 2.14.3: Absolute and Hölder Continuity of Harmonic Measure on F.C.J.D.

If $\partial\Omega$ consists of finitely many pairwise disjoint Jordan curves of class $C^{1+\alpha}$, where $\alpha > 0$, then

$$d\omega(z,\zeta) = -\frac{\partial g(z,\zeta)}{\partial \vec{n_\zeta}} \frac{ds(\zeta)}{2\pi}. \tag{2.41}$$

In other words, harmonic measure for $z \in \Omega$ is absolutely continuous with respect to the arc length on $\partial\Omega$, and the density

$$\frac{d\omega}{ds} = \frac{-1}{2\pi}\frac{\partial g(\zeta,z)}{\partial \vec{n_\zeta}} = P_\zeta(z)$$

is of class $C^\alpha(\partial\Omega)$ and satisfies

$$c_1 < \frac{d\omega}{ds} < c_2. \tag{2.42}$$

for positive constants c_1 and c_2.

Proof:

Let φ be a conformal map from Ω^* onto Ω, where $\partial\Omega^*$ consists of analytic Jordan curves. If $\zeta = \varphi(\zeta^*)$ and $z = \varphi(z^*)$, then

$$\frac{\partial g_\Omega(\zeta,z)}{\partial \vec{n_\zeta}} = \frac{\partial g_{\Omega^*}(\zeta^*,z^*)}{\partial \vec{n_{\zeta^*}}} \cdot \frac{1}{|\varphi'(\zeta^*)|}$$

and

$$|\varphi'(\zeta^*)| := \frac{ds(\zeta)}{ds(\zeta^*)}$$

by **Corollary 2.14.2** and the uniqueness of Green function (by Lindelöf's **Lemma 1.1**). By **Remark 2.12**, harmonic measure is conformally invariant and now **(2.42)** follows from the case when $\partial\Omega$ is analytic, i.e., by the result **Corollary 2.6.1**. Finally, using **Corollary 2.14.2** once more,

$$\frac{\partial g}{\partial \vec{n_\zeta}} \in C^\alpha(\partial\Omega).$$

Therefore, **(2.42)** holds because $|\varphi'| > 0$ on $\partial\Omega^*$ by **Theorem 2.14**.

□

When $\partial\Omega$ is of class C^1, harmonic measure is absolutely continuous with respect to the arc length, but the density may not be continuous, bounded, or even bounded below.

Example 2.4: Harmonic Measure $\ll$ Arc Length $\not\Rightarrow$ Bounded Density

There is a simply connected Jordan domain Ω such that $\partial\Omega \in C^1$ but such that no conformal map $\varphi : \mathbb{D} \to \Omega$ is of class C^1 on $\overline{\mathbb{D}}$. Worse yet, $|\varphi'|$ can have infinite non-tangential limit at some point on $\partial\mathbb{D}$. ⋄

The following example illustrates how Green's theorem can be applied when $\partial\Omega$ is of class $C^{1+\alpha}$, without first mapping to a domain with real analytic boundary.

Example 2.5: Green's Theorem Applied to F.C.J.D. without Real Analytic Boundary

Green's theorem can be used on a finitely connected domain bounded by a finite number of pairwise disjoint $C^{1+\alpha}$ curves with functions u and v in $C^1(\partial\Omega) \cap C^2(\Omega)$. ⋄

Summary of Chapter 2

In this chapter, we solve the Dirichlet problem on a domain bounded by a finite number of Jordan curves. Since solving the Dirichlet problem on a domain Ω is equivalent to constructing a harmonic measure on $\partial\Omega$, our aim is to generalize the harmonic measure to a broader class of boundary.

In the first section we introduced the Schwartz alternating method (see **Remark 2.2**) to prove **Solution to Dirichlet Problem on Finitely Connected Jordan Domain with Bounded Piecewise Continuous Boundary Function**. This result relaxes the boundary condition from continuous to bounded piecewise continuous, while extending the Dirichlet problem from a simply connected domain to a Jordan domain bounded by finitely many Jordan curves. Since the Dirichlet problem on Ω is solved, the construction of a **Harmonic Measure (over Finitely Connected Jordan Domain)** is immediate. In particular, the harmonic measure we constructed is Borel there and satisfies Harnack's inequality in conjunction with the uniform boundedness. The Schwartz alternating method relies on Lindelöf's maximal principle **Lemma 1.1**, hence the piecewise continuity, which is necessary for **Lemma 1.1**, cannot be relaxed, as **Remark 2.5** demonstrated.

In the second section, we study the Green function and the Poisson integral. We first introduced the **Green Function with Pole (over Bounded Domain)** along with some elementary properties in **Remark 2.6**. We extended the definition to unbounded domain **Green Function with Pole (over Unbounded Domain)**, and finally extended the definition to **Green Function with Pole (under Conformal Mapping)**. In particular, the Green function in all modes are uniquely determined by (iii), (iv), and (v) in **Remark 2.6**, via an application of Lindelöf's maximal principle **Lemma 1.1**. Moreover, **Remark 2.7** tells us that the Green function is conformal invariant on finitely connected Jordan domain. We then established the connection between Green function and conformal mapping via **Green Function as Log of Conformal Mapping**, we defined **Analytic Arc** and **Jordan Analytic Curve** so that we can work with Green function under conformal mapping. We showed that **Finitely Connected Jordan Domain Has Partition and Homeomorphism Extension on the Boundary**. The fact that the Green function is symmetric in the space variables is proved in **Theorem 2.4**. We then proved **Sufficiency for Harmonic Extension to Analytic Curve over Finitely Connected Jordan Domain**, from which the formula for harmonic measure that generalizes the Poisson integral on $\mathbb{D}$ is provided via Green function in **Theorem 2.6**. The formal definition for **Poisson Kernel (over Finitely Connected Jordan Domain)** is immediate, which generalizes our previous definition in the unit disc $\mathbb{D}$. Next, we proved that the harmonic measure is absolute continuous with respect to the arc length on $\partial\Omega$ and the density is real analytic there in **Corollary 2.6.1**. Finally, the **Solution to Dirichlet Problem over Finitely Connected Jordan Domain with Bounded Borel Boundary Data** is proved. This result relaxes piecewise continuity to Borel, hence results in an alternative definition for harmonic measure.

The rest of this chapter is to answer two questions that generalize our result in **Corollary 2.6.1**. The first is to translate the boundary condition to the boundary smoothness and the second is to relax the real-anlyticity. The main result is Kellogg's **Theorem 2.14**, which relies on the study of modes of continuity in **Section 2.3**.

Heuristically, the harmonic minorant (respectively, harmonic majorant) serves as the harmonic correction for continuity. In the third section, we defined the harmonic correction for differentiability, namely, **Harmonic Conjugate (Conjugate Function)**. We proved that **Non-Tangential Limit for Harmonic Conjugate Exists Almost Everywhere** and established **Connection Between Harmonic Conjugate and Conformal Map**. The harmonic conjugate does not behave well, as **Example 2.1** suggest, a bounded continuous boundary function may have unbounded harmonic conjugate. This forces us to control the unbounded nature for harmonic conjugate, and one of the attempts is **Zygmund's Exponential Integrability for Harmonic Conjugate**. We defined the **Alpha-Hölder Class and Alpha-Hölder Continuous Function**, and **Alpha-Hölder Norm**, **Herglotz Integral of Alpha-Hölder Continuous Class** to understand the room between continuously differentiability and continuity. We proved **Criterion for Alpha-Hölder Continuous Class with Norm Bound**, defined **k times Continuously Differentiable**, and proved **Criterion for Hölder Extension to Boundary of Unit Disc**. We defined **Zygmund Class** and **Zygmund Norm and Zygmund Function** and proved **Criterion for Zygmund Boundary Data with Zygmund Norm Bound**. Finally, the previous classes, in conjunction with **Lipschitz Function, Class of Lipschitz Function** and **Norm on Class of Lipschitz Function** enables us to understand the room for between continuously differentiability and continuity, illustrated in **Remark 2.16**.

The main result of the last section, and perhaps this chapter, is the promised Kellogg's **Theorem 2.14**. For us to state and prove it, we first proved **Analytic Continuation of Riemann Maps Across Shared Arcs in Nested Jordan Domains**. We defined modes of tangent: **Tangent of Arc**, **Unit Tangent Vector of Arc**, and **Continuous Tangent of Arc**. Then we proved **Criterion for Tangent and Continuous Tangent on Jordan Boundary**. In particular, Conformal map with continuous tangent may not be continuously differentiable, as illusrated in **Example 2.3**. We defined the **Alpha-Hölder Class for Arc** and proved **Kellogg's Theorem** via **Hölder Continuity Equivalence for Analytic Functions and Its Inverse**. The Kellogg's theorem is so important that the smoothness of the boundary function can be understood via the smoothness of curves on the boundary; and the degree of smoothness on arc can be found via **Corollary 2.14.1**. The answer to the first question is for the change of degree of smoothness on finitely connected Jordan domain, which is answered in **Corollary 2.14.2**. Finally, the answer to the second question, which relaxes the real analyticity hence generalized **Corollary 2.6.1**, is demonstrated in **Corollary 2.14.3**. We should keep in mind that the density between harmonic measure and arc length may be unbounded even they are absolutely continuous with respect to each other, the counter-example is given in **Example 2.4**.

3. Potential Theory

The goal of this chapter is to solve the Dirichlet problem on an arbitrary domain Ω. There are three traditional ways to solve the problem:

(i) The Wiener method is to approximate Ω from inside sub-domains Ω_n of the type studied in **Chapter 2** and to show that the harmonic measure $\omega(z, E, \Omega_n)$ converges weak* to a limit measure on $\partial\Omega$. With Wiener's method one must prove that the limit measure $\omega(z, E, \Omega)$ does not depend on the approximating sequence $\{\Omega_n\}_{n\geq 1}$.

(ii) The Perron method associates to any bounded function f on $\partial\Omega$ a harmonic function $\mathscr{P}_f$ on Ω. The function $\mathscr{P}_f$ is called the upper Perron envelip of a family of subharmonic functions constrained by f on $\partial\Omega$. Perron's method is elegant and general. With Perron's method the difficulty is linearity; one must prove that

$$\mathscr{P}_{-f} = -\mathscr{P}_f,$$

at least for f continuous.

(iii) The Brownian motion approach, originally from Kakutani in 1944, identifies $\omega(z, E, \Omega)$ with the probability that a random moving particle, starting at z, first hit $\partial\Omega$ in the set E. This method has considerable intuitive appeal, but it leaves many theorems hard to reach.

We shall follow Wiener's method and use the energy integral to prove that the limit $\omega(z, E, \Omega)$ is unique. This leads to the notion of capacity, equilibrium distribution, and regular point and to the characterization of regular points by Wiener series.

For the Perron method see Ahlfors 1979 or Tsuji 1959. Appendix F includes Kakutani's theorem for the discrete version of Brownian motion.

We conclude this chapter with some potential theoretic estimates for harmonic measures.

3.1 Capacity and Green Function

Let E be a compact plane set such that $\Omega := \mathbb{C}^\infty \backslash E$ is a finitely connected Jordan domain. By **Chapter 2** and a conformal mapping, we see that Ω has Green function $g_\Omega(z, \infty)$ with pole at ∞, and if $a \notin \overline{\Omega}$, by **(2.6)**, one has

$$g_\Omega(z, \infty) = \log|z - a| + h(z, \infty)$$

where $h(z, \infty)$ is harmonic on Ω by **Remark 2.6** (iv), by **Remark 2.6** (i) $h(z, \infty)$ is continuous on $\partial\Omega$, and by **Remark 2.6** (iii) one has

$$h(\zeta, \infty) = -\log|\zeta - a|, \zeta \in \partial\Omega.$$

(Recall that $u(z)$ is harmonic at ∞ if $u(1/z)$ is harmonic on a neighborhood of 0).

Our goal in this section is to (i) extend the definition for Green function to the finitely connected Jodan doamin, (ii) show that the capacity for approximating sequence $\{\Omega_n\}_{n\geq 1}$ converges to the capacity for Ω and the result is independent of the choice of

$\{\Omega_n\}_{n\geq 1}$, and (iii) calculate the logarithmic capacity for some elementary sets that are often encountered.

Definition: Robin's Constant

The quantity

$$\gamma := \gamma(E) := h(\infty, \infty)$$

is called Robin's constant for E, and we have

$$g_\Omega(z, \infty) = \log|z| + \gamma + o(1) \text{ as } z \to \infty. \tag{3.1}$$

Note that as $z \to \infty$, the Robin's constant is the harmonic correction with both argument being infinity. It may happen that the Robin's constant is ∞, hence we take the exponential to define the capacity. Here it is understood that $e^{-\infty} = 0$.

Defintion: Logarithmic Capacity

Define the logarithmic capacity of E to be

$$\mathrm{Cap}(E) := e^{-\gamma(E)}.$$

Thus $\mathrm{Cap}(E) > 0$ in the case at hand. When $\mathrm{Cap}(E) = 0$, E is called polar.

Remark 3.1: Log Capacity Scaling under Univalent Conformal Map

Let Ω_1 and Ω_2 be finitely connected Jordan domains such that $\infty \in \Omega^j$ and $E_j := \mathbb{C}^\infty \backslash \Omega_j$ for $j = 1,2$. Assume there is a conformal map ψ from Ω_1 onto Ω_2 such that for $|z|$ sufficiently large, one has

$$\psi(z) = az + b_0 + \frac{b_1}{z} + \cdots$$

with $a > 0$. Then

$$g_{\Omega_1}(z, \infty) = g_{\Omega_2}\big(\psi(z), \infty\big),$$

so that by **(3.1),**

$$\gamma(E_1) = \gamma(E_2) + \log a$$

and

$$\mathrm{Cap}(E_1) = a\mathrm{Cap}(E_2). \tag{3.2}$$

◇

Proposition 3.1: Logarithmic Capacity of Closed Disc

The capacity of a closed disc is the radius of the disc, i.e., $\mathrm{Cap}\big(\overline{\mathscr{B}}_r(x)\big) = r$ for $\mathscr{B}_r(x) := \{y : |x - y| < r\}$ for some $r > 0$.

Proof:

The desired result follows from

$$g_{\mathbb{C}^\infty\backslash\overline{\mathbb{D}}}(z, \infty) = \log|z|$$

by **Theorem 2.2**.

□

Now let E be any compact plane set and unite Ω for the component of $\mathbb{C}^\infty\backslash E$ such that $\infty \in \Omega$. Fix a sequence $\{\Omega_n\}_{n\geq 1}$ of finitely connected domains such that

$$\infty \in \Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega,$$

such that

$$\Omega = \bigcup_{n\geq 1} \Omega_n,$$

and such that $\partial\Omega_n$ consists of $C^{1+\alpha}$ Jordan curves for some $\alpha > 0$. Define $E_n := \mathbb{C}\backslash\Omega_n$.

Theorem 3.2: Capacity and Robin Constant Independent of Approximating Sequence

The definition for logarithmic capacity and Robin's constant we have constructed are independent of the choice of the approximating sequence $\{\Omega_n\}_{n\geq 1}$.

Proof:

Because $\overline{\Omega}_n \subset \Omega_{n+1}$, it follows from an ordinary maximum principle for harmonic functions that

$$g_{\Omega_{n+1}}(z,\infty) > g_{\Omega_n}(z,\infty) \text{ on } \Omega_n,$$

and hence that

$$\gamma(E_{n+1}) > \gamma(E_n) \text{ and } \mathrm{Cap}(E_{n+1}) < \mathrm{Cap}(E_n).$$

Now define

$$\mathrm{Cap}(E) := \lim_{n\to\infty} \mathrm{Cap}(E_n). \tag{3.3}$$

Because $g_\Omega(z,w)$ is an increasing function of Ω, an interlacing of the domains Ω_n shows that the definition (3.3) does not depend on the choice of the sequence $\{\Omega_n\}_{n\geq 1}$. Note that if $\widehat{E} := \mathbb{C}\backslash\Omega$, then by definition,

$$\mathrm{Cap}(\partial E) = \mathrm{Cap}(E) = \mathrm{Cap}(\widehat{E}) = \mathrm{Cap}(\partial\widehat{E}). \tag{3.4}$$

By definition, the Robin constant

$$\gamma(E) \equiv \log\left(\frac{1}{\mathrm{Cap}(E)}\right) = \lim_{n\to\infty}\gamma(E_n)$$

is Robin's constant for the arbitrary set E.

□

Now we can talk about some elementary properties for logarithmic capacity and Robin's constant. Scaling under univalent conformal mapping has already been offered in **Remark 3.1**.

Proposition 3.3: Monotonicity for Capacity and Robin's Constant

If $E \subset F$ then

$$\mathrm{Cap}(E) \leq \mathrm{Cap}(F) \text{ and } \gamma(E) \geq \gamma(F). \tag{3.5}$$

Proof:

This is an immediate consequence from (3.3).

□

If $\mathrm{Cap}(E) > 0$, then

$$\lim_{n\to\infty}\gamma(E_n) = \gamma(E) < \infty,$$

and by Harnack's principle

$$g_\Omega(z,\infty) = \lim_{n\to\infty} g_{\Omega_n}(z,\infty)$$

defines a harmonic function Ω having expansion

$$g_\Omega(z,\infty) = \log|z| + \gamma(E) + o(1) \tag{3.6}$$

at infinity. When $z = \infty$, the symmetry (2.9) in **Theorem 2.4** shows that

$$g_\Omega(z,w) = \lim_{n\to\infty} g_{\Omega_n}(z,w) \tag{3.7}$$

exists for all $z, w \in \Omega$ for $z \neq w$, and $g_\Omega(z,w)$ satisfies condition (iii), (iv), and (v) in

Remark 2.6, hence uniquely identified.

Definition: Green Function with Pole (over F.C.J.D.)

The function $g_\Omega(z, w)$ satisfies **(3.7)**, (iii), (iv), and (v) in **Remark 2.6** is the Green function for Ω with pole at w.

Using our definition for capacity in conjunction with the univalent conformal mapping, the following result

Proposition 3.4: Logarithmic Capacity for Interval

Let $\alpha < \beta$ be two real numbers. Then $\operatorname{Cap}\big([\alpha, \beta]\big) = \dfrac{\beta - \alpha}{4}$.

Proof:

Suppose E is compact and connected. Let Ω be the component of $\mathbb{C}^\infty \backslash E$ such that $\infty \in \Omega$ and let $\psi : \Omega \to \mathbb{C}^\infty \backslash \overline{\mathbb{D}}$ be the conformal mapping such that for $|z|$ sufficiently large, one has

$$\psi(z) = az + b_0 + \frac{b_1}{z} + \cdots$$

with $a > 0$. For $r > 1$,

$$\Omega_r := \{z : |\psi(z)| > r\}$$

is bounded by an analytic Jordan curve and

$$g_{\Omega_r}(z, \infty) = \log\left|\frac{\psi(z)}{r}\right|.$$

Then by **(3.7)**, as $r \downarrow 1$, $g_\Omega(z, \infty) = \log|\psi(z)|$. Then using **Proposition 3.1** in the first equality and **(3.2)** in the second, one has

$$1 = \operatorname{Cap}(\overline{\mathbb{D}}) = a\operatorname{Cap}(E).$$

Consequently, for $[\alpha, \beta] \subset \mathbb{R}$, the normalized conformal map has scaling factor a with respect to $\overline{\mathbb{D}}$ being $\dfrac{\beta - \alpha}{4}$. It follows that

$$\operatorname{Cap}\big([\alpha, \beta]\big) = a\operatorname{Cap}(\overline{\mathbb{D}}) = \frac{\alpha - \beta}{4}$$

as desired.

□

Proposition 3.5: Log Capacity Lower Bound for Subset of Unit Disc

If $E \subset \partial\mathbb{D}$ then $\operatorname{Cap}(E) \geq \sin\left(\dfrac{\operatorname{Leb}(E)}{4}\right)$, where $\operatorname{Leb}(E)$ denotes the usual Lebesgue measure.

Proof:

If E is an arc on $\partial\Omega$, then after a conformal mapping, **Proposition 3.4** gives

$$\operatorname{Cap}(E) = \sin\left(\frac{\operatorname{Leb}(E)}{4}\right).$$

Because of **(3.3)**, we may assume that E is a finite union of arcs. Define

$$F(z) := \frac{1}{4}\int_E \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta,$$

and let $\omega(z) = \omega(z, E, \mathbb{D})$. Then by definition of harmonic conjugate and **(2.23)**, one has

$$\overline{F(1/\overline{z})} = -F(z) \text{ and } F = \frac{\pi}{2}(\omega + i\widetilde{\omega}) \text{ on } \mathbb{D}.$$

Therefore,

$$\frac{-\pi}{2} \le \operatorname{Re}(F) \le \frac{\pi}{2}$$

on $\Omega := \mathbb{C}^\infty \backslash E$ and $F(0) = -F(\infty) = \dfrac{\operatorname{Leb}(E)}{4}$. Hence $H(z) := e^{iF(z)}$ maps Ω into the right half plane and $H(\infty) = \overline{H}(0)$. Now

$$G(z) = \log\left| z \frac{H(z) + \overline{H}(0)}{H(z) - H(0)} \right|$$

is superharmonic on Ω and

$$\lim_{z \in \Omega \to \partial\Omega} G(z) \ge 0.$$

By the maximum principle, $G(z) > 0$ in Ω so that

$$G(z) \ge g_{\Omega_n}(z, \infty) \text{ for all } n.$$

For $|z|$ sufficiently large,

$$G(z) = \log|z| - \log \sin\left(\frac{\operatorname{Leb}(E)}{4}\right) + \cdots$$

so that by (3.1),

$$-\log \sin\left(\frac{\operatorname{Leb}(E)}{4}\right) \ge \gamma(\partial\Omega_n) \to \gamma(E).$$

It follows that

$$\operatorname{Cap}(E) \ge \sin\left(\frac{\operatorname{Leb}(E)}{4}\right),$$

as desired.

□

3.2 The Logarithmic Potential

Let μ be a finite, compactly supported signed measure.

Definition: Logarithmic Potential

The logarithmic potential of μ is the function

$$U_\mu(z) := \int \log \frac{1}{|\zeta - z|} d\mu(\zeta).$$

Note that the logarithmic potential can also be defined as $\int \log|\zeta - z| \, d\mu(\zeta)$, the only difference is that in our definition the log potential is a superharmonic function, and the alternative definition makes the log potential become subharmonic.

Remark 3.2: Log Potential Converges Absolutely Lebesgue-Almost Everywhere

By Fubini's theorem, the integral U_μ is absolutely convergent for Leb-almost everywhere z. In particular, since we are working with the plane, the integral U_μ is absolutely convergent for area-almost every z. ◇

Note that the name for the following lemma is not always correct! It depends on the way we define the log potential, as prescribed.

Lemma 3.6: Log Potential as Superharmonic Function (In Our Setting)

If $\mu > 0$, the log potential $U_\mu(z)$ is lower semicontinuous and superharmonic.

Proof:

The lemma holds because for ζ fixed, the function

$$z \mapsto \log \frac{1}{|\zeta - z|}$$

is lower semicontinuous and superharmonic. Moreover, since for positive measure μ, the integration preserves subharmonicity and lower semicontinuity, result follows.

□

The next theorem connects the notions of log potential, Green function, capacity, and harmonic measure. Suppose $\Omega := \mathbb{C}^\infty \backslash E$ is bounded by a finite family of disjoint $C^{1+\alpha}$ Jordan curves, $\alpha > 0$, write μ_E for the harmonic measure of ∞ relative to Ω,

$$d\mu_E := d\omega(\infty, \cdot, \Omega) = \frac{-1}{2\pi} \partial g(\zeta, \infty) \partial \overrightarrow{n_\zeta} ds = P_\infty(\zeta) ds \qquad (3.8)$$

where the second and the third relation holds by **Corollary 2.14.3**.

Theorem 3.7: Fundamental Identity for Log Potential

If $\Omega := \mathbb{C}^\infty \backslash E$ is connected and bounded by finitely many pairwise disjoint $C^{1+\alpha}$ Jordan curves, then

(a) The integral $U_{\mu_E}(z)$ is absolutely convergent at every $z \in \mathbb{C}$.

(Absolutely convergent Log Potential)

(b) The potential U_{μ_E} is continuous on $\mathbb{C}$.

(Continuity of Log Potential)

(c.1) For $z \in \overline{\Omega}$,

$$g(z, \infty) = \gamma(E) - U_{\mu_E}(z). \qquad (3.9)$$

(c.2) For $z \in \Omega$,

$$U_{\mu_E}(z) < \gamma(E). \qquad (3.10)$$

(c.3) For $z \in \mathbb{C}^\infty \backslash \Omega$,

$$U_{\mu_E}(z) = \gamma(E). \qquad (3.11)$$

The identity **(3.9)** is also known as the fundamental identity for Green function. The identity reads as the Green function with hole at infinity equals to the Robin's constant of the exterior domain subtracts the harmonic measure of infinity relative to the domain. Moreover, the reason that the identity is fundamental since it reflects the way we solve the Dirichlet problem: in Wiener and Brownian motion approaches, we use Robin's constant as the subtractor; in Perron's approach, the subtractor is Perron's envelop.

Remark 3.3: Harmonic Measure of ∞ Relative to Ω as Equilibrium Measure

Later we shall see that μ_E is the unique probability measure on E such that U_μ is constant on E. For this reason μ_E is called the equilibrium distribution of E.

In particular, every equilibrium measure is a harmonic measure but the converse is not true. This concept is helpful in two ways, we make a remark for two interpretations:

Remark 3.4: Harmonic Measure as Inner Measure

The equilibrium measure μ_E has constant log potential on exterior domain by **Remark 3.2**. Then the fundamental identity for Green function **(3.9)** tells us that: At equilibrium state, the Green function equals to the Robin's constant of exterior domain minus a constant term. Moreover, since Robin's constant is the magnitude of the log energy (different in a minus sign), this tells us that at equilibrium state, the Green function equals to the log energy of the exterior domain. Therefore, the equilibrium measure, as a 'mass distribution' to the harmonic functions approximating $\partial\Omega$ from inside, can be regarded as an inner measure. Substituting Robin's constant by Perron's envelop, the same holds for Perron's approach. ⋄

The definition for harmonic measure $\omega(z, E, \Omega)$ is then clear: Note that the harmonic measure is a transition density, i.e. it is a harmonic in z and a probability measure in E. When equilibrium, then the probability measure for E is a constant, hence the harmonic measure is itself the solution to the Dirichlet problem, which behave like an inner measure. On the other hand, when the equilibrium state is not obtained, the harmonic measure still behave like an inner measure, but with conditioning on the exterior domain. In all scenarios, our interpretation is correct.

The second interpretation is that if we are given a collection of harmonic measure indexed by time $t \in \mathbb{R}$, then $\{\omega^t(z, E, \Omega)\}_{t\in\mathbb{R}}$ should converge to an equilibrium measure when $t \to \infty$. Thus, the equilibrium measure can be regarded as the asymptotic behavior of the harmonic measures. Moreover, consider a collection of harmonic measures that starts at equilibrium, then not-equilibrium, and finally converge to equilibrium; then the equilibrium state is recurrence. This interpretation opens the way to the study of stochastic solution to Dirichlet problems.

Proof of **Theorem 3.7**:

We can assume $0 \notin \overline{\Omega}$ (otherwise scaling and translating $\overline{\Omega}$).

Step I: Assertion (a)

Clearly, the integral $U_\mu(z)$ is absolutely convergent at all $z \notin \partial\Omega$ since

$$g(z, \infty) = \log|z| - \int_{\partial\Omega} \log|\zeta|\, d\omega(z, \zeta)$$

by the definition of log potential and **(3.1)**, and

$$\gamma(E) = -\int_{\partial\Omega} \log|\zeta|\, d\omega(z, \zeta)$$

by the definition of Robin's constant.

Step II: (c.1) for $z \notin \partial\Omega$ and (c.2)

For fixed $z_0 \in \Omega$,

$$g(z, z_0) = \log\left|\frac{z}{z - z_0}\right| - \int_{\partial\Omega} \log\left|\frac{\zeta}{\zeta - z_0}\right| d\omega(z, \zeta), \tag{3.12}$$

by definition of log potential and Green function. Because the right hand side of **(3.12)** satisfies (iii), (iv), and (v) in **Remark 2.6**, they uniquely determine a Green function. Now, sending $z \to \infty$ yields

$$g(\infty, z_0) = \gamma(E) - \int \log \frac{1}{|\zeta - z_0|} d\mu_E(\zeta) = \gamma(E) - U_{\mu_E}(z_0),$$

where the first relation holds by (3.1), definition for Robin's constant, and (3.8). The second relation holds by the definition of log potential.
For $z \in \Omega$, (c.1) is then a consequence of the symmetry of Green function via **Theorem 2.4**, i.e., $g(\infty, z) = g(z, \infty)$. Then because $g(z, \infty) > 0$ on Ω by **Remark 2.6** (ii), (c.1) implies (c.2).
Step III: (c.3) for $z \notin \partial\Omega$
For $z \notin \overline{\Omega}$, $v(\zeta) = \log\left(\frac{|\zeta|}{|\zeta - z|}\right)$ is harmonic on a neighborhood of $\overline{\Omega}$ and one has

$$
\begin{aligned}
0 &= v(\infty) \quad \text{(Definition of } v\text{)}\\
&= \int \log|\zeta| d\mu_E(\zeta) + \int \log|\frac{z}{|\zeta - z|} d\mu_E(\zeta)\\
&= -\gamma(E) + U_{\mu_E}(z) \quad \text{(Definition of Robin's constant and log potential)}
\end{aligned}
$$

where the second equality holds by integrating both sides of the first equality with respect to μ_E and using the property of log $\log(a/b) = \log a + \log(1/b)$. Therefore (c.3) holds for $z \notin \overline{\Omega}$.
Step IV: (b) and (c.1), (c.3) for $z \in \partial\Omega$.
If $z \in \partial\Omega$, then

$$U_\mu(z) \leq \liminf_{\Omega \ni \omega \to z} U_\mu(z) \leq \gamma(E) < \infty$$

where the first inequality holds by the lower semicontinuity (l.s.c.) of U_μ by **Lemma 3.6**, the second inequality holds by the fundamental identity for Green function in (3.9) in conjunction with the positiveness of Green function by **Remark 2.6** (ii), and the last inequality holds since $\mathrm{Cap}(E) > 0$.
Since the integrand is bounded below, that means the integral $U_\mu(z)$ converges absolutely. By Cantor's tour, $\partial\Omega$ consists of $C^{1+\alpha}$ curves with $\mathrm{Leb}(\partial\Omega) = 0$ where Leb is the Lebesgue measure. Then by the superharmonicity of U_μ in **Lemma 3.6** and the continuity of $g(z, \infty)$,

$$U_\mu(z) \geq \limsup_{\delta \to 0} \left(\int_{\Omega \backslash \mathscr{B}_\delta(z)} U_\mu(z) \frac{d\xi d\eta}{\pi\delta^2} + \int_{\mathscr{B}_\delta(z) \cap \overline{\Omega}} \gamma \frac{d\xi d\eta}{\pi\delta^2} \right) = \gamma$$

where the first relation holds by taking $\zeta := \xi + i\zeta$ and using (c.1), supermean inequality of superharmonic function U_μ, and continuity of Green function. The second relation holds by the definition of Robin's constant and continuity of Green function. The notation $\mathscr{B}_\delta(z)$ denotes the ball centered at z with radii $\delta > 0$. Consequently, (c.1) and (c.3) holds at $z \in \partial\Omega$, and it follows that U_μ is continuous on $\mathbb{C}$.

$\square$

Let E be a compact set with $\mathrm{Cap}(E) > 0$ and let $E_n := \mathbb{C}^\infty \backslash \Omega_n$ be as in the first section. Then by (3.7) and **Theorem 3.7**, any weak* limit point μ_E of the sequence $\{\mu_{E_n}\}_{n \geq 1}$ satisfies (c.1) and (c.2) on Ω. In the fourth section we will use the energy integral to show that there is a unique weak* limit μ_E independent of the sequence E_n

and to establish a version of (c.3) for U_{μ_E} on E. A different proof of the uniqueness of the weak* limit $\{\mu_{E_n}\}_{n\geq 1}$ for a bounded domain Ω is given in Exercise 4.

3.3 The Energy Method

Let ν be a signed measure with compact support.

Definition: Finite Energy

If

$$\iint \left| \log \frac{1}{|z-\zeta|} \right| d|\nu|(\zeta) d|\nu|(z) < \infty \tag{3.13}$$

we say that ν has finite energy.

Definition: Energy Integral

If ν has finite energy we define the energy integral $I(\nu)$ by

$$I(\nu) := \iint \log \frac{1}{|z-\zeta|} d\nu(\zeta) d\nu(z) = \int U_\nu(z) d\nu(z).$$

Remark 3.5: Positive Definiteness of Energy Integral and Zero Total Mass

The energy integral has a very important property: It is positive definite on the space of signed measures $\mathscr{M}$ with positive energy and zero intetal, i.e., the total mass $\nu(\mathbb{C}) = 0$. ⋄

Theorem 3.8: Positive Definiteness for Energy Integral and Vanishing Condition

If (3.13) holds and if $\nu(\mathbb{C}) = 0$, then $I(\nu) \geq 0$. Moreover, if $I(\nu) = 0$ then $\nu \equiv 0$.

Proof:

Denote $L(z) := \log \dfrac{1}{|z|}$, by Green's theorem

$$f(z) = \frac{-1}{2\pi} \iint L(z-w) \Delta f(w) du dv \tag{3.14}$$

whether $f \in C^\infty$ has compact support, $w = u + iv$. The proof is divided into two cases, in the first case we consider $\nu \ll$ Leb, i.e., ν is a compactly supported absolutely continuous signed measure with respect to the Lebesgue measure. In the second case we prove the signed measure, and use mollification argument to recover the absolute continuity via a kernel and use the special case to derive the desired result.

Case I: Special case for absolutely continuous signed measure

First consider the special case of an absolutely continuous signed measure

$$d\nu := h(z) dx dy,$$

where $h \in C^\infty$ has compact support and satisfies

$$\iint h(x) dx dy = 0. \tag{3.15}$$

Note that h serves as the kernel of ν, hence the convolution

$$U_\nu := L * h \in C^\infty.$$

For $|z|$ sufficiently large, one has

$$
\begin{aligned}
|U_\nu(z)| &= \left|(L*h)(z)\right| \\
&= \left|\iint \log\frac{1}{|z-w|}\, h(w)\, du\, dv\right| \quad \text{(Definition of convolution)} \\
&\le \iint \left|\log\frac{|z|}{|z-w|}\right| |h(w)|\, du\, dv \\
&\le \frac{C}{|z|}\iint |h(w)|\, du\, dv \quad \text{(Taylor expansion for large } |z|\text{)} \\
&= \frac{C'}{|z|} \quad \text{(since } h \text{ compactly supported and integrable)}
\end{aligned}
\tag{3.16}
$$

and

$$
|\nabla U_\nu(z)| \le \frac{C'}{|z|^2} \tag{3.17}
$$

by the same argument. For any $f \in C^\infty$ with compact support, one has

$$
\begin{aligned}
\iint \Delta U_\nu f dxdy &= \iint U_\nu \Delta f dxdy \quad \text{(Green's theorem and Fubini)} \\
&= -2\pi \iint hf dxdy
\end{aligned}
$$

where the last equality holds by (3.14), definition $d\nu := h(z)dxdy$, and the fact that h and f both have compact support. Therefore,

$$
\Delta U_\nu = -2\pi h.
$$

Now, combining (3.16), (3.17), and Green's theorem using in the last equality, we have

$$
\begin{aligned}
I(\nu) &= \iint U_\nu h dxdy \quad \text{(Definition of energy integral)} \\
&= \frac{-1}{2\pi}\iint U_\nu \Delta U_\nu dxdy \quad (\Delta U_\nu = -2\pi h) \\
&= \frac{1}{2\pi}\iint |\nabla U_\nu|^2 dxdy.
\end{aligned}
$$

This shows that $I(\nu) \ge 0$ in this special case when ν is an absolutely continuous signed measure. Moreover, if $I(\nu) = 0$, then $\nabla U_\nu = 0$ and

$$
h = \frac{-1}{2\pi}\Delta U_\nu = 0,
$$

hence $\nu \equiv 0$, as desired.

Case II: General case via mollification argument

To derive the full **Theorem 3.8** from the special case we apply a standard mollification argument. Let ν be a signed measure not necessarily with a compact support. Let $K \in C_c^\infty(\mathbb{C})$ be a compactly supported analytic function such that

(i) K is radical, i.e., $K(z) = K(|z|)$.
(ii) K is positive definite, i.e., $K \geq 0$.
(iii) K has Lebesgue integral 1, i.e., $\displaystyle\int K dxdy = 1$.

Note that K is a probability density kernel. Set

$$K_\varepsilon(z) := \varepsilon^{-2} K\left(\frac{z}{\varepsilon}\right)$$

and let ν_ε be the absolutely continuous measure with density

$$h_\varepsilon(z) := K_\varepsilon * \nu(z) = \int K_\varepsilon(z-w)d\nu(w)$$

(since K_ε and ν are compactly supported, so is their convolution). Then for all continuous function f,

$$\lim_{\varepsilon\to 0} \int f d\nu_\varepsilon = \int f d\nu, \tag{3.18}$$

that is,

$$\nu_\varepsilon \xrightarrow[\varepsilon\downarrow 0]{\text{weak*}} \nu.$$

Furthermore, $h_\varepsilon \in C^\infty$ has compact support and satisfies (3.15) and it is then the kernel for the absolutely continuous measure ν_ε. Thus

$$I(\nu_\varepsilon) = \iint K_\varepsilon * K_\varepsilon * L(z-\zeta)d\nu(z)d\nu(\zeta)$$

where

$$K_\varepsilon * K_\varepsilon(z) = \int K_\varepsilon(z-w)K_\varepsilon(w)dudv$$

and consequently

$$K_\varepsilon * K_\varepsilon * L(w) = \int \big(K_\varepsilon * K_\varepsilon(z)\big)L(w-z)dxdy.$$

Now because K_ε is assumed to be radical, $K_\varepsilon * K_\varepsilon$ is also radical. Moreover, since $L(z)$ is superharmonic by definition, one has

$$K_\varepsilon * K_\varepsilon * L(z) \leq L(z) \tag{3.19}$$

by supermean inequality. Now, the lower semicontinuity (l.s.c.) of $L(z)$ viewed as a map to $(-\infty, \infty]$ gives

$$K_\varepsilon * K_\varepsilon * L(z) \to L(z) \text{ as } \varepsilon \to 0$$

by bounded convergence. Since $\nu(\mathbb{C}) = 0$ by assumption, one has

$$\iint \log \frac{1}{|\alpha z - \beta\zeta|} d\nu(\zeta)d\nu(z) = \iint \log \frac{1}{|z-\zeta|} d\nu(\zeta)d\nu(z)$$

for every $\alpha, \beta > 0$. Therefore, without loss of generality, one may assume that ν is supported on $\{z : |z| < 1/2\}$. It follows from (3.13), (3.19), and classical Lebesgue's dominated convergence theorem (LDCT) that

$$\lim_{\varepsilon\to 0} I(\nu_\varepsilon) = I(\nu)$$

and the positive definiteness is passed to $I(\nu)$ by equality.

Finally, suppose that $I(\nu) = 0$ and write $U_\varepsilon := U_{\nu_\varepsilon}$. One has

$$\iint |\nabla U_\varepsilon|^2 dxdy = I(\nu_\varepsilon) \to 0 \tag{3.20}$$

where the first relation holds by (3.16), (3.17), and Green's theorem; the second relation holds since $\lim_{\varepsilon\to 0} I(\nu_\varepsilon) = I(\nu)$.

We also have

$$U_\varepsilon(z) = O\left(\frac{1}{|z|}\right) \text{ uniformly in } \varepsilon$$

because all h_ε, $\varepsilon < 1$, satisfying (3.15) and vanishing outsied a common compact set. Then by (3.20) and Lemma 3.9 below, one has

$$\lim_{\varepsilon\to 0} \iint |U_\varepsilon(z)|^2 dxdy = 0.$$

Let $f \in C^\infty$ with compact support. Then (3.18) in conjunction with Green's theorem yields

$$\int f d\nu \underset{\varepsilon\downarrow 0}{\leftarrow} \int f d\nu_\varepsilon \underset{\varepsilon\downarrow 0}{\leftarrow} \frac{-1}{2\pi} \int \Delta f U_\varepsilon dxdy = 0.$$

This results in the desired result $\nu \equiv 0$.

□

The equation

$$\Delta U_\nu = -2\pi h \tag{3.21}$$

is the usual Poisson's equation. Before we prove the promised lemma to conclude the proof of Theorem 3.8, we use a remark to demonstrate the mollification argument.

Remark 3.6: Mollification Argument

Mollification argument is used to construct an absolutely continuous measure (with respect to Lebesgue measure, for example) with compact support from a signed measure that is not necessarily compactly supported.

We start with given a signed measure ν.

Step I: Define a Probability Density Kernel

We start with defining the probability density kernel $K \in C_c^\infty(\mathbb{C})$ that is

(i) compactly supported and analytic.
(ii) K is radical, i.e., $K(z) = K(|z|)$.
(iii) K is positive definite, i.e., $K \geq 0$.
(iv) K has Lebesgue integral 1, i.e., $\int K dxdy = 1$.

Step II: Construct a Kernel that is uniformly bounded.

In our proof, we are aiming to prove the absolute continuity with respect to the Lebesgue measure, so we set

$$K_\varepsilon(z) := \varepsilon^{-2} K\left(\frac{z}{\varepsilon}\right).$$

In practice, by changing the normalizing terms, we can include other geometric measures as well.

Step III: Construct absolutely continuous measure

Convolute new kernel with given measure to get absolutely continuous measures, that is, let ν_ε be the absolutely continuous measure with density

$$h_\varepsilon(z) := K_\varepsilon * \nu(z) = \int K_\varepsilon(z-w)d\nu(w).$$

Step IV: Derive weak convergence and compactly support result.
It can be shown that

$$\nu_\varepsilon \xrightarrow[\varepsilon\downarrow 0]{\text{weak*}} \nu$$

and by our design in the second step, ν is compactly supported in

$$\{z : |z| < 1/2\}.$$

In particular, the mollification argument tells us that a signed measure is the weak limit of a sequence of compactly supported absolutely continuous signed measures. ◇

Lemma 3.9: Bounded Log Potential with Vanishing Gradient Vanishes on Compacts
Assume $U_n(z) \in C^\infty(\mathbb{C})$ satisfy

(i) $|U_n(z)| \leq \dfrac{C}{|z|}$ for some constant $C > 0$.

(ii) $\|\nabla U_n\|_{L^2} \to 0$.

Then

$$\iint_K |U_n|^2 dxdy \to 0 \tag{3.22}$$

for every compact set K.

Proof:

Without loss of generality, we may assume that $K \subset [-L,L]\times[-R,R]$ for $L,R \in \mathbb{C}$. Then

$$\int_{[-L,L]} |U_n(x,y)|^2 dx \leq 2\int_{-L}^{L} \Big|U_n(x,y) - U_n(x,-L)\Big|^2 dx + \frac{2C^2}{L}$$

(Fundamental Theorem for Calculus)

$$\leq 2\int_{-L}^{L} \Big(\int_{-R}^{y} \Big|\frac{\partial U_n}{\partial y}\Big| dy\Big)^2 dx + \frac{2C^2}{L}$$

(Cauchy-Schwartz inequality)

$$\leq 4L\|\nabla U_n\|_{L^2}^2 + \frac{2C^2}{L} \quad \text{(Triangle Inequality in } L^2\text{)}$$

$$\to \frac{2C^2}{L} \quad \text{(Assumption (ii))}$$

Integrating over $[-R,R]$ when L is large compared to R (vice versa) gives the desired (3.22).

□

Index of Definitions:
DP denotes Dirichlet problem
F.C.J.D. denotes finitely connected Jordan Domain

1.1
Harmonic Measure (for Set of Finite Union in Half Plane)
Dirichlet Problem (over Half Plane)
Harmonic Measure (for Measurable Set on Half Plane)
Poisson Kernel (over Half Plane)
Poisson Integral (over Half Plane)
Harmonic Measure (for Set of Finite Union over Unit Disc)
Poisson Integral (over Unit Disc)
Poisson Kernel (over Unit Disc)
Solution to the Dirichlet Problem over Unit Disc

1.2
Cone (over Unit Disc)
Non-Tangential Limit (over Unit Disc)
Non-Tangential Maximal Function (over Unit Disc)
Solution to the Dirichlet Problem with Leb-a.e. Non-Tangential Limit
Weak Type 1-1
Hardy-Littlewood Maximal Function

1.3
Jordan Curve and Jordan Domain
Solution to DP over Jordan Domain for Bounded Boundary Functions
Harmonic Measure (over Jordan Domain)

1.4
Hyperbolic Distance (over Unit Disc)
Hyperbolic Metric
Koebe Function
Hyperbolic Distance (over Simply Connected Domain)
Quasi-Hyperbolic Distance
Whitney Square

1.5
Pseudohyperbolic Metric (over Unit Disc)

Pseudohyperbolic Metric (over Simply Connected Domain)

2.1
Finitely Connected Jordan Domain
Piecewise Continuous Function
Harmonic Measure (over Finitely Connected Jordan Domain)
Solution to DP over F.C.J.D. with Bounded Borel Boundary Data

2.2
Green Function with Pole (over Bounded Domain)
Green Function with Pole (over Unbounded Domain)
Green Function with Pole (under Conformal Mapping)
Analytic Arc
Jordan Analytic Curve
Poisson Kernel (over Finitely Connected Jordan Domain)

2.3
Harmonic Conjugate (Conjugate Function)
Alpha-Hölder Class and Alpha-Hölder Continuous Function
Alpha-Hölder Norm
Herglotz Integral of Alpha-Hölder Continuous Class
k times Continuously Differentiable
Zygmund Class
Zygmund Norm and Zygmund Function
Lipschitz Function, Class of Lipschitz Function
Norm on Class of Lipschitz Function

2.4
Tangent of Arc
Unit Tangent Vector of Arc
Continuous Tangent of Arc
Alpha-Hölder Class for Arc

3.1
Robin's Constant
Logarithmic Capacity
Green Function with Pole (over F.C.J.D.)

3.2
Logarithmic Potential

3.3
Finite Energy
Energy Integral

Index of Results:

Lemma 1.1: Lindelöf's Maximum Principle
Theorem 1.2: Existence and Uniqueness for Solution to Dirichlet Problem on ℍ
Theorem 1.3: Poisson Integral Formula over Unit Disc
Theorem 1.4: Fatou's Theorem
Lemma 1.5: Hardy-Littlewood Max as Upper Bound for Non-Tangential Max
Lemma 1.6: Measure Bound for Open Intervals via Disjointed Subintervals
Lemma 1.7: Hardy-Littlewood Maximum Function Is Weak Type 1-1
Corollary 1.4.1: Solution to DP over Unit Disc for Bounded Boundary Condition
Theorem 1.8: Carathéodory's Theorem
Theorem 1.9: Koebe One-Quarter Theorem
Lemma 1.10: Area Theorem
Theorem 1.11: Koebe's Estimate for Conformal Image
Corollary 1.11.1: Koebe's Estimate for Invariant Simply Connected Domain
Theorem 1.12: Growth, Distortion, and Angular Distortion for Univalent Maps
Theorem 1.13: Hayman-Wu Theorem

Theorem 2.1: Solution to DP on F.C.J.D. with Bounded Piecewise Continuous Data
Theorem 2.2: Green Function as Log of Conformal Mapping over F.C.J.D.
Lemma 2.3: F.C.J.D. Has Partition and Homoemorphism Extension on Boundary
Theorem 2.4: Green Function with Pole is Symmetric over F.C.J.D.
Lemma 2.5: Sufficiency for Harmonic Extension to Analytic Curve over F.C.J.D.
Theorem 2.6: Harmonic Measure as Generalization of Poisson Integral Formula
Corollary 2.6.1: Absolute Continuity and Analyticity of Harmonic Measure on F.C.J.D
Theorem 2.7: Solution to DP over F.C.J.D. with Bounded Borel Boudnary Data
Proposition 2.8: Non-Tangential Limit for Conjugate Function Exists A.E.
Theorem 2.9: Zygmund's Exponential Integrability for Harmonic Conjugate
Theorem 2.10: Criterion of Alpha-Hölder Continuous Class with Norm Bound
Corollary 2.10.1: Criterion for Alpha-Hölder Class Extension to Boundary of Unit Disc
Theorem 2.11: Criterion for Zygmund Boundary Data with Zygmund Norm Bound
Theorem 2.12: Analytic Continuation of Riemann Maps Across Shared Arcs in Nested Jordan Domains
Theorem 2.13: Criterion for Tangent and Continuous Tangent on Jordan Boundary
Theorem 2.14: Kellogg's Theorem
Lemma 2.15: Hölder Continuity Equivalence for Analytic Functions and Its Inverse

Corollary 2.14.1: Change of Hölder Class Coefficients under Conformal Map on Arc
Corollary 2.14.2: Change of Hölder Coefficients under Conformal Maps on F.C.J.D.
Corollary 2.14.3: Absolute and Hölder Continuity of Harmonic Measure on F.C.J.D.

Proposition 3.1: Logarithmic Capacity for Closed Disc
Theorem 3.2: Capacity and Robin Constant Independent of Approximating Sequence
Proposition 3.3: Monotonicity for Capacity and Robin's Constant
Proposition 3.4: Logarithmic Capacity for Interval
Proposition 3.5: Log Capacity Lower Bound for Subset of Unit Disc
Lemma 3.6: Log Potential as Superharmonic Function (In Our Setting)
Theorem 3.7: Fundamental Identity for Log Potential
Theorem 3.8: Positive Definiteness for Energy Integral and Vanishing Condition
Lemma 3.9: Bounded Log Potential with Vanishing Gradient Vanishes on Compacts

Index of Examples and Remarks:

Remark 1.1: Some Elementary Properties for Harmonic Measure on Half Plane
Remark 1.2: Harmonic Measure for Interval over $\mathbb{H}$
Remark 1.3: Harmonic Measure as Transition Density and Harmonic Function
Remark 1.4: Harmonic Measure Satisfies Harnack's Inequality
Remark 1.5: Definition in (1.6) Does Not Depend on the Conformal Mapping
Remark 1.6: Some Elementary Properties of Cones
Example 1.1: Example of Non-Tangential Limit over Unit Cone
Remark 1.7: Proof for Fatou's Theorem via Approximate Identity Argument
Remark 1.8: Hardy-Littlewood Max Is Simpler than Non-Tangential Max
Remark 1.9: Harmonic Measure as Indicator Along Non-Tangential Limit over $\mathbb{D}$
Remark 1.10: Equivalent Question for Harmonic Measure over Jordan Domain
Remark 1.11: Hyperbolic Distance (over Unit Disc) Is Conformally Invariant
Remark 1.12: Hyperbolic Shortest Arc and Hyperbolic Length
Remark 1.13: Hyperbolic Distance Bound over Simply Connected Domain
Remark 1.14: Whitney Squares As Substitute for Hyperbolic Balls
Remark 1.15: Whitney Squares Are ALMOST Conformal Invariant
Remark 1.16: Equalities in **Theorem 1.12** Holds $\Leftrightarrow$ ψ Is Koebe

Remark 2.1: In Proving Solution to DP We Can Assume Bounded Domain
Remark 2.2: Schwarz Alternating Method
Remark 2.3: Harmonic Measure over Finitely Connected Jordan Domain Is Borel
Remark 2.4: Harmonic Measure Satisfies Harnack's Inequality
Example 2.1: Boundary Data over F.C.J.D. Can Be Weakened to Bounded Borel

Remark 2.5: Condition for Applying Schwarz Alternating Method
Remark 2.6: Some Elementary Properties of Green Function with Pole
Remark 2.7: Green Function with Pole over F.C.J.D. Is Conformal Invariant
Remark 2.8: Schwarz Reflection Principle Extends Harmonic Locally on Boundary
Remark 2.9: Comparing Harmonic Measure to Geometric Measure
Remark 2.10: Isometry Between Space of Bounded Harmonic Functions and L^∞
Remark 2.11: Alternative Proof for **Theorem 2.7** and Conformal Estimate
Remark 2.12: Equivalent Definition for Harmonic Measure on F.C.J.D.
Remark 2.13: Connection Between Harmonic Conjugate and Conformal Map
Example 2.2: Bounded Continuous Function with Unbounded Harmonic Conjugate
Remark 2.14: Locally Bounded Harmonic Function Has Local Bounded PDE
Remark 2.15: **Theorem 2.10** Fails when $\alpha = 1$, **Corollary 2.10.1** Fails when $\alpha = 0$
Remark 2.16: Zygmund Class, Lipschitz Class, and Alpha-Hölder Class
Example 2.3: Conformal Map with Continuous Tangent But Not Continuously Differentiable
Remark 2.17: $\alpha = 0$ in **Theorem 2.14** Ruins Equivalence
Example 2.4: Harmonic Measure $\ll$ Arc Length $\Rightarrow$ Bounded Density
Example 2.5: Green's Theorem Applied to F.C.J.D. without Real Analytic Boundary

Remark 3.1: Log Capacity Scaling under Univalent Conformal Map
Remark 3.2: Log Potential Converges Absolutely Lebesgue-Almost Everywhere
Remark 3.3: Harmonic Measure of ∞ Relative to Ω as Equilibrium Measure
Remark 3.4: Harmonic Measure as Inner Measure
Remark 3.5: Positive Definiteness of Energy Integral and Zero Total Mass
Remark 3.6: Mollification Argument