Notes on Topology

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1.1 Introduction to Topology

There are many terminologies named after the 'topology-properties' in the learning of math, such as closedness, openness, compactness, etc. Before our formal definition for a topological space, we here present a big picture for some key concepts we shall encounter:

Topological Space: A topological space is a set equipped with a collection of subsets (open sets) that satisfy certain axioms. These open sets define the notion of "nearness" and help establish the structure of the space.

Continuity: In topology, continuity refers to a mapping between two topological spaces that preserves the notion of closeness. A function is continuous if the preimage of an open set is open.

Homeomorphism: A homeomorphism is a bijective mapping between two topological spaces that is continuous, with a continuous inverse. It essentially represents a one-to-one correspondence between two spaces that preserves their topological properties.

Compactness: A topological space is compact if every open cover (a collection of open sets whose union covers the space) has a finite subcover. Compactness captures the idea of being "finite" or "bounded" in a topological sense.

Connectedness: A space is connected if it cannot be divided into two disjoint nonempty open sets. Intuitively, it means that the space is not "broken" into separate parts.

Metric Space: A metric space is a type of topological space where distances between points are defined using a metric (a function that satisfies certain properties). Euclidean spaces are examples of metric spaces.

Topology vs. Geometry: While geometry focuses on distances, angles, and measurements, topology is concerned with more qualitative properties, like continuity and neighborhoods. Topology studies shapes and spaces up to continuous transformations.

Now we shall give a formal definition for the topological space, but before that, we shall introduce a terminology we have been using all the time without knowing its formal definition:

Definition: metric

A metric d on a set X is a function $d : X \times X \to [0,\infty)$ such that:

(i) $d(x, y) = d(y, x)$ (Symmetric)

(ii) $d(x, y) \geq 0$ with equality $\Leftrightarrow x = y$ (Positive Homogeneous)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

holds $\forall x, y, z \in X$.

A metric, also known as a distance function, is a fundamental concept that quantifies the distance or "closeness" between elements in a set. It provides a formal way to measure how far apart two points are from each other. Metrics are used in various mathematical contexts, including metric spaces, which are mathematical structures where distances between points are defined.

Example 1.1: Euclidean metric

We have used many and many times the distance function in the $\mathbb{R}^n$ space, the so-called Euclidean metric given by the formula:

$$d(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}, \tag{1.1}$$

which offers the magnitude of the distance between two points $x, y \in \mathbb{R}^n$. ||

For those who already familiar with the notion 'norm', we want to talk about is very subtle difference: A norm is a generalization of a metric for vector spaces, which measures the "size" of vectors. While a metric measures distances between points, a norm measures the magnitude of vectors.

It is now natural to give the formal definition for metric space:

Definition: Metric Space

A metric space is a set X equipped with a metric d. Metric spaces provide a framework to discuss concepts like continuity, convergence, compactness, and open/closed sets. Topology and analysis heavily rely on the notion of metric spaces.

In basic analysis books, we have already seen the open/closed balls, which are defined via interior points and limit points. Now we shall introduce the same terminology but via the language of topology:

Definition: open ball (in metric space)

Let (X, d) be a metric space. For $x \in X$ an arbitrary point and $r > 0$ a constant, an open ball $B_r(x)$ (or $B(x, r)$) centered at x of radius r is

$$B_r(x) := \{y \in X \mid d(x, y) < r\} = \{y \in X \mid d(y, x) < r\}. \tag{1.2}$$

In later discussion, (X, d) will automatically represent a metric space with set X equipped with metric d well-defined. Moreover, the last equality in (1.2) holds by the symmetricity of metric. Furthermore, it shall make no confusion for closed balls in metric space being defined with the same approach:

Definition: closed balls (in metric space)

Let (X, d) be a metric space. For $x \in X$ an arbitrary point and $r > 0$ a constant, an open ball $B_r(x)$ (or $B(x, r)$) centered at x of radius r is

$$B_r(x) := \{y \in X \mid d(x, y) \le r\} = \{y \in X \mid d(y, x) \le r\}. \tag{1.3}$$

Lemma 1.1:

Let (X, d) be a metric space, for $x \in X$ and $r > 0$. $B_r(x)$ is an open ball centered at x with radius r. Then $\forall y \in B_r(x)$, $\exists \rho > 0$ such that $B_\rho(y) \subseteq B_r(x)$.

Proof:

Since $y \in B_r(x)$, $r > d(x, y) \Rightarrow \rho = r - d(x, y) > 0$.

Then $z \in B_\rho(y) \Leftrightarrow d(y, z) < \rho \Rightarrow d(y, z) < r - d(x, y)$

$$\Rightarrow r > d(y, z) + d(x, y) \ge d(x, z) \qquad \text{(Triangle Inequality)}$$

which means $z \in B_r(x)$ by (1.2). Thus $z \in B_\rho(y) \Rightarrow z \in B_r(x)$. Since z is arbitrarily chosen, we have the desired result $B_\rho(y) \subseteq B_r(x)$.

□

Under the definition of open ball, we could therefore derive the definition for a subset being open in a metric space:

Definition: open set (in metric space)

Let (X, d) be a metric space, a subset $U \subseteq X$ is said to be open if $\forall x \in U$, $\exists r > 0$ such that $B_r(x) \subseteq U$.

Since the complement of an open set in the whole space is closed, we shall not give the definition of closed sets.

Properties: open set in metric space

Let (X, d) be a metric space, then

(i) $\varnothing, X$ are open.

(ii) If U, V are open in X, then $U \cap V$ is open in X.

(iii) If $\{U_\alpha\}_{\alpha \in I}$ are a collection of open sets in X with the index set I chosen arbitrarily, then $\bigcup_{\alpha \in I} U_\alpha$ is also open.

Note that, the second properties is valid when such a intersection is done among finite many open sets. We may also express that, openness is 'closed' under finite intersections and arbitrary unions.

Properties: closed sets in metric space

Let (X, d) be a metric space, then

(i) $\varnothing, X$ are closed.

(ii) If U, V are closed in X, then $U \cup V$ is closed in X.

(iii) If $\{U_\alpha\}_{\alpha \in I}$ are a collection of closed sets in X with the index set I chosen arbitrarily, then $\bigcap_{\alpha \in I} U_\alpha$ is also closed.

On the contrary, being closed is 'closed' under finite unions and arbitrary intersections. Moreover, observing that we have $\varnothing, X$ being closed and open at the same time, this could be very counter-intuitive, but the fact is that being closed and being open are not mutually excluded!

We now offer a proof of property (ii) of open sets in metric spaces, the proof of others could be done with the similar approach:

Topology is usually define by the collection of open sets that could be found in a given "space", using the above statements, we give the criterion for being a topology.

Definition: Topology (on a set)

A topology T on a set X is a collection of subsets of X such that

(i) $\varnothing, X \in T$.

(ii) Finite intersections of open sets are open: If $U, V \in T$ then $U \cap V \in T$.

(iii) Arbitrary unions of open sets are open: If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets such that $U_\alpha \in T$ $\forall \alpha$ then $\bigcup_{\alpha \in I} U_\alpha \in T$.

In topology, the concept of power sets plays a significant role in defining the structure of open sets and capturing the properties of a topological space. The power set of a set is a collection of all possible subsets of that set, and it is sometimes used to define the topology on the set.

Definition: Power sets

Given a set X, the power set of X, denoted by $\mathscr{P}(X)$, is the collection of all subsets of X, including $\varnothing$ and X itself, is given by $\mathscr{P}(X) := \{A \mid A \subseteq X\}$.

Remark:

(i) A topology T over a set X satisfies $T \subseteq \mathscr{P}(X)$.

(ii) Elements of T are called open sets.

Moreover, the three conditions above are regarded as axioms; as an alternative, we could state, instead, with the terms of closed sets:

Definition: Alternative defintion for Topology (on a set)

A topology T on a set X is a collection of subsets of X such that

(i) $\varnothing, X \in T$.

(ii) If U, V are closed in X, then $U \cup V$ is closed in X.

(iii) If $\{U_\alpha\}_{\alpha \in I}$ are a collection of closed sets in X with the index set I chosen arbitrarily, then $\bigcap_{\alpha \in I} U_\alpha$ is also closed.

Now we are able to introduce some familiar terminologies we have encountered in the basic analysis course:

Definition: closed, neighbourhood

Let (X, T) be a topological space.

(i) $A \subset X$ is called **closed** when $X \backslash A$ is open.

(ii) $U \subset X$ is called a **neighbourhood** of $x \in X$ if there is an open set V such that $x \in V \subset U$.

The term "open sets" carries over from the idea of open intervals and continuity in real analysis. While the term might not immediately seem intuitive in more abstract spaces, it reflects the foundational concept that open sets capture the notion of closeness and neighborhoods, preserving the properties of continuity and convergence.

Still, if there is no confusion, we shall always use (X, T) to represent a topological space with a topology T over the set X.

The definition of a topological space that is now standard was a long time in being formulated. Various mathematicians — Fréchet, Hausdorff, and others — proposed different definitions over a period of years during the first decades of the twentieth century, but it took quite a while before mathematicians settled on the one that seemed most suitable. They wanted, of course, a definition that was as broad as possible, so that it would include as special cases all the various examples that were useful in mathematics — Euclidean space, infinite-dimensional Euclidean space, and function spaces among them — but they also wanted the definition to be narrow enough that the standard theorems about these familiar spaces would hold for topological spaces in general. This is always the problem when one is trying to formulate a new mathematical concept, to decide how general its definition should be. The definition finally settled on may seem a bit abstract, but as you work through the various ways of constructing topological spaces, you will get a better feeling for what the concept means.

Example 1.2: Metric topology

A metric d on a set X defines a topology T_d on X, $u \in T_d \Leftrightarrow \forall x \in U \, \exists r > 0$ such that $B_r(x) \subseteq U$. T_d is called the topology induced/defined by the metric d.

As we see in **Example 1.2**, a metric induces a topology. Since the concept of a metric is closely connected to the idea of distance, and the metric topology captures the concept of "closeness" and provides a natural way to define open sets in a topological space.

Definition: Metric topology

Given a metric space (X, d). The metric topology induced by the metric d is: The open sets in the metric topology are the sets U such that for every point x in U there exists a positive real number $\varepsilon > 0$ such that the open ball $B_\varepsilon(x) \subseteq U$, that is to say:

$$U \text{ is open} \Leftrightarrow \forall x \in X\, \exists \varepsilon > 0 \text{ such that } B_\epsilon(x) \subseteq U.$$

The metric topology is a fundamental example of a topological space induced by a metric. It provides a natural way to define open sets and preserve the notions of continuity, convergence, and closeness within a space.

Example 1.3: Finite Complement Topology

Let X be a set, let T_f be the collection of all subsets U of X such that $X \backslash U$ either is finite or is all of X. Then T_f is a topology on X, called the finite complement topology. Let us now check that T_f is indeed a topology:

(i) $\varnothing$ and X are in T_f, since $X - X$ is finite and $X - \varnothing$ is all of X.

(iii) If $\{U_\alpha\}$ is an indexed family of non-empty elements of T_f, to show that $\cup U_\alpha$ is in T_f is to show that:

$$X \backslash \cup U_\alpha = \cup (X \backslash U_\alpha).$$

The latter set is finite because each set $X \backslash U_\alpha$ is finite.

(ii) If $U_1, \cdots, U_n$ are non-empty elements of T_f, to show that $\cap U_i$ is T_f is to show that:

$$X \backslash \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \backslash U_i).$$

The latter set is a finite union of finite sets and, therefore, finite. ||

The finite complement topology is a specific type of topology defined on a set that involves the complements of finite subsets. It's an interesting example of a topology that highlights the interplay between open and closed sets.

Definition: Finite Complement Topology

Given a set X, the finite complement topology on X is defined by considering the open sets to be those that are either the empty set $\varnothing$ or have finite complements (i.e., their complements are finite or the entire space X):

A subset U of X is open in the finite complement topology if $U = \varnothing$ or if $X \backslash U$ is finite or equal to X itself.

Definition: standard topology on $\mathbb{R}^n$

The standard topology on $\mathbb{R}^n$ is the topology induced by the Euclidean distance, i.e. $d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$.

Example 1.4: Different metric may define the same topology

We see that a metric could induce a topology, we now give a fact that different metric could provide the same topology!

Let $X = \mathbb{R}^2$ and $d_\infty : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$. We have

$$d_\infty(x,y) := \max\{\,|x_1 - y_1|, |x_2 - y_2|\,\}.$$

(i) It is trivial that d_∞ is a metric.

(ii) Open balls with respect to d_∞ are open squares.

(iii) But open sets for this metric are exactly the same as for the Euclidean one. ||

Example 1.5: Discrete topology and indiscrete topology

Let X be a set, then the indiscrete topology $T_{\min}$ on X is defined to be $\{\varnothing, X\}$.

The discrete topology, on the other hand, is defined to be $T_{\max} = \mathscr{P}(X)$. ||

Remark:

(i) For any topology T on X, $T_{\min} \subseteq T$.

(ii) For any topology T on X, $T \subseteq T_{\max}$.

(iii) For x any element of X, $\{x\} \in T_{\max}$.

Above we use the terminology $\subseteq$, which is valid only if the topologies we are comparing are comparable, this naturally induces the following definition:

Definition: comparable topology

Suppose that T and T' are two topologies on a given set X. If $T' \supseteq T$, we say that T' is finer/larger/stronger than T; if T' properly contains T, i.e. $T' \supset T$, we say that T' is strictly finer than T. We also say that T is coarser/smaller/weaker than T', or strictly coarser, in these two respective situations. We say that T is comparable with T' if either $T' \supseteq T$ or $T \supseteq T'$.

Remark:

Two topologies on X need not to be comparable! ||

Exercise 1.1:

There is a topology on $\mathbb{R}$ that does not come from any metric.

Proof:

Consider $T := \{\varnothing\} \cup \left\{ U \subseteq \mathbb{R} \,\middle|\, \mathbb{R}\backslash U \text{ is finite} \right\}$.

[Claim]: T is a topology

(i) $\varnothing \in T$ by definition, $\mathbb{R}\backslash\varnothing \in T$.

(ii) If $U, V \in T$ are open, without loss of generality, we may assume that they are all non-empty and both $\mathbb{R}\backslash U$ and $\mathbb{R}\backslash V$ are finite.

Want To Show: $\mathbb{R}\backslash(U \cap V)$ is finite

$\mathbb{R}\backslash(U \cap V) = (\mathbb{R}\backslash U) \cup (\mathbb{R}\backslash V)$ which is finite union of finite set which is finite $\Rightarrow U \cap V \in T$.

(iii) $\{U_\alpha\}_{\alpha\in A} \subseteq T$, $\{U_\alpha\}_{\alpha\in A}$ non-empty, then $\mathbb{R}\backslash \bigcup_{\alpha\in A} U_\alpha = \bigcap_{\alpha\in A} (\mathbb{R}\backslash U_\alpha) \in T$ since finite intersection of finite set is also finite.

Thus, T is a topology coming from no metric in $\mathbb{R}$.

□

Comment:

When we define a new space, there are many features as well as properties fall into our interest. We need to identify the objects of this space (in this book, they are open sets or closed sets; in most literature, mathematicians prefer the former one), we need to see the relations between objects (such as binary relation, which attracts the most), as comparability makes sense, it is natural to see operations within objects (e.g. when studying numbers, we need to define addition; in topological terms, however, other than basic set operations, we also care about the extension and restriction). Moreover, we need to specify the generating objects. That is, when we have unlimited objects to study, we wish the existence of some small set of objects containing all the information, which is called basis, and there is an extension called subbasis, which is useful when we need to use even less objects to describe the basis. These summarize most features we care in a given space, when the number of spaces increases, it is important to see if there are possible transformations within them (for example, in topological sense, such a transformation is the continuous maps). Transformation itself could be as concrete as it can be.

To summarize, we shall concern with:

(1) Objects of topological spaces, the construction and properties;
(2) Relations among objects;
(3) Operations among objects;
(4) Generating Basis;
(5) Transformations within spaces.
(6) Properties and key features.

The order may vary from case to case and the presence may also change. The order is not related to their importance. In (6) we do apply the terminologies of properties and features again, since in studying more concrete materials, we often care about the topological aspects and functional aspects; for the former one we usually study the openness, closedness, compactness, separability, etc. while the latter one offers treatments upon the transformations. Furthermore, do remember that even these properties and features share the same name in different areas (such as topology and functional space), their formal definition and behaviour may vary! This is beacuse these terms form a structure of the studying mathematical terms, but different terms share different building blocks. Therefore same definition may fail in different areas. Now let us go back to topology, we now start our discussion with respect to (4), the generating basis:

1.2 Basis of Topology

For each of the examples in the preceding section, we were able to specify the topology by describing the entire collection T of open sets. Usually this is too difficult. In most cases, one specifies instead a smaller collection of subsets of X and defines the topology in terms of that.

Definition: Basis

If X is a set, a basis for a topology on X is a collection $\mathscr{B}$ of subsets of X (called basis elements) such that

(i) For each $x \in X$, there is at least one basis element B containing x.

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathscr{B}$ satisfies these two conditions, then we define the topology T generated by $\mathscr{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of T) if for each $x \in U$, there is a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of T.

Let us now check that the collection T generated by the basis $\mathscr{B}$ is indeed a topology on X, this description follows from James R. Munkres:

[Claim]: T is a topology on X.

(i) If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in T, since $\forall x \in X\ \exists$ some basis elements $B \in \mathscr{B}$ containing x and contained in X.

(ii) Now let us take an indexed family $\{U_\alpha\}_{\alpha \in A}$ where A is an arbitrary index set, countable or not, finite or not, of elements of T and we wish to arrive at the fact that $U = \bigcup_{\alpha \in A} U_\alpha$ is indeed an element of T.

Given $x \in U\ \exists \alpha$ such that $x \in U_\alpha$. Since U_α is open, $\exists$ a basis element $B \in \mathscr{B}$ such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, hence U is open.

(iii) Lastly we take two elements U_1 and U_2 of T and we wish to show that $U_1 \cap U_2$ belongs to T as well. Without loss of generality, let us assume that the intersection is not empty. Given $x \in U_1 \cap U_2$, choose a basis $B_1 \in \mathscr{B}$ containing x such that $B_1 \subset U_1$; choose also a basis element $B_2 \in \mathscr{B}$ containing x such that $B_2 \subset U_2$. The second condition for a basis enables us to choose a basis B_3 containing x such that $B_3 \subset U_3 := U_1 \cap U_2$, therefore, $U_1 \cap U_2$ belongs to T by definition.

(iv) Finally, we generalize (iii) to the case of finite intersection $U_1 \cap \cdots \cap U_n$ of elements of T also lives in T. This fact is trivial for $n = 1$, hence by induction, we suppose it is valid for $n - 1$ and prove it for n:

We have $(U_1 \cap \cdots \cap U_n) = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$. By hypothesis the RHS belongs to T; by the result we just proved, taking the intersection $U_1 \cap \cdots \cap U_{n-1}$ as U_1 in (iii) and U_n as U_2, we conclude the result.

||

Therefore, this definition is well-defined. Another way of describing the topology generated by a basis is given by the following lemma:

Lemma 1.2: From Basis to Topology

Let X be a set; let $\mathscr{B}$ be a basis for a topology T on X. Then T equals to the collection of all unions of elements of $\mathscr{B}$.

Proof:

To prove the statement is equivalent to prove that $\bigcup_{B \in \mathscr{B}} B = T$:

"$\subseteq$":

Given a collection of elements of $\mathscr{B}$, they are also elements of T. Because

T is a topology, their union is in T.

"$\supseteq$":

Conversely, given $U \in T$, choose for each $x \in U$ an element B_x of $\mathscr{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathscr{B}$.

□

This lemma states that every open set U in X can be expressed as a union of basis elements. This expression for U is not, however, unique. Thus the use of the term "basis" in topology differs drastically from its use in linear algebra, where the equation expressing a given vector as a linear combination of basis vectors is unique.

We have discussed so far how to go from a basis to the topology it generates, now we offer a technique where we could approach the reversed direction:

Lemma 1.3: From Topology to Basis

Let X be a topological space. Suppose that $\mathscr{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathscr{C}$ such that $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology of X.

Proof:

The proof is divided into two parts, in the first part, we must show that $\mathscr{C}$ is indeed a basis, i.e. we need to prove that $\mathscr{C}$ satisfies the two conditions for being a basis. While in the second part, we have to show that the topology T generated by $\mathscr{C}$ is a topology of X.

WTS I: $\mathscr{C}$ is a basis.

(i) Given $x \in X$, since X is itself an open set, there is, by hypothesis an element $C \in \mathscr{C}$ such that $x \in C \subset X$.

(ii) To check the second condition, without loss of generality, we may assume that $C_1 \cap C_2 \neq \varnothing \forall C_1, C_2 \in \mathscr{C}$. Let $x \in C_1 \cap C_2$ be arbitrarily chosen. Since C_1 and C_2 is open then so is $C_1 \cap C_2$. Therefore, there exists, by construction, an element C_3 in $\mathscr{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

WTS II: topology generated by $\mathscr{C}$ is a topology of X.

Let T be the collection of oepn sets of X; we want to prove that the topology T' generated by $\mathscr{C}$ equals to the topology T.

First, note that if $U \in T$ and if $x \in U$, then there is an element $C \in \mathscr{C}$ such that $x \in C \subset U$. It follows that U belongs to the topology T'. Conversely, if W belongs to the topology T', then W equals a union of elements of $\mathscr{C}$, by the preceding lemma. Since each element of $\mathscr{C}$ belongs to T and T is assumed to be a topology, $W \in T$.

□

We introduced the partial ordering in topological spaces, i.e. we denote $T_1 \subseteq T_2$ for two topologies T_1 and T_2 if T_1 is smaller than T_2. Now we shall introduce a criterion interms of the bases for determining this partial relationship:

Lemma 1.4: From Basis to Comparison of Topologies

Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies T and T', respectively, on a set X. Then the following are equivalent:

(i) T' is finer than T.

(ii) $\forall x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

Proof:

(ii) $\Rightarrow$ (i):

Let U be an element of T and $x \in U$. Since $\mathscr{B}$ generates $T \Rightarrow B \in \mathscr{B}$ such that $x \in B \subset U$. By (ii), there is an element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$, which means $U \in T'$.

(i) $\Rightarrow$ (ii):

We are given $x \in X$ and $B \in \mathscr{B}$ with $x \in B$. Now B belongs to T by definition and $T \subset T'$ by (i); therefore, $B \in T'$. Since T' is generated by $\mathscr{B}'$, there is an element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

□

While not as fundamental as a basis, a subbasis provides a more flexible starting point for constructing a topology by allowing unions and finite intersections of its elements. Subbases are particularly useful when dealing with more complex spaces or when generating a topology from multiple sources.

Definition: subbasis

A subbasis for a topology on a set X is a collection $\mathcal{S}$ of subsets of X such that the collection of all possible finite intersections of elements from $\mathcal{S}$ forms a basis for the topology on X. That is, if (X, T) is a topological space. A subset $\mathcal{S}$ of T is a subbasis of T if $\mathscr{B} := \left\{ S_{i_1} \cap \cdots \cap s_{i_k} \middle| k > 0, S_{i_1}, \cdots, S_{i_k} \in \mathcal{S} \right\}$ is a basis of T.

Remark:

A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$. ||

1.3 Continuous Mappings

For readers who are familiar with category theory, they may regard the topologies as objects and the continuous maps between them as morphisms; thus, it is necessary to introduce the terminology of continuous mapping. Before that, let us briefly recall what a category means:

Definition: Category

A category $\mathscr{C}$ consists of:

(i) A collection of subjects $\mathscr{C}_0$.

(ii) For each pair of objects $a, b \in \mathscr{C}_0$, a collection of morphisms $\mathrm{Hom}(a, b)$ (it may be empty if $a \neq b$). We write $a \xrightarrow{f} b$ or $b \xleftarrow{f} a$ if $f \in \mathrm{Hom}(a, b)$.

(iii) For each object $a \in \mathscr{C}$, a morphism $Id_a \in \mathrm{Hom}(a, a)$.

(iv) For each triple of objects $a, b, c \in \mathscr{C}_0$, a composiition map given by

$$\circ : \mathrm{Hom}(b, c) \times \mathrm{Hom}(a, b) \rightharpoonup \mathrm{Hom}(a, c)$$

$$\left(c \overset{g}{\leftarrow} b, b \overset{f}{\leftarrow} a\right) \mapsto c \overset{g\circ f}{\leftarrow} a.$$

Remark:

These data are subject to two conditions:

(1) $\forall a, b \in \mathscr{C}_0$ and $\forall f \in \mathrm{Hom}(a, b)$ we have $f \circ Id_a = f = Id_b \circ f$.

(2) The composition is associative, i.e. $\forall d \overset{h}{\leftarrow} c \overset{g}{\leftarrow} b \overset{f}{\leftarrow} a$ we have that $h \circ (g \circ f) = (h \circ g) \circ f$. ||

A topological spaces and continuous mappings form a category known as the category of topological spaces, often denoted as TOP. In this category, the objects are the topological spaces, and the morphisms (arrows) between objects are the continuous mappings between those spaces.

Definition: TOP

We claim that the topological spaces along with the continuous mappings among them being a category, denoted as TOP, defined by:

(i) The objects of the category TOP are the topological spaces.

(ii) Given two topological spaces (X, T_X) and (Y, T_Y), a morphism (arrow) from X into Y in TOP is a continuous mapping $f : (X, T_X) \to (Y, T_Y)$.

(iii) The composition of morphisms in TOP is defined as the usual composition of functions. If $f : X \to Y$ and $g : Y \to Z$ are continuous mappings, then their composition $g \circ f : X \to Z$ is also a continuous mapping.

(iv) For each topological space (X, T_X), the identity morphism $Id_X : X \to X$ is the continuous mapping defined as the identity function on (X, T_X).

Remark:

As we mentioned above, the composition of morphisms is associative in TOP, i.e. $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$. ||

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in \mathbb{R}\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous if it is continiuous $\forall x_0 \in \mathbb{R}$.

Definition: continuous maps (in metric spaces)

Let (X, d_X) and (Y, d_Y) be two metrci spaces. A function $f : X \to Y$ is continuous at $x_0 \in X$ if $\forall \varepsilon > 0 \exists \delta > 0$ such that $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$. We say such an f is continuous if it is continuous at every point of X.

Lemma 1.5:

$$d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon \Leftrightarrow B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x_0)).$$

Proof:

LHS holds $\Leftrightarrow x \in B_\delta(x_0) \Rightarrow f(x) \in B_\delta(f(x_0))$

$\Leftrightarrow x \in B_\delta(x_0) \Rightarrow x \in f^{-1}(B_\varepsilon(f(x_0))) \Leftrightarrow B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$.

The other side is analogous.

□

Lemma 1.5 tells us that a map between metric spaces is continuous if and only if the preimage of open set is open.

Lemma 1.6:

Let (X, d_X) and (Y, d_Y) be two metric spaces, a function f is continuous in the sense of $f : X \to Y$, i.e. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$, if and only if $\forall U \subseteq Y$ open with respect to the topology T_{d_Y}, $f^{-1}(U)$ is open in X with respect to T_{d_X}.

Proof:

"$\Rightarrow$":

Suppose that $U \subseteq Y$ is open, if $f^{-1}(U) = \emptyset$ then openness is guaranteed. Without loss of generality, we may assume that $f^{-1}(U) \neq \emptyset$. Let $x_0 \in f^{-1}(U)$, then $f(x_0) \in U$. Since U is oepn, then $\exists \varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subseteq U$ due to continuity, $\exists \delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))) \subseteq f^{-1}(U)$. Therefore, $f^{-1}(U)$ is open since x_0 is chosen arbitrarily.

"$\Leftarrow$":

Suppose that $\forall U \subseteq Y$ open, $f^{-1}(U)$ is open. Choose $x_0 \in X$ and $\varepsilon > 0$, we know that open balls are open sets, thus $B_\varepsilon(f(x_0))$ is open in Y $\Rightarrow f^{-1}(B_\varepsilon(f(x_0)))$ is oepn and $x_0 \in f^{-1}(B_\varepsilon(f(x_0))) \Rightarrow \exists \delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$. By **Lemma 1.5**, $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$ $\Rightarrow f$ is continuous at x_0. Moreover, since x_0 is chosen arbitrarily, we can therefore conclude that f is continuous at everywhere in its domain.

□

One important consequence of **Lemma 1.6** is that continuity does not depend on metric, it depends on the topology. More precisely, continuity of a function depends not only on f but also its domain and co-domain topologies. Furthermore, one may turn this lemma into the defintion of continuity, and it should not be very hard to see that this definition is equivalent to the open-set definition. We now offer the open-set perspective:

Definition: continuous maps (open-set perspective)

Let (X, T_X) and (Y, T_Y) be two topological spaces. A function $f : X \to Y$ is continuous (with respect to T_X and T_Y) if $\forall U \subseteq Y$ open, $f^{-1}(U)$ is open in X.

We now give some important results on continuity. Before that, we need a terminologies taken from [6].

Remark:

(i) The preimage of the intersection is the intersection of the preimages.
(ii) The preimage of the union is the union of the preimages.
(iii) The image of the union is the union of the images.
(iv) The image of the intersection is *a subset of* the intersection of the images.

Moreover, the preimage may not coincide with the inverse function. If certain special conditions are satisfied, then the inverse function exists and we use the same notation to denote that function. ||

Theorem 1.7:

Let (X, T_X) and (Y, T_Y) be two topological spaces and let $f : X \to Y$ be a function. Then the following statements are equivalent:

(i) f is continuous;
(ii) Inverse image of every basis element of T_Y is open;
(iii) Inverse image of every subbasis element of T_Y is open.

Proof:

To prove the equivalent relations is to prove that (i) ⇔ (ii) and (ii) ⇔ (iii):
(i) ⇒ (ii):
Let f be continuous. Since every basis element of T_Y is open, its inverse image is also open.
(ii) ⇒ (i):
To prove the inverse. Let $\mathscr{B}_Y$ be a basis for T_Y and let the inverse of every basis element $B \in \mathscr{B}_Y$ be open in X, i.e. $f^{-1}(B) \in T_X$. Note that any open set $V \subseteq Y$ can be, according to the definition, written as a union of the basis elements, i.e. $V = \bigcup_{\alpha\in A} B_\alpha$ hence $f^{-1}(V) = f^{-1}\big(\bigcup_{\alpha\in A} B_\alpha\big)$, according to the above remark, we have $f^{-1}\big(\bigcup_{\alpha\in A} B_\alpha\big) = \bigcup_{\alpha\in A} f^{-1}(B_\alpha)$ for some $\{B_1, \cdots, B_{|A|}\} \subseteq \mathscr{B}_Y$. Since the union of open sets is open hence $f^{-1}(V)$ is open.
(ii) ⇒ (iii):
Since every subbasis element is in the basis it generates, inverse image of every subbasis elements of Y is open in X.
(iii) ⇒ (ii):
Let now $\mathscr{S}_Y$ be subbasis of Y which generates the basis $\mathscr{B}_Y$. Let the inverse image of every subbasis element $S \in \mathscr{S}_Y$ be open in X, i.e. $f^{-1}(S) \in T_X$. Since any basis element can be written as a finite intersection of subbasis elements, i.e. $B = \bigcap_{i=1}^{n} S_i$ and $f^{-1}(B) = \bigcap_{i=1}^{n} f^{-1}(S_i)$. Since finite intersection of open sets is open, hence $f^{-1}(B)$ is open in X.

□

Remark:

Thus, to test the continuity of a function it suffices to check the openness of inverse images of elements of only a subset of T_Y, namely, its subbasis. ||

Theorem 1.8:

Let $f : X \to Y$ be a map where (X, T_X) and (Y, T_Y) are two topological spaces. Then the following statements are equivalent:
(1) f is continuous;
(2) Inverse image of every closed set of Y is closed in X.
(3) For each $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof: consult [6].

It is natural to derive from the continuous maps to the isomorphism between topological spaces. The isomorphisms in topological spaces is called homeomorphism, the choice of another name other than isomorphism has a particular reason, for which, along with detailed description of homeomorphism, will be left to the discussion of product topology.

Comment:

In real analysis, continuity is primarily concerned with functions between real numbers or subsets of the real line. In topology, continuity is a more general concept that applies to functions between topological spaces. This definition extends the real analysis definition to more abstract spaces. Continuous functions in topology preserve the underlying topological structure and open sets. In functional analysis, continuity often deals with linear operators between vector spaces equipped with a suitable topology. This concept is crucial in studying the properties of linear transformations and their relationships with topological structures.

Though applications vary from case to case, the definition is quite the same, one may recall the $\varepsilon - \delta$ langauge in basic analysis course to describe the continuity, or consult the continuity we just defined in this section. In either case, the most important information by admitting continuity is that it captures the idea of smoothness, preservation of structure, and gradual change, making it a crucial tool for understanding mathematical relationships and their implications.

In the next few subsections, we shall introduce the product topology, the subspace topology, the order topology, the metric topology, the quotient topology.

1.4 Product Topology

The product topology is a construction that allows us to define a topology on the Cartesian product of two or more topological spaces. It captures the idea of "coordinate-wise" openness and is a fundamental concept in topology, especially when studying products of spaces and their properties.

We would like to introduce the product of pairs of topological spaces (X, T_X) and (Y, T_Y), we would like a topology T on $X \times Y$ so that the Universal Property holds:

Definition: Universal Property

Let (X, T_X) and (Y, T_Y) be two topological spaces. Consider two continuous maps $p_X : (X \times Y, T) \to (X, T_X)$ and $p_Y : (X \times Y, T) \to (Y, T_Y)$, where T is the desired topology on the space $X \times Y$. Then for any topological space (Z, T_Z) and any two continuous functions

$$f_X : (Z, T_Z) \to (X, T_X) \text{ and } F_Y : (Z, T_Z) \to (Y, T_Y).$$

There exists a unique continuous function $f : Z \to X \times Y$ such that

$$p_X \circ f = f_X \text{ and } p_Y \circ f = f_Y.$$

Now We start to construct such a topology. Denote $T = T_{\text{prod}}$. Then:

$$\forall U \subseteq X \text{ open}, p_X^{-1}(U) = U \times Y \in T,$$
$$\forall V \subseteq X \text{ open}, p_Y^{-1}(V) = X \times V \in T.$$

Let now $\mathcal{S} := \left\{p_X^{-1}(U) \,\middle|\, U \in T_X\right\} \cup \left\{p_Y^{-1}(V) \,\middle|\, V \in T_Y\right\}$. Take $T = T_{\mathcal{S}}$.

[Claim]: $\mathcal{S}$ is a subbasis of T_{prod}.

Note that $\cup \mathcal{S} = X \times Y$ by this construction. Now $\forall U_1, U_2 \in T_X$ and $V_1, V_2 \in T_Y$, one has

$p_X^{-1}(U_i) \cap p_Y^{-1}(V_i) = (U_i \times Y) \cap (X \times V_i) = U_i \cap V_i$, for $i = 1,2$.

Then $\mathcal{S}$ is a subbasis of T_{prod}.

Moreover, we could obtain the basis $\mathscr{B} = \left\{U \times V \middle| U \in X, V \in Y\right\}$ for U, V being open in X and Y, respectively.

[Claim]: Universal Property holds

Let now (Z, T_Z) be a topological space. Consider the continuous functions:

$$f_X : (Z, T_Z) \to (X, T_X) \text{ and } f_Y : (Z, T_Z) \to (Y, T_Y).$$

We obain the unique function $f : Z \to X \times Y$ such that $f(z) = \big(f_X(z), f_Y(z)\big)$.

Moreover,

$f^{-1}(U \times V) = \left\{z \in Z \middle| \big(f_X(z), f_Y(z)\big) \in U \times V\right\} = f_X^{-1}(U) \cap f_Y^{-1}(V) \in T_Z$.

Therefore, f is unique and continuous, as we desired.

Question:

How arbitrary is T? That is, can we choose a different T' such that $p_X : (X \times Y, T') \to (X, T_X)$ and $p_Y : (X \times Y, T') \to (Y, T_Y)$ being continuous and the Universal Property fails to be false?

Answer:

This T, in fact, is unique.

[Claim]: $T = T'$

"$\subseteq$":

Since we want p_X and p_Y to be continuous, then the subbasis $\mathcal{S} = \left\{p_X^{-1}(U) \middle| U \in T_X\right\} \cap \left\{p_Y^{-1}(V) \middle| V \in T_Y\right\} \in T'$. Thus $T_{\mathcal{S}} \subseteq T'$.

"$\supseteq$":

The universal property of $\big((X \times Y, T'), p_X, p_Y\big)$ tells us that if we take $(Z, T_Z) = (X \times Y, T_{\mathcal{S}})$, take $f_X = p_X$ and $f_Y = p_Y$. Then $f : (X \times Y, T) \to (X \times Y, T')$ is continuous, and $f(x, y) = (x, y)$. That is to say, the identity mapping gives us different T, a contradiction.

Therefore $\forall W \in T', f^{-1}(W) = W \in T \Rightarrow T' \subseteq T$.

Now we offer the formal definition for the product topology:

Definition: product topology

Let (X, T_X) and (Y, T_Y) be two topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U is an open set of X and V is an open set of Y.

Remark:

Note that the collection $\mathscr{B}$ is not a topology on $X \times Y$. ||

Theorem 1.9: Basis for Product Topology

If $\mathscr{B}$ is a basis for the topology of X and $\mathscr{C}$ is a basis for the topology of Y. Then the collection $\mathscr{D} := \{B \times C \,|\, B \in \mathscr{B}, C \in \mathscr{C}\}$ is a basis for the topology of $X \times Y$.

Proof:

We apply **Lemma 1.3**. Given an open set W of $X \times Y$ and a point $x \times y$ of W, by definition of the product topology there is a basis element $U \times V$ such that

$x \times y \in U \times V \subset W$. Because $\mathscr{B}$ and $\mathscr{C}$ are bases for X and Y, respectively, we can choose an element B of $\mathscr{B}$ such that $x \in B \subset U$, and an element C of $\mathscr{C}$ such that $y \in C \subset V$. Then $x \times y \in B \times C \subset W$. Thus the collection $\mathscr{D}$ meets all the requirements for applying **Lemma 1.3**, result follows.

□

Similarly, since we have introduced the product topology, we want to study its basis and subbasis as we did to the basic topology. It is sometimes useful to express the product topology in terms of subbasis. To this end, we first introduce certain functions called projections:

Definition: projection function

Let $\pi_1 : X \times Y \to X$ be defined by the equation $\pi_1(x, y) = x$ and let $\pi_2 : X \times Y \to Y$ be defined by the equation $\pi_2(x, y) = y$. The maps π_1 and π_2 are called the projections of $X \times Y$ onto its first and second factors, respectively.

Theorem 1.10: Subbasis for Product Topology

The collection $S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$ is a subbasis for the product topology on $X \times Y$.

Proof:

Let T denote the product topology on $X \times Y$ and let T' be the topology generated by $\mathcal{S}$. We wish to show that $T = T'$.

"$\supseteq$":

Since every element of $\mathcal{S}$ belongs to T, so do arbitrary unions of finite intersections of elements of $\mathcal{S}$ implies $T' \subseteq T$.

"$\subseteq$":

Conversely, every basis element $U \times V$ for the topology T is a finite intersection of elements of $\mathcal{S}$ since $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$. Therefore, $U \times V \in T' \Rightarrow T \subseteq T'$.

□

The product topology is the finest (largest) topology that makes all projection maps continuous. This means that it is the strongest topology that fits the "coordinate-wise" openness criterion. It is also used to define and study continuous mappings between product spaces. A mapping between product spaces is continuous if and only if each of its component mappings is continuous.

Definition: homeomorphism

A continuous map $f : (X, T_X) \to (Y, T_Y)$ is a homeomorphism if there exists a continuous function $g : (Y, T_Y) \to (X, T_X)$ such that $g \circ f = Id_X$ and $f \circ g = Id_Y$ where $Id_X : X \to X$ and $Id_Y : Y \to Y$. With $Id_X(x) = x$ and $Id_Y(y) = y$.

Remark:

Every homeomorphisms are open and closed maps. Moreover, any homeomorphism is a continuous bijection, but a continuous bijection may not have a continuous inverse. Now we give a counter-example: ||

Example 1.5: Continuous bijection may not have a continuous inverse

Let $X = [0,2\pi) \subseteq \mathbb{R}$ with subspace topology, let

$$Y = S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$$

with subspace topology. Then the mapping $f : [0,2\pi) \to S^1$ for $f(\theta) = (\cos\theta, \sin\theta)$ is continuous. (Check it!) However, the inverse function, which tries to map points on the unit circle back to angles, is not continuous. This is because, for points close to 1 and 0 (where the angle wraps around), small perturbations in S^1 lead to large changes in the angle. ||

Question:

What does it mean when we say that two topological spaces are the same?

Definition: homeomorphic topological spaces

We say two topological spaces X and Y are homeomorphic (the same) if there exists a homeomorphism $f : X \to Y$.

Lemma 1.11:

Let (X, T_X) and (Y, T_Y) be two topological spaces. Suppose that (W, T_W) is a topological space together with two continuous maps:

$$g_X : W \to X \text{ and } g_Y : W \to Y$$

so that for any topological space (Z, T_Z) and any two continuous mappings

$$f_X : Z \to X \text{ and } f_Y : Z \to Y.$$

There exists a unique continuous function $f : Z \to W$ such that

$$g_X \circ f = f_X \text{ and } g_Y \circ f = f_Y.$$

Then (W, T_W) is homeomorphic to $(X \times Y, T_{\text{prod}})$.

Proof:

Left as an exercise (hint: Universal Property). □

We are now exposed to enough materials for the product topology between the topological spaces (X, T_X) and (Y, T_Y); in fact, this could be extended to arbitrary products.

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of sets indexed by an arbitrary index set A (countable or not, finite or not). That is, we have a function $f : A \to X_\alpha$ defined by $f(\alpha) = X_\alpha$. There then exists a set, namely the product $\prod_{\alpha \in A} X_\alpha$, together with a collection of maps $\{p_\alpha : \prod_{\beta \in A} X_\beta \to X_\alpha\}_{\alpha \in A}$, so that for an arbitrarily chosen set Z and a family of maps $\{f_\alpha : Z \to X_\alpha\}_{\alpha \in A}$, there exists a unique function $f : Z \to \prod_{\alpha \in A} X_\alpha$ such that $p_\beta \circ f = f_\beta$ holds $\forall \beta \in A$ (β behaves the same as α, its employment here is to aviod ambiguity).

Exercise 1.4: Cartesian Product

Let $A := \{1,2\}$. Then there exist two sets X_1 and X_2 such that the product is

$$\prod_{\alpha \in A} X_\alpha = \left\{ g : \{1,2\} \to X_1 \cup X_2 \text{ such that } g(1) \in X_1, g(2) \in X_2 \right\}$$

$$= \left\{ \big(g(1), g(2)\big) \,\middle|\, g(1) \in X_1, g(2) \in X_2 \right\} = X_1 \times X_2,$$

where $X_1 \times X_2$ denotes the cartesian product.

Proof:

This result is trivial but the proof is not.

We set $\prod_{\alpha \in A} X_\alpha = \left\{ g : A \to \bigcup_{\alpha \in A} X_\alpha \,\middle|\, g_\alpha := g(\alpha) \in X_\alpha \forall \alpha \in A \right\}$. We define $p_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$ by $p_\beta(g) = g(\beta)$ since $A \overset{g_\alpha}{\to} \bigcup_{\alpha \in A} X_\alpha \overset{p_\beta}{\to} X_\beta$. Given now $\{f_\alpha : Z \to X_\alpha\}_{\alpha \in A}$, we may define $f : Z \to \prod_{\alpha \in A} X_\alpha$ by $f(Z) = \big(f_\alpha(Z)\big)_{\alpha \in A}$.

Then, $p_\beta\big(f(z)\big) = f_\beta(z) \forall z \in Z \Rightarrow p_\beta \circ f = f(\beta) = f_\beta$.

Next we shall prove the existence, which follows:

If $h : Z \to \prod_{\alpha \in A} Z_\alpha$ is another function such that $p_\beta \circ h = f_\alpha$, then $\forall z \in Z$ and $\forall \beta$, one has $h(z)_\beta = (p_\beta \circ h)(z) = f_\beta(z) = \big(f(z)\big)_\beta$, which implies that $h(z) = f(z) \forall z$ hence uniqueness follows and the result.

□

This exercise tells us that the cartesian product is unique. The next exercise, with proof left empty, shows that the reverse is also valid:

Exercise 1.5:

Let A be a set and consider the collection of topological spaces indexed by elements of A defined by $\big\{(X_\alpha, T_\alpha)\big\}_{\alpha \in A}$. Then there is a topological space (W, T_W) together with a family $\big\{p_\alpha : (W, T_W) \to (X_\alpha, T_\alpha)\big\}_{\alpha \in A}$ so that for any topological spaces (Z, T_Z) and their corresponding family of continuous mappings, there exists a unique contunuous map $f : (Z, T_Z) \to (X_\alpha, T_\alpha)$ such that $p_\alpha \circ f = f_\alpha \forall \alpha$. Moreover, W is unique up to a homeomorphism.

Notation:

The usual notation for (W, T_W) is $\big(\prod_{\alpha \in A} X_\alpha, T_{\text{prod}}\big)$.

One may notice in **Exercise 1.5**, we did say the variation of W is unique up to a homeomorphism, recall that the term "homeomorphism", is often replaced by "isomoprhism", our usage here is to aviod ambiguity: Just like the continuous functions in topological spaces inherit the same properties as continuous maps in other mathematical structures, the isomophisms, which is a structure-preserving bijection between objects. In the context of topological spaces, however, the term "isomorphism" is typically not used directly because it can be too rigid. A topological isomorphism would imply a bijective map that preserves not only open sets but also other topological properties (like compactness, connectedness, etc.), which may not always be a meaningful or interesting concept.

Instead, the concept of a homeomorphism is introduced here, which focuses on preserving the topology's key properties while allowing for more flexibility and generalization. Homeomorphisms enable us to study topological spaces in a way that respects their topological structure while potentially allowing for deformations or changes that do not alter the topological properties.

So far, we have introduced most of the important concepts in topological spaces. When we define a space, it is important to talk about its elements and the maps between its elements, there are many other important terminologies we would like to explore, but lack of tools, we often wish to learn information from the maps, that is why the continuity and differentibility of a function is so important in analysis. But still, there are some other concepts we can discuss without heavy machinery, which are often called the topological properties, which are fundamental characteristics of topological spaces that capture the behavior and structure of the spaces under consideration. These properties are often studied in topology to classify, compare, and analyze different types of spaces. For example, openness, closedness, compactness, separability, etc, are key features we would like to use in further exploration.

Let us now pause a while and discuss the methodology we mentioned in the above paragraph.

Algorithm 1.1: Defining Mathematical Spaces

Defining a new space in mathematics involves a systematic approach that combines creativity, rigor, and clear communication. Whether you're introducing a new topological space, metric space, vector space, or any other mathematical structure, here's a general methodology to consider:

I. Motivation and Inspiration:

Clearly identify the motivation for defining the new space. What problem are you trying to address? What concepts or structures does this space generalize or capture? Look for inspiration from existing mathematical spaces, objects, or structures that have relevant properties or behaviors.

II. Defining the Set (Object):

Begin by defining the underlying set of the new space. Determine what kind of elements the set should contain based on the desired properties of the space. Specify any constraints, conditions, or requirements on the elements that belong to the set.

III. Defining Operations and Relations:

If applicable, define any operations (addition, multiplication, etc.) that should be defined on the elements of the set. These operations should align with the properties you want the space to have. Define any relevant relations (equivalence, order, etc.) that help establish the structure of the space.

IV. Topology and Open Sets (if relevant):

If defining a topological space, specify the topology on the set. This involves determining the collection of open sets that satisfy the desired topological properties. If the space has a metric, define the metric function that measures distances between elements.

V. Properties and Axioms:

List the key properties, axioms, or characteristics that you want the space to possess. Ensure that these properties align with the motivation and desired behaviors of the new space.

VI. Examples and Counterexamples:

Provide examples of elements and subsets in the space that illustrate its key properties. Consider providing counterexamples that highlight the limitations or boundary cases of the space's properties.

VII. Connections to Existing Concepts:

Establish connections between your new space and existing mathematical concepts. This can help others understand the context and significance of the new space.

||

We wish our discussion of the topology follows this step clear enough. Now let us continue our discussion in product topology. Since this is an introductory book on topics of topology, we do not want to include too many advanced materials even in basic category theory. So we shall close this subsection with an example:

Example 1.6: Box Topology

There is another "natural" topology on $\prod_{\alpha\in A} X_\alpha$ called the box topology, namely, T_{box}, defined by $T_{\text{box}} := \left\{\prod_{\alpha\in A} U_\alpha \,\middle|\, U_\alpha \subseteq X_\alpha \text{ are open}\right\}$. This is apparantly a product topology different from T_{prod}, concidence occur when A is finite. If A is not finite, then $T_{\text{box}} \supsetneq T_{\text{prod}}$. Note that since $T_{\text{box}} \supseteq T_{\text{prod}}$, then the maps $p_\beta : \left(\prod X_\alpha, T_{\text{box}}\right) \to (X_\beta, T_\beta)$ are continuous. However, the universal property fails for T_{box}, hence it is not a "good" topology. ||

1.5 The Subspace Topology

With now enough tools and notions, we can get to the discussion of subspace topologies. The subspace topology is an essential concept in topology that allows us to study subsets of topological spaces while inheriting the topology from the original space. It provides a way to understand the topology of a subset in terms of the topology of the larger space.

Definition: Subspace Topology

Let (X, T) be a topological space. If $Y \subseteq X$, then $T_Y := \{Y \cap U \mid U \in T\}$ is a topology on Y, called the subspace topology.

In other words, the open sets in the subspace topology are the intersections of open sets in the original space X with the subset Y.

We care about the basis for subspace topology too. In fact, the new basis is taken the same way as we take the subspace topology, this is the result of the following lemma:

Lemma 1.12:

If $\mathscr{B}$ is a basis for the topology of X then the collection $\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$ is a basis for the subspace topology on Y.

Proof:

Given U open in X and given $y \in U \cap Y$, we can choose an element $B \in \mathscr{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. Then by **Lemma 1.3** that $\mathscr{B}_Y$ is a basis for the subspace topology on Y.

□

When dealing with a space X and a subspace Y, one needs to be careful when one uses the term "open set". Does one mean an element of the topology of Y or an element of the topology on X? We make the following definition:

Notations:

If Y is a subspace of X, we say that a set U is open in Y (or open relative to Y) if it belongs to the topology of Y; this implies in particular that it is a subset of Y. We say that U is open in X if it belongs to the topology of X.

There is a special situation in which every open set in Y is also open in X.

Lemma 1.13:

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is also open in X.

Proof:

Since U is open in Y, $U = Y \cap V$ for some open set V open in X. Since Y and V are both open in X, then so is $Y \cap V$.

□

The properties that characterize the subspace topology are more important than the definition above. In his notes [4] in Fall 2014, John Terilla offered two characterizations of the subspace topology. The following two characterizations are drawn from his lecture notes: The first one characterizes that the subspace topology is the coarsest topology on Y for which the inclusion map $i : Y \to X$ is continuous. The second one is a universal property that characterizes the subspace topology on Y by characterizing which functions into Y are continuous. We shall use one lemma to conclude these results:

Theorem 1.14:

Let (X, d_X) be a topological space and let $Y \subseteq X$ be a subset. We claim that $T^Y := \left\{U \subseteq Y \middle| \, \exists \tilde{U} \subseteq X \text{ open with } U = \tilde{U} \cap Y\right\}$ is a topology on Y. Moreover, T^Y is the smallest topology on Y, and the inclusion $i : Y \hookrightarrow X$ is continuous.

Proof:

We first prove that T^Y under this setting is indeed a topology on Y, that means we need to check the three conditions for being a topology:

(i) $Y = X \cap Y \in T^Y$ by definition. Similarly, $\varnothing = \varnothing \cap Y \in T^Y$.

(ii) For arbitrarily chosen open sets $U, V \in T^Y$, there exists $\tilde{U}$ and $\tilde{V} \subseteq X$ being open such that $U = \tilde{U} \cap Y$ and $V = \tilde{V} \cap Y$. Moreover, one has $(U \cap V)\backslash\big((\tilde{U} \cap Y) \cap (\tilde{V} \cap Y)\big) = (\tilde{U} \cap \tilde{V}) \cap Y \in T^Y$ since both $\tilde{U}$ and $\tilde{V}$ are open in X.

(iii) Similarly, if $\{U_\alpha\}_{\alpha \in A}$ is an arbitrarily chosen collection of open sets in T^Y then $\bigcup_{\alpha \in A} U_\alpha \in T^Y$.

Now we know that T^Y is indeed a topology on Y. It remains to check the behaviour of the inclusion mapping $i : Y \hookrightarrow X$:

First, let T' be any other topology such that $i : Y \hookrightarrow X$ being continuous.

Let now $U \in T^Y$ be an open set, then by definition of T^Y one has $U = \tilde{U} \cap Y$ for some open set $\tilde{U} \subseteq X$, but then $\tilde{U} \cap Y = i^{-1}(\hat{U}) \in T'$ hence $T^Y = T'$.

Result follows from the uniqueness.

□

This type of result follows from [2], and the proof adapts no materials from category theory. As we promised, let us dive deep into its result and interprete it:

In order to describe the first characterization, let us illustrate a general fact: Let (X, T_X) be a topological space and let S be any open set inside. Conside the function:

$$f : S \to X.$$

It makes no sense to ask if f is continuous unless S is equipped with a topology. There do exist topologies on the set S such that f is continuous. If T_f is the intersection of all topologies on S for which f being continuous, then T_f will be the coarsest topology for which f being continuous. Note that T_f has a simple explicit description as

$$T_f := \left\{ f^{-1}(U) \,\middle|\, U \subseteq X \text{ being open} \right\}.$$

This leads to the following alternative definition of the subspace topology:

Definition: Subspace Topology (alternative definition)

Let (X, T_X) be a topological space and let $S \subseteq X$ be any subset of X. The subspace topology on S is defined to be the coarsest topology on S for which the canonical inclusion $f : S \hookrightarrow X$ is continuous.

Remark:

There is an astonishing result: the coarsest topology on S, having the function f being continuous, may not be a subset of X! Here is why: Since f is injective, S is isomorphic as a set to its image $f(S) \subseteq X$; and the set S with the subspace topology determined by the injection $f : S \to X$ is homeomorphic to the set $f(S) \subseteq X$ with the subspace topology determined by the inclusion $i : f(S) \subseteq X$. If f is not injective, then the topology T_f is not referred to as the subspace topology. ||

We pause a moment to see this important result, notice that this terminology could be very useful in the construction of weak topologies. For example, suppose that X is a set without any structure and let $\{Y_\alpha\}_{\alpha\in A}$ is a collection of topological spaces. We are given a collection of maps $\{\varphi_\alpha\}_{\alpha\in A}$ such that $\forall \alpha \in A$, $\varphi_\alpha : X \to Y_\alpha$ and we wish to construct a topology on X such that all the maps $\{\varphi_\alpha\}_{\alpha\in A}$ are continuous. In practice, we wish to use as less open sets as possible to build such a topology. This is the strategy in constructing a weak topology, which provides a way to study convergence and continuity in functional space when the original space is too small to contain the limit points. Detailed description could be found in [5].

There is a principle in mathematics that if one can understand the morphisms in a category, then one can understand the objects. Without making this principle more precise now, let us give an illustration:

Suppose that you want to understand a topological space (X, T_X). One approach is to study the continuous functions $f : Z \to X$ or $f : X \to Z$ where (Z, T_Z) is another topological space. Now, the subspace topology has an important universal property which characterizes precisely which functions $f : Z \to Y$ are continuous for all topological spaces (Z, T_Z). This property completely determines the subspace topology on Y, which is the second characterization of subspace topology:

Universal Property for the Subspace Topology:

For every topological space (Z, T_Z) and every function $f : Z \to Y$, f is continuous if and only if $i \circ f : Z \to X$ is continuous, where $i : Y \to X$ is the natural inclusion.

One should think of the universal property stated above as a property that may be attributed to a topology on Y. At this point, one may think that some topologies have this properties and some do not. Furthermore, the subspace topology is the only topology on Y with this property! The proof of this observation may involve too much heavy machinery from category theory, hence we shall not present it here, for those who are interested in may consul [4].

1.6 Order Topology

Relation is one of the most important features of mathematical objects. In order to offer a better understanding of the order topology, we adapt some background terminologies from [7]. Before that, we will give the following notations for the common number systems:

Notations: Common Number Systems

(i) The natural number is given by $\mathbb{N} := \{0,1,2,\cdots\}$. Note that 0 is not always an element of $\mathbb{N}$.

(ii) The integers $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.

(iii) The rational numbers $\mathbb{Q} = \left\{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.

Definition: Relation

Let A be a set and let $n \in \mathbb{N}$ with $n \geq 1$. An n-ary relation R on A is a subset $R \subseteq A^n$, where A^n is the cartesian product. Given $\overline{a} = \langle a_0, \cdots, a_{n-1} \rangle \in A^n$, we say that R holds for $\overline{a}$ if $\overline{a} \in R$, and otherwise we say R does not hold for $\overline{a}$.

If $n = 1$, we call R a unary relation; if $n = 2$, we call R a binary relation; and if $n = 3$, we call R a ternary relation. In most scenarios, we consider R as a binary relation. Moreover, sometimes when a and b has the binary relation R, we may say aRb, but this does not necessarily mean bRa, with $aRb \Leftrightarrow bRa$ when the binary relation is said to be reflexive. This leads to the necessity in defining the features of a given binary relation:

Definition: features of binary relation

Let R be a binary relation on a set A. Then we define the followings:

(1) R is **reflexive** if $\forall a \in A$, $\langle a, a \rangle \in R$.

(2) R is **irreflexive** if $\forall a \in A$, $\langle a, a \rangle \notin R$.

(3) R is **symmetric** if $\forall a, b \in A$, $\langle a, b \rangle \in R \Rightarrow \langle b, a \rangle \in R$.

(4) R is **antisymmetric** if $\forall a, b \in A$, $\langle a, b \rangle \in R \not\Rightarrow \langle b, a \rangle \in R$.

In the above, (1) and (2), (3) and (4) are dual to each other, respectively.

(5) R is **transitive** if $\forall a, b, c \in A$, $\langle a, b \rangle \in R \wedge \langle b, c \rangle \in R \Rightarrow \langle a, c \rangle \in R$.

(6) R is **total** if $\forall a, b \in A$, either $\langle a, b \rangle \in R$ or $\langle b, a \rangle \in R$.

(7) R is **trichotomy** if $\forall a, b \in A$, exactly one of $a = b$, $\langle a, b \rangle \in R$, or $\langle b, a \rangle \in R$ is valid.

For the most part, our examples come from the following classes of algebraic structures:

Definition: Equivalence Relation

An equivalene relation is a pair $(E, \sim)$ such that E is a set and $\sim$ is a binary relation on E which is reflexive, symmetric, and transitive. That is, a relation $\sim$ on a set E satisfies, $\forall a, b, c \in E$:

(i) $a \sim a$. (reflexive)

(ii) $a \sim b \Rightarrow b \sim a$. (Symmetric)

(iii) $a \sim b \wedge b \sim c \Rightarrow a \sim c$. (Transitive)

Definition: Equivalence Class

Given an equivalence relation $\sim$ on a set X and an element $x \in X$, the equivalence class of x in X, denoted by $[x]$, is defined to be the set of all elements in X that is equivalent to x; namely, $[x] := \{y \in X \mid y \sim x\}$.

Definition: Preorder

A preorder on a set X is a binary relation $\leq$ such that it is reflexive and transitive.

Definition: Partial Order (Partially ordered set, or Poset)

A partial order is a pair $(P, \leq_P)$ such that P is a set and $\leq_P$ is a binary relation on P which is reflexive, antisymmetric, and transitive.

Definition: Ordered Sets

An ordered set is a set X equipped with a binary relation $\leq$ that satisfies certain properties:

(i) $\forall x, y \in X, x \leq y$ or $y \leq x$. (Comparability, or, Total Order)

(ii) $\forall x, y, z \in X, x \leq y$ and $y \leq z \Rightarrow x \leq z$. (Transitivity)

(iii) $\forall x, y \in X, x \leq y$ and $y \leq x \Rightarrow x = y$. (Antisymmetricity)

Definition: linear order

A linear order is a pair $(L, \leq_L)$ such that L is a set and $\leq_L$ is a binary relation on L which is reflexive, antisymmetric, transitive, and total.

Remark:

Thus, a linear order is a partial order which is total. ||

Definition: strict linear order

Similarly, we could define the strict linear order with the only difference being $\leq_L$ being replaced by $<_L$, which is irreflexive, transitive, and trichotomy.

We will also use the following terminologies for functions:

Definition: Injective, Surjective, and Bijective

Let A and B be two sets and let $f : A \to B$ be a function. Then the range of f is the set $\text{range}(f) := \left\{ b \in B \,\middle|\, \exists a \in A \text{ such that } f(a) = b \right\}$.

We say that f is one-to-one (or **injective**) if $f(a_1) \neq f(a_2) \forall a_1 \neq a_2 \in A$. That is to say, $b \in \text{range}(f) \Rightarrow \exists! a \in A$ such that $f(a) = b$.

We say that f is onto B (or **surjecitve**) if $\text{range}(f) = B$. That is, if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

Moreover, we say that f is one-to-one and onto (or bijective) if it is both injective and surjective.

The order topology is a concept in topology that arises from the order structure of a set. It's a way of defining a topology on a totally ordered set that captures the order relationships between elements. The order topology is particularly useful when studying ordered sets and their continuity properties.

Notations:

Let $(L, \leq)$ be a linear order, and let $a, b \in L$. We define the following notions:

(i) $(a, \infty) = \{x \in L \mid a < x\}$.

(ii) $(-\infty, b) = \{x \in L \mid x < b\}$.

(iii) $(a, b) = \{x \in L \mid a < x < b\}$.

(iv) $(a, b] = \{x \in L \mid a < x \leq b\}$.

Definition: Order Topology

Let $(L, \leq)$ be a linear order with at least two elements. Define the set

$$\mathcal{S} := \{(-\infty, b) \mid b \in L\} \cup \{(a, \infty) \mid a \in L\}.$$

Then $\mathcal{S}$ is a subbasis on L, and the topology generated by it is called the order topology on L.

This essentially (but not precisely) means that the order topology on L is the one generated by the basis of open intervals, in the sense described above, since $\forall a, b \in L \wedge a < b$, $(a, b) = (-\infty, b) \cap (a, \infty)$. Therefoer, (a, b) is in the basis generated by $\mathcal{S}$.

Example 1.7: Order Topology

(i) The order topology on $(\mathbb{R}, \leq)$ is the same as the usual topology. We already know this since the usual topology is generated by the basis of open intervals.

(ii) The order topology on $(\mathbb{Q}, \leq)$ is the same as its subspace topology inherited from the usual topology on $\mathbb{R}$. This is also easy to see, but not trivial, since, e.g. $(-\pi, \pi) \cap \mathbb{Q}$ is an open set in the subspace topology inherited from the usual topology on $\mathbb{R}$, but is not a basic open interval in the order topology on $\mathbb{Q}$. It is, of course, a union of basic open intervals.

(iii) The order topology on $(\mathbb{N}, \leq)$ is also the same as its subspace topology inherited from the usual topology on $\mathbb{R}$, which is to say that it is discrete. Indeed, for example, we have $\{7\} = (6,8)$.

Definition: Order Topology (Alternative)

Let X be a set with a simple order relation; assume that X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(i) All open intervals (a, b) in X.

(ii) All intervals of the form $[a, b)$, where a is the smallest element (if any) of X.

(iii) All intervals of the form $(a, b]$, where b is the largest element (if any) of X.

Then the collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

It is easy to see the difference between these two definitions, the first one is established by the subbasis while the second one is generated by basis.

Definition: Rays

$(a, +\infty)$ and $(-\infty, a)$ are called open rays while $[a, +\infty)$ and $(-\infty, b]$ are called closed rays.

Theorem 1.15:

Let X be a set with a simple order relation. The open rays form a subbasis for the order topology T_X on X.

Comment:

The order topology reflects the order relationships in the ordered set. Open intervals correspond to open sets in the topology, and their unions and intersections capture intervals of elements in the order. More literature on topics of orders could be accessed via [8], the readers are also encouraged to view materials in [1].

1.7 Quotient Topology

The motivation behind the quotient topology arises from the need to study spaces that are formed by identifying or "gluing" certain points together in a way that respects the original topology. The quotient topology provides a way to study the resulting space in a manner that captures the relationships between points that have been identified as equivalent.

In various mathematical contexts, it's common to want to treat certain points as being the same or equivalent for the purpose of analysis. For example, in geometry, one might want to consider a square where points along its boundary are treated as equivalent to form a topological circle. The quotient topology allows one to study the circle while keeping track of the underlying square's topology.

Definition: Quotient Set

The quotient set $X/\sim$ is the set of all equivalence classes of X with respect to the equivalence relation $\sim$. In other words, each element of $X/\sim$ is itself an equivalence class of elements in X. Therefore, $X/\sim := \{[x] \mid x \in X\}$.

Given an equivalence relation $\sim$ on a set X, we have the quotient set $X/\sim$. There is an canonical map which is surjective. More specifically,

$$q : X \to X/\sim \text{ such that } x \mapsto [x].$$

Let us now generalize this description and generalize it into the term quotient maps:

Definition: quotient map

A quotient map is a function $p : X \to X/\sim$ that assigns each element of X to its equivalence class in the quotient set $X/\sim$. It respects the equivalence relation, meaning that $x \sim y \Rightarrow p(x) = p(y)$. Hence p is surjective.

When we say that a function is "constant on an equivalence class," it means that the function takes the same value for all elements within that equivalence class. In the context of equivalence relations and quotient sets, if two elements are related by the equivalence relation, they belong to the same equivalence class. Therefore, a function being constant on an equivalence class means that it assigns the same value to all elements in that class. That is to say, $x \sim y \Rightarrow f(x) = f(y)$.

Now let us move to the discussion of the universal property of the quotient maps:

Universal Property of Quotient Map:

For all set Y , $\forall f : X \to Y$ which is constant on the equivalent classes of $\sim$ lie on $f(x) = f(x')$ if $x \sim x'$. Then there exists a unique $\bar{f} : X/\sim \to Y$ given by

$\bar{f}(p[x]) \cong \bar{f}([x]) = f(x)$. That is to say, $f = \bar{f} \circ p$.

Example 1.8:

Let $X = \mathbb{R}$. Define the equivalence relation $x \sim x' \Leftrightarrow x - x' \in \mathbb{Z}$. Then $f : \mathbb{R} \to \mathbb{C}$ by sending $x \mapsto e^{2\pi i x} =: f(x)$, which is constant on equivalence classes $\Rightarrow \exists ! \bar{f} : \mathbb{R}/\sim \to \mathbb{C}$ such that $\bar{f}([x]) = e^{2\pi i x} = f(x)$. ||

Lemma 1.16:

Let (X, T_X) be a topological space and denote $\sim$ as an equivalence relation on X. Let $q : X \to X/\sim$ be the quotient map. There exists a topology T_{quot}, which denotes the quotient topology, so that $q : (X, T_X) \to (X/\sim, T_{\text{quot}})$ is continuous, and so that for any topological spaces (Z, T_Z) and any continuous map $f : (X, T_X) \to (Z, T_Z)$, which is constant on the equivalence classes of $\sim$, there exists a unique continuous map $\bar{f} : (X/\sim, T_{\text{quot}}) \to (Z, T_Z)$ so that $\bar{f} \circ q = f$.

Proof:

Define the quotient topology $T_{\text{quot}} := \left\{ U \subseteq X/\sim \,\middle|\, q^{-1}(U) \text{ open} \right\}$. We first check that T_{quot} is indeed a topology:

(i) $q^{-1}(\emptyset) = \emptyset$ and $q^{-1}(X/\sim) = X$, hence $\emptyset, X \in T_{\text{quot}}$.

(ii) If $U, V \in T_{\text{quot}}$, then $q^{-1}(U)$ and $q^{-1}(V)$ are both open by definition, then $q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V)$ is also open $\Rightarrow U \cap V \in T_{\text{quot}}$.

(iii) Let $\{U_\alpha\}_{\alpha \in A} \subseteq T_{\text{quot}}$ be a collection of open sets. Then under the map $q^{-1}(\bigcup_{\alpha \in A} U_\alpha) = \bigcup_{\alpha \in A} q^{-1}(U_\alpha)$, which is open in X, hence $\bigcup_{\alpha \in A} U_\alpha \in T_{\text{quot}}$.

Next we prove that the Universal Property does hold:

Suppose that $f : (X, T_X) \to (Z, T_Z)$ is continuous and constant on the equivalence classes of X, the uniqueness is guaranteed by the surjectivity; it left us to prove the existence:

If there exists a unique map $\bar{f} : X/\sim \to Z$ such that $\bar{f} \circ q = f$, we need to check the continuity of $\bar{f}$. Given $V \in T_Z$, since

$q^{-1}\left((\bar{f})^{-1}(V)\right) = (\bar{f} \circ g)^{-1}(V) = f^{-1}(V)$ which is open, hence result follows.

□

We can turn **Lemma 1.16** into a definition of the quotient topology, which uses the notion of quotient maps to construct a topology on a set.

Definition: Quotient Topology

Let (X, T_X) be a topological space, and let $\sim$ be an equivalence relation on X. The quotient space $X/\sim$ is the set of equivalence classes of X under $\sim$. The quotient topology on $X/\sim$ is defined in a way that makes the canonical projection $p : X \to X/\sim$ continuous. That is, $U \subseteq X/\sim$ is open if and only if $p^{-1}(U)$ is open in X.

In topology, the fiber of a function is a concept that helps us understand how a function behaves locally with respect to its target space. The notion of a fiber is particularly important when dealing with continuous maps between topological spaces.

Definition: Fiber

A fiber of a function $f : (X, T_X) \to (Y, T_Y)$ is a set of the form $f^{-1}(y)$ for some $y \in Y$. Namely, it is the set $\{x \in X \,|\, f(x) = y\}$.

Remark:

If such an f is continuous, then the fiber $f^{-1}(y)$ inherits a topology from X. It's a subset of X with the topology induced by X. ||

Fibers are used to describe the inverse images of points in the target space. They help in understanding which points in the domain map to a given point in the codomain. Moreover, note that the preimage and the inverse of the same function could be very different. Consider a function $f : X \to Y$, the preimage of a set B under f is the set of all elements in the domain X that map to elements in B, is denoted by $f^{-1}(B) = \{x \in X \,|\, f(x) \in B\}$. Even we use the same notation as the inverse function, it is important to see that an inverse function is bijective while the preimage function may not be, therefore it is possible for the fibre to be an empty set.

Example 1.9: Fiber

Let $q : X \to X/\sim$ and the fiber

$$q^{-1}([x]) = \{x' \in X \,|\, [x'] = [x]\} = \{x' \in X \,|\, x \sim x'\} = [x].$$ ||

Remark:

In general, the fibers are a function $f : X \to Y$ that are equivalent classes of $\sim$ defined by $x \sim x' \Leftrightarrow f(x) = f(x')$. Hence, fiber indicates equivalent classes in general. ||

Now we state a result using the fiber to find the homeomorphic space of a given quotient topology.

Lemma 1.17:

Let $f : (X, T_X) \to (Y, T_Y)$ be a continuous and surjective map and let $\sim$ be an equivalence relation on X defined by $x \sim x' \Leftrightarrow f(x) = f(x')$. Assume T' is the largest topology on Y making f continuous, i.e. $\forall U \subseteq Y, f^{-1}(U)$ is open implies the openness of U. Then (Y, T') is homeomorphic to $(X/\sim, T_{\text{quot}})$.

Proof:

We first prove that f is a continuous bijection: Since f is constant on the equivalence classes, then there exists a unique continuous map $\bar{f} : (X/\sim, T_{\text{quot}}) \to (Y, T')$ such that $\bar{f}([x]) = f(x)$. If $\bar{f}$ is injective, $\bar{f}([x]) = \bar{f}([x'])$ implies that $f(x) = f(x')$ hence $x = x' \Rightarrow [x] = [x']$, therefore, $\bar{f}$ is a continuous bijection.

It remains to prove that $h := (\bar{f})^{-1} : (Y, T') \to (X/\sim, T_{\text{quot}})$ is continuous. Given $U \subseteq X/\sim$ open, since $h^{-1}(U) = \bar{f}(U)$, then $h^{-1}(U) \subseteq Y$ is open if and only if $f^{-1}(h^{-1}(U))$ is open in X. That is, $f^{-1}(\bar{f}(U))$, since $q \circ \bar{f} = f$, then

$$f^{-1}(\bar{f}(U)) = (\bar{f} \circ q)^{-1}(\bar{f}(U)) = q^{-1}\big((\bar{f})^{-1}(\bar{f}(U)\big) = q^{-1}(U).$$

Since q is continuous and $q^{-1}(U)$ is open in X. Then by assumption, on T', $h^{-1}(U) = \bar{f}(U)$ is open in Y, hence $h := (\bar{f})^{-1}$ is continuous $\Rightarrow \bar{f}$ is the desired homeomorphism.

□

Example 1.10: Homeomorphism on $\mathbb{R}^2/\sim$

Consider $\mathbb{R}^2$ is the standard topology, $f : \mathbb{R}^2 \to \mathbb{R}$ under the map $(x, y) \mapsto x$. Then for all $U \subseteq \mathbb{R}, f^{-1}(U) = U \times \mathbb{R}$, which is open in $\mathbb{R}^2$ hence U is open. Therefore, the quotient space $\mathbb{R}^2/\sim$, where $(x, y) \sim (x', y') \Leftrightarrow x = x'$, is homeomorphic to $\mathbb{R}$ and the map $\bar{f}([x, y]) = f(x)$ is a homeomorphism. ||

We have already noted that the subspaces do not behave well: if $p : X \to Y$ is a quotient map and A is a subspace of X, then the map $q : A \to p(A)$ obtained by restircting p need not to be a quotient map. One has, however, the following theorem:

Theorem 1.18: Quotient Map and Subspace

Let $p : X \to Y$ be a quotient map; let A be a subspace of X that is saturated with respect to p; let $q : A \to p(A)$ be the map obtained by restricting p. Then

(i) If A is either open or closed in X, then q is a quotient map.

(ii) If p is either an open map or a closed map, then q is a quotient map.

Proof:

Step I:

We verify the following two equations:

$$q^{-1}(V) = p^{-1}(V) \text{ if } V \subseteq p(A);$$
$$p(U \cap A) = p(U) \cap p(A) \text{ if } U \subseteq X.$$

To chcek the first equation, we note that since $V \subseteq p(A)$ and A is saturated, $p^{-1}(V)$ is contained in A. It follows that both $p^{-1}(V)$ and $q^{-1}(V)$ equal all points of A that are mapped by p into V. To check the second equation, we note that for any two subsets U and A of X, we have the inclusion

$$p(U \cap A) \subseteq p(U) \cap p(A).$$

To prove the reverse inclusion, suppose $y = p(u) = p(a)$, for $u \in U$ and $a \in A$. Since A is saturated, A contains the set $p^{-1}(p(a))$, so that in particular A contains u. Then $y = p(u)$, where $u \in U \cap A$.

Step II:

Now suppose that A is open or p is open. Given the subset V of $p(A)$, we assume that $q^{-1}(V)$ is open in A and show that V is open in $p(A)$.

Suppose first that A is open. Since $q^{-1}(V)$ is open in A and A is open in X, the set $q^{-1}(V)$ is open in X. Since $q^{-1}(V) = p^{-1}(V)$, the latter set is open in X, so that V is open in Y because p is a quotient map. In particular, V is open in $p(A)$.

Now suppose that p is open. Since $q^{-1}(V) = p^{-1}(V)$ and $q^{-1}(V)$ is open in A, we have $p^{-1}(V) = U \cap A$ for some set U open in X. Now $p(p^{-1}(V)) = V$ since p is surjective; then $V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$. The set $p(U)$ is open in Y because p is an open map; hence V is open in $p(A)$.

Step III:

The proof when A or p is closed is obtained by replacing the term "open" by the term "closed" throughout *Step II*.

□

Now we consider other concepts introduced previously. Composites of maps behave nicely; it is easy to check that the composite of two quotient maps is a quotient map; this fact follows from the equation

$$p^{-1}(q^{-1}(U)) = (p \circ q)^{-1}(U). \tag{1.4}$$

On the other hand, products of maps do not behave well; the Cartesian product of two quotient maps need not to be a quotient map. One needs further conditions on either the maps or the spaces in order for this statement to be true. One such, a condition on the spaces, is called local compactness, which we shall study later. Another, a condition on the maps, is the condition that both the maps p and q be open maps. In that case, it is easy to see that $p \times q$ is also an open map, so it is a quotient map.

1.8 Openness, Closedness, and Limitedness

Recall in basic analysis course, we have encountered the terminologies called limit points, closedness, and closure. All of them could offer a perspective corresponding to the convergence of a sequence; in particular, Cauchy sequence, where the convergence of all Cauchy sequences lead to completeness. In this subsection, we shall dive deeper in the treatment of closed sets, closures, and limit poins. These lead naturally to consideration of a certain axiom for topological spaces called the Hausdorff axiom. Moreover, after this subsection, we will be equipped with enough background for the study of not only the separation properties but also the compactness as well as connectedness.

Definition: Neighbourhood

A neighbourhood of a point x in a topological space X is a subset $N \subseteq X$ such that there exists an open set such that $U \subseteq X$ with $x \in U \subseteq N$.

Proposition 1.19: Neightbourhood and Open Sets

A subset U of a topological space X is open if and only if U is a neighbourhood of every points $x \in U$.

Proof:

If U is open and $x \in U$, then U is an open neighbourhood of x. Suppose that $U \subseteq X$ is a subset, then U is a neighbourhood of every $x \in U$. Then $\forall x \in U$, there exists an open set V_X such that $x \in V_X \subseteq U$, hence

$U = \bigcup_{x \in U} \{x\} \subseteq_{x \in U} V_X \subseteq U \Rightarrow U = \bigcup_{x \in U} \{x\} = \bigcup_{x \in U} V_X \Rightarrow U$ is open,

□

Definition: Limit Points

Let X be a topological space and let $A \subseteq X$ be a subset, a point $x \in X$ is called a limit point of A if for all neighbourhood N of x, $N \cap (A \backslash \{x\}) \neq \varnothing$.

Definition: Interior

Let (X, T_X) be a topological space and let $A \subseteq X$ be a subset, then the interior of A, denoted by A°, is the largest open set that contains A.

Remark:

(i) $A^\circ := \bigcup_{U \subseteq A \text{ open}} U \subseteq A.$

(ii) $A \subseteq X$ is open $\Leftrightarrow A = A^\circ$.

(iii) It may happen that $A^\circ = \varnothing$. For example, if $X = \mathbb{R}$ and $A = \mathbb{Q}$. Then $\forall x \in \mathbb{Q}$, $\forall \varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon)$ containing irrationals implies $(x - \varepsilon, x + \varepsilon) \subsetneq \mathbb{Q} \Rightarrow \mathbb{Q}^\circ = \varnothing$.

Definition: Direct Set

A directed set is a set Λ with a preorder $<$ such that $\forall \lambda_1, \lambda_2 \in \Lambda$, $\exists \lambda_3 \in \Lambda$ such that $\lambda_1 < \lambda_3$ and $\lambda_2 < \lambda_3$.

Example 1.11: Directed Sets

$(\mathbb{N}, \leq)$ is a directed set.

Let X be a topological space, let $x \in X$ be a point. Let $\Lambda := \{N \subseteq X \mid N \text{ is a neighbourhood of } x\}$. Define $<$ on Λ by inverse inclusion: $N_1 < N_2 \Leftrightarrow N_1 \supseteq N_2$. Given $N_1, N_2 \in \Lambda$, $N_1 \supseteq N_1 \cap N_2$ and $N_2 \supseteq N_1 \cap N_2$ hence Λ is a directed set. ||

Recall in calculus, the convergence is built up by the accumulation performance of a sequence. In topology, there is also such a sequence, made up, however, by the open sets, this is the reason we introduced relations and directed sets. Such a sequence, which is fundamental in our understanding of the convergence and the continuity, is called "nets".

In topology, a net is a generalization of sequences that provides a way to analyze convergence and continuity in topological spaces that might not be first-countable, as well as to study properties such as compactness and closure. Nets offer a broader framework for understanding the behavior of points in a topological space, particularly when sequences may not be sufficient.

Definition: Net

A net in a topological space X is a function $x : \Lambda \to X$ by sending $\lambda \mapsto x$, where Λ is a directed set. We denote that $x : \Lambda \to X$ by $\{x_\lambda\}_{\lambda \in \Lambda}$.

Observation of this definition tells us that a net is a function generalizes the concept of sequences, allowing indexing by any directed set, making them applicable in more general topological spaces.

Definition: Convergence of Net

A net $(x_\lambda)_{\lambda \in \Lambda}$ in a topological space X converges to $y \in X$ if and only if for all neighbourhood W of y, $\exists \lambda_0 \in \Lambda$ such that $\lambda_0 < \lambda \Rightarrow x_\lambda \in W$. If $x_\lambda \to y$, we say y is a limit point of $(x_\lambda)_{\lambda \in \Lambda}$. A net $(x_\lambda)_{\lambda \in \Lambda}$ is convergent if it has a limit.

Recall that the closure of a set is a fundamental concept that characterizes the points that are "close" to the set. The closure of a set includes the set itself along with its limit points, providing a way to describe the extent to which the set fills its surrounding space. The closure is an important tool for analyzing the behavior of sets within a given topological space. Loosely speaking, the closure of $A \subseteq X$ is the smallest closed set $\overline{A}$ contains A. Now we introduce the relationships between closures and interiors:

Definition: Closure

If X is a topological space, $A \subseteq X$ is a subset, the closure $\overline{A}$ of A is the smallest closed set that contains A.

Lemma 1.20: Existence and Uniqueness

The closure $\overline{A}$ of a subset A of a topological space X exists and is unique.

Proof:

Let $\overline{A} := \bigcap_{C \subseteq X \text{ closed}, A \subseteq C} C$. Then (i) $\overline{A}$ is closed. (ii) By construction, $A \subseteq \overline{A}$.

(iii) $C' \supseteq A$ where C' is closed implies $\overline{A} = \bigcap_{C \subseteq X \text{ closed}, A \subseteq C} C \subseteq C'$. ($C'$ denotes the set containing all limit points)

□

Definition: Closed

$A = \overline{A} \Leftrightarrow A$ is closed.

Proposition 1.21:

Let X be a topological space and $A \subseteq X$ is a subset. Then $\overline{A} = A \cup A'$. In particular, $x \in \overline{A} \Leftrightarrow$ for all neighbourhood N of x $N \cap A \neq \emptyset$.

Proof:

$x \notin A \cup A' \Leftrightarrow$ there is a neighbourhood N of x such that $N \cap A = \emptyset$.

[Claim]: $x \notin \overline{A} \Leftrightarrow \exists$ neighbourhood N of x such that $N \cap A = \emptyset$.

"$\Rightarrow$":

$x \notin \overline{A} \Rightarrow x \in (\overline{A})^c \Rightarrow X \backslash \overline{A} = N$ is open in X. Say N is a neighbourhood of x since $x \in N \Rightarrow \emptyset = N \cap \overline{A} \supseteq N \cap A$.

"$\Leftarrow$":

Suppose that there exists a neighbourhood N of x such that $N \cap A = \emptyset$. Then there exists an open set $U \subseteq X$ such that $x \in U \subseteq N$.

Hence $A \subseteq X \backslash U \Rightarrow \overline{A} \subseteq X \backslash U \Rightarrow U \cap \overline{A} = \emptyset$ but $x \in U$, thus $x \notin \overline{A}$.

By *[Claim]*, it suffices to show that $x \notin A' \cup A$then $x \notin \overline{A}$ by definition.

□

Hausdorff space is a fundamental concept in topology that captures a strong separation property within a topological space. Named after the mathematician Felix Hausdorff, this property ensures that distinct points in the space can be separated by disjoint open neighborhoods. Limits of sequences or nets in a Hausdorff space are unique. If a sequence or net converges to a point, then it can only converge to that particular point.

Definition: Hausdorff space

A topological space X is called Hausdorff (T_2) if $\forall x, y \in X$ such that $x \neq y$, there exist open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

Example 1.12: Hausdorff space

(i) Any metric space (X, d) is Hausdorff. Since $\forall x, y \in X$ with $x \neq y$, $d(x, y) > 0$ then $B_{\frac{r}{2}}(x) \cap B_{\frac{r}{2}}(y) = \emptyset \Rightarrow$ Hausdorff.

(ii) Let $X := \{a, b, c\}$ and $T := \{\emptyset, X, \{a, b\}, \{a, c\}, \{b\}\}$, this is not Hausdorff.

Now we give one of the most important results derived from Hausdorff:

Proposition 1.22:

In Hausdorff space, limits of sequences when exist then unique.

Proof:

Suppose that X is Hausdorff, consider two points $y, z \in X$, $\{x_n\}$ is a sequence with $x_n \to y$ and $x_n \to z$.

[Claim]: $y = z$.

If $y \neq z \Rightarrow$ there exists an open neighbourhood U of y and V of z such that $U \cap V = \varnothing$. Since $x_n \to y$ there exists $N \in \mathbb{N}$ such that $x_n \in U \; \forall n > N$. Since $U \cap V = \varnothing$ then $x_n \notin V \; \forall n > N$. Then $x_n \not\to z$ hence contradiction.

Therefore, $y = z$ and uniqueness follows.

□

Proposition 1.23:

For any topological space (X, T_X), for any subset $A \subseteq X$, we have

(i) $X \backslash A^\circ = \overline{X \backslash A}$.

(ii) $X \backslash \overline{A} = (X \backslash A)^\circ$.

Proof:

Recall that a subset $U \subseteq X$ is open $\Leftrightarrow U^c$ is open.

$$X \backslash A^\circ = X \backslash \Big(\bigcup_{U \subseteq A, U \text{ open}} U \Big) = \bigcap_{X \backslash A \subseteq X \backslash U, U \text{ open}} (X \backslash U) = \bigcap_{C \subseteq X \text{ closed}, C \supseteq (X \backslash A)} = \overline{X \backslash A}.$$

Similarly,
$$\overline{X \backslash A} = X \backslash \bigcap_{A \subseteq C \text{ closed}} C = \bigcup_{X \backslash A \supseteq X \backslash C, X \backslash C \text{ open}} (X \backslash C) = \bigcup_{U \text{ open}, U \subseteq X \backslash A} U = (X \backslash A)^\circ.$$

□

Proposition 1.24:

Let (X, T_X) be a topological space, let $A, B \subseteq X$ be subsets, we have

(i) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$ and $\overline{A} \subseteq \overline{B}$.

(ii) $\overline{(\overline{A})} = \overline{A}$, $(A^\circ)^\circ = A^\circ$.

(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(iv) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proof: Elementary.

Remark:

(i) Consider $(0,1), (1,2) \subseteq \mathbb{R}$ and $(0,1) \cap (1,2) = \varnothing$ hence $\overline{[(0,1) \cap (1,2)]} = \varnothing$. But $\overline{(0,1)} \cap \overline{(1,2)} = \{1\}$. In general, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, equality may not happen.

(ii) Since $\mathbb{Q} \subseteq \mathbb{R}$ and $\overline{\mathbb{Q}} \subseteq \mathbb{R}$, $(\overline{\mathbb{Q}})^\circ = \mathbb{R}^\circ = \mathbb{R}$. $\mathbb{Q}^\circ = \varnothing$ and $\overline{\mathbb{Q}^\circ} = \varnothing$. So $\overline{A^\circ} \subseteq (\overline{A})^\circ$ and the inclusion may be strict. ||

Definition: Boundary

Let X be a topological space and $A \subseteq X$ is a subset. The boundary (or frontier) of A is the set $\partial A := \overline{A} \cap \overline{X \backslash A}$.

Example 1.13: Boundary

(i) Consider $X = \mathbb{R}$ and $A = (0,1)$. Then
$$\partial A = \overline{(0,1)} \cap \overline{\mathbb{R} \backslash (0,1)} = [0,1] \cap (\mathbb{R} \backslash (0,1)) = \{0,1\}.$$

(ii) Consider again $X = \mathbb{R}$ but $A = \mathbb{Q}$. Then $\partial \mathbb{Q} = \overline{\mathbb{Q}} \cap \overline{\mathbb{R} \backslash \mathbb{Q}} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

Proposition 1.25:

Let X be a topological space and let $A \subseteq X$ be a subset. Then

(i) $\overline{A} = A \cup \partial A$.

(ii) $A^\circ = A \backslash \partial A$.

(iii) $X = A^\circ \cup \partial A \cup (X \backslash A)^\circ$.

Proof:

(i):

$A \cup \partial A = A \cup (\overline{A} \cap \overline{(X \backslash A)}) = (A \cup \overline{A}) \cap (A \cup \overline{(X \backslash A)}) = \overline{A} \cap X = \overline{A}$.

(ii):

$$A \backslash \partial A = A \backslash (\overline{A} \cap \overline{(X \backslash A)}) = (A \backslash \overline{A}) \cup (A \backslash \overline{(X \backslash A)}) = \emptyset \cup (A \cap (X \backslash \overline{(X \backslash A)}))$$
$$= \emptyset \cup (A \cap A^\circ) = A^\circ \text{ since } X \backslash \overline{B} = (X \backslash B)^\circ.$$

(iii):

By **Proposition 1.23** we have $(X \backslash A)^\circ = X \backslash \overline{A}$. Since

$$X = \overline{A} \cup (X \backslash \overline{A}) = \overline{A} \cup (X \backslash A)^\circ.$$

By (i), $$= (A \cup \partial A) \cup (X \backslash A)^\circ = (A \backslash \partial A) \cup \partial A \cup (X \backslash A)^\circ$$
$$= A^\circ \cup \partial A \cup (X \backslash A)^\circ.$$

□

In topology, a neighborhood basis (or local basis) at a point in a topological space provides a systematic way to describe the open sets around that point. It is a collection of open sets that serve as building blocks for the neighborhoods of the point. A neighborhood basis is crucial for understanding the local structure of a topological space, analyzing convergence, and defining continuity.

Definition: Neighbourhood Basis

A neighborhood basis $\mathscr{B}_x$ of a point $x \in X$ where X is a topological space is a collection of neighborhood of x so that for all neighbourhood W, there exists $B \in \mathscr{B}_x$ such that $x \in B \subseteq W$.

Example 1.14: Neighbourhood Basis

Let $X = \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, set $\mathscr{B}_x := \{B_r(x) \,|\, r \in \mathbb{Q}^+\}$, wher $B_r(x) := \{y \in X \,|\, d(x, y) < r\}$ is an open ball. We claim that $\mathscr{B}_x$ is a neighbourhood basis of X, If N is a neighbourhood of $x \in \mathbb{R}^n$, then there exists an $R > 0$ such that $B_R(x) \subseteq N$. Moreover, there exists an $r \in \mathbb{Q}^+$ such that $0 < r \leq R$. Then $x \in B_r(x) \subseteq B_R(x) \subseteq N$ as we desire. ||

Not all topological spaces have a countable neighborhood basis at every point. However, first-countable spaces are those where each point has a countable neighborhood basis. For example $\mathbb{R}^n$ is first countable according to the following definition:

Definition: First Countable

A topological space X is said to be first countable if every point $x \in X$ has a countable neighbourhood basis.

In a first-countable space, sequences and nets can be characterized more easily. Convergence of sequences can be described using neighborhoods from the local basis. However, this may not be valid in general topologies:

Proposition 1.26:

Let X be a first countable topological space. Consider a subset $A \subseteq X$ and a point $y \in \overline{A}$. Then there exists a sequence $\{x_n\}_n \subseteq A$ with $x_n \xrightarrow{n \to \infty} y$.

Proof:

Let $\{N_i\}_{i=0}^{\infty}$ be a countable neighbourhood basis of y. By replacing N_i (if necessary) by $N_1 \cap N_2 \cap \cdots \cap N_i$, we may assume that $N_0 \supseteq N_1 \supseteq \cdots$.. Since $y \in \overline{A}$ and N_i is a neighbourhood of y, then $N_i \cap A \neq \emptyset$. Pick $x_i \in N_i \cap A$, since $N_i \supseteq N_k \forall k \geq i$, let W be a neighbourhood of y, since $\{N_i\}$ is a neighbourhood basis, there exists $i \in \mathbb{N}$ such that $y \in N_i \subseteq W$, then for $k \geq i$ one has $x_k \in N_k \subseteq N_i \subseteq W \Rightarrow x_n \to y$.

□

Remark:

Note that this result does not apply to general topologies. For example, it fails when applying to box topology T_{box}. ||

Now let us introduce the convergence results of nets:

Proposition 1.27:

Let X be a topological space. Consider a subset $A \subseteq X$ and a point y. Then $y \in \overline{A} \Leftrightarrow \exists (x_\lambda)_{\lambda \in \Lambda}$ in A such that $x_\lambda \to y$.

Proof:

"$\Leftarrow$":

Suppose that there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in A such that $x_\lambda \to y$. Then for any neighbourhood W of y, by definition, $W \cap \{x_\lambda \mid \lambda \in \Lambda\} \neq \emptyset \Rightarrow W \cap A \neq \emptyset$ since $\forall \lambda \in \Lambda$, $x_\lambda \in A$ hence by **Proposition 1.21** $y \in \overline{A}$.

"$\Rightarrow$":

Left as exercise (Hint: apply **Proposition 1.21**).

□

Nets provide a way to define continuity for functions between topological spaces, even in cases where sequences might not suffice.

Proposition 1.28:

Let $f : X \to Y$ be a function between topological spaces X and Y. Then f is continuous if and only if for all net $(x_\lambda)_{\lambda \in \Lambda}$ in X with $x_\lambda \to w$ then $f(x_\lambda) \to f(w)$.

Proof:

"$\Rightarrow$":

Suppose f is continuous, $(x_\lambda)_{\lambda \in \Lambda}$ is a net, $x_\lambda \to w$. Let U be a neighbourhood of $f(w)$ in Y, since f is continuous, $f^{-1}(U)$ is a neighbourhood of w since $x_\lambda \to w$. There exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in f^{-1}(U)$ $\forall \lambda$ with $\lambda_0 < \lambda \Rightarrow f(x_\lambda) \in U$ $\forall \lambda$ with $\lambda_0 < \lambda$. Hence $f(x_\lambda) \to f(w)$.

"$\Leftarrow$":

Left as exercise.

□

Proposition 1.29:

A topological space X is Hausdorff $\Leftrightarrow$ limits of nets in X are unique.

Proof: Exercise.

Proposition 1.30:

A topological space X is Hausdorff $\Leftrightarrow$ the diagonal

$\Delta_x := \{(x, y) \in X \times X \mid x = y\}$ is closed in $X \times X$ where $X \times X$ has product topology.

Proof:

For $x, y \in X$ such that $x \neq y \Leftrightarrow (x, y) \notin \Delta_x$.

$\Delta_x \subseteq X \times X$ is closed $\Leftrightarrow X \times X \backslash \Delta_x$ is open $\Leftrightarrow \forall (x, y) \in (X \times X \backslash \Delta_x)$ exists open neighbourhood W of (x, y) such that $W \cap \Delta_x = \varnothing \Leftrightarrow$ exists $U \subseteq X, V \subseteq X$ open subsets such that $(x, y) \in U \times V$ and $U \times V \cap \Delta_x = \varnothing$ $\Leftrightarrow U \cap V = \varnothing \Leftrightarrow X$ is Hausdorff.

□

Recall in analysis we have the terminology "compact", for which we admit any convergent sequence has a convergent subsequence. It is natural to ask the properties for the "subnet" even we have not formally introduced the compactness.

Before that we need introduce another term called "cofinal". In topology, the term "cofinal" refers to a relationship between two directed sets. This concept is used primarily in the context of nets, which are generalized sequences that provide a way to study convergence and continuity in topological spaces. Cofinality captures the idea of one directed set being "larger" than another in a specific sense.

Definition: Cofinal

Given two directed sets A and B, where $f : A \to B$ is a function that preserves the order of the sets, we say that B is cofinal in A if $\forall a \in A\ \exists b \in B$ such that $f(b) \geq a$.

When dealing with nets, cofinal subsets of the directed set are used to construct subnets that capture specific convergence patterns of the original net. Moreover, in the context of compactness, a directed set is cofinal in another if the net indexed by the latter directed set is used to construct a convergent subnet of the net indexed by the former directed set.

Example 1.15: Cofinal

Consider $A = \mathbb{N}$ and $B = \{2n \mid n \in \mathbb{N}\}$. The function $f : A \to B$ defined by $f(n) = 2n$ makes B cofinal in A. Every natural number n has an associated even number $2n$ that is greater than or equal to n. ||

Definition: Subnet

Given a net $f : A \to X$ and a cofinal subset B of the directed set A, the composition $f \circ g : B \to X$ is called a subnet of f, where $g : B \to A$ is a function that preserves the order of B and is cofinal.

Definition: Convergence of Subnet

A subnet $g : B \to X$ of the net $f : A \to X$ is said to converge to a point $x \in X$ if for every neighborhood U of x there exists an index $b_0 \in B$ such that for all $b \geq b_0$ $g(b) \in U$.

Proposition 1.31:

Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net on a topological space X that converges to $w \in X$. Then any subset $(x_{\varphi(\mu)})_{\mu \in B}$ of (x_λ), $x_{\varphi(\mu)} \to w$.

Proof:

Let N be a neighbourhood of w, since $x_\lambda \to w$ then $\exists \lambda_0 \in \Lambda$ such that for

$\lambda_0 < \lambda, x_\lambda \in \mathbb{N}$. Since φ is cofinal, there exists μ_0 such that $\lambda_0 < \varphi(\mu_0)$. Then $\forall \mu \in B$ with $\mu_0 < \mu$ one has $\varphi(\mu_0) < \varphi(\mu)$; therefore, $\mu_0 < \mu$, and $\lambda_0 < \varphi(\mu_0) < \varphi(\mu) \Rightarrow \varphi(\mu) \in N$ by definition.

□

Comment:

A subnet retains the convergence properties of the original net. If the original net f converges to a point x then any subnet g also converges to x. Moreover, convergence of subnets provides a tool for characterizing the convergence of the original net without necessarily considering all the elements of the original directed set. Furthermore, even not mentioned here, compactness of a topological space can often be characterized by the convergence of certain types of subnets of a net.

1.9 Compactness and Convergence

Compactness is a fundamental concept in topology that captures the idea of "closeness" and "boundedness" in a topological space. A compact space is a space where every open cover has a finite subcover, meaning that it's possible to select a finite number of open sets from the cover that still cover the entire space.

Definition: Cover and Subcover

Let X be a topological space, a collection $\{U_\alpha\}_{\alpha\in A}$ is a cover of X if and only if $X = \bigcup_{\alpha\in A} U_\alpha$. $\{U_\alpha\}_{\alpha\in A}$ is an open cover if it is a cover and each U_α is open (we can define closed by the same way), a subcover of a cover $\{U_\alpha\}_{\alpha\in A}$ is a subcollection $\{U_\beta\}_{\beta\in B}$ for some $B \subseteq A$.

Example 1.16: Cover and Subcover

Let (X, d) be a metric space, then $\{B_r(x) \mid x \in X, r \geq 0\}$ is a cover of X and if we let $\{B_r(x) \mid x \in X, r \in \mathbb{Q}^+\}$, then this is a subcover. ||

Definition: Compact

A topological space X is compact if every open cover of X has a finite subcover. That is, given an open cover $\{U_\alpha\}_{\alpha\in A}$, $\exists k \in \mathbb{N}$ such that for $\alpha_1, \cdots, \alpha_k \in A$ one has $X = \bigcup_{i=1}^{k} U_i$, which is the subcover.

Remark:

(i) If X is compact, then every net in X has a convergent subnet.

(ii) $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ K is closed and bounded. This does not hold in general spaces.

(iii) Tychonoffs Theorem tells us that a product of compact spaces is compact. ||

Proposition 1.32:

Let X be a topological space and let $Y \subseteq X$ be a subspace.
Then Y is compact $\Leftrightarrow$ for all collection $\{U_\alpha\}_{\alpha\in A}$ of sets open in X, with $Y \subseteq \bigcup_{\alpha\in A} U_\alpha$ of sets open in X, with $Y \subseteq \bigcup_{\alpha\in A} U_\alpha$, there exists $k \in \mathbb{N}$ such that

for $\alpha_1, \cdots, \alpha_k$ one has $Y \subseteq \bigcup_{i=1}^{k} U_{\alpha_i}$.

Proof:

"⇒":

Suppose that Y is compact, then $\{V_\alpha := U_\alpha \cap Y\}_{\alpha \in A}$ is an open cover of Y. Since Y is compact, there exists $k \in \mathbb{N}$ such that for $\alpha_1, \cdots, \alpha_k$ one has

$Y = V_{\alpha_1} \cup \cdots \cup V_{\alpha_k} \Rightarrow Y \subseteq \bigcup_{i=1}^{k} U_{\alpha_i}$.

"⇐":

Let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of Y, by the definition of subspace topology, there exists $U_\alpha \subseteq X$ open such that $V_\alpha = Y \cap U_\alpha \, \forall \alpha \in A$

$\Rightarrow Y = \bigcup_{\alpha \in A} V_\alpha \subseteq \bigcup_{\alpha \in A} U_\alpha$. By definition, there exist $\alpha_1, \cdots, \alpha_n$ such that

$Y \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k} \Rightarrow Y = Y \cap \left(\bigcup_{i=1}^{k} U_{\alpha_i}\right) = (Y \cap U_{\alpha_1}) \cup \cdots \cup (Y \cap U_{\alpha_k})$. □

Lemma 1.33:

Images of compact spaces under continuous maps are compact.

Proof:

Let X be a compact topological space and consider $f : X \to Y$ being continuous for Y another topological space.

[Claim]: $f(X) \subseteq Y$ is compact.

This is to show that given any collection $\{U_\alpha\}_{\alpha \in A}$ of sets open in Y with $f(X) \subseteq \bigcup_{\alpha \in A} U_\alpha$, there exist $U_{\alpha_1}, \cdots, U_{\alpha_k}$ such that $f(X) \subseteq \bigcup_{i=1}^{k} U_{\alpha_i}$. Since f is continuous, U_α is open and $f(X) \subseteq \bigcup_{\alpha \in A} U_\alpha$, $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is a collection of open sets and $X = \bigcup_{\alpha \in A} f^{-1}(U_\alpha)$. Since X is compact there exist $\alpha_1, \cdots, \alpha_k$ such that $X = f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_k})$. Therefore, $f(X) \subseteq \bigcup_{i=1}^{k} f^{-1}(U_{\alpha_i})$. According to **Proposition 1.32**, $f(X)$ is compact.

□

Lemma 1.34:

A closed subset of a compact space is compact.

Proof:

Suppose that X is compact, take a closed subset $K \subseteq X$. $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets in X, such that $K \subseteq \bigcup_{\alpha \in A} U_\alpha$. Then $\{U_\alpha\}_{\alpha \in A} \cup \{X \backslash K\}$ is an open cover of $X \Rightarrow \exists k \in \mathbb{N}$ such that for $U_{\alpha_1}, \cdots, U_{\alpha_k}$ one has

$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_k} \cup (X \backslash K) \Rightarrow K \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_k}$, thus K is compact by **Proposition 1.32**.

□

Lemma 1.35:

A compact subset of a Hausdorff space is closed.

Proof:

Let X be Hausdorff and let $K \subseteq X$ be a compact subset.

[Claim]: $\forall x \in X \backslash K$, there exists an open set U such that $x \in U$.

$U \subseteq X \backslash K$, i.e. $U \cap K = \varnothing$. Since X is Hausdorff, then $\forall k \in K$, there exist open neighbourhoods U_k of x , V_k of k, such that $U_k \cap V_k = \varnothing$. Since $\{V_k\}_{k \in K}$ is a collection of sets open in X with $K \subseteq \bigcup_{k \in K} V_k$. Since K is compact, $\exists n \in \mathbb{N}$ and $V_{k_1}, \cdots, V_{k_n}$ such that $K \subseteq V_{k_1} \cup \cdots \cup V_{k_n}$. Let now $U = U_{k_1} \cap \cdots \cap U_{k_n}$ and $U \cap \left[V_{k_1} \cap \cdots \cap V_{k_n}\right] \subseteq U_k$.

[Claim]: $U \cap \left[V_{k_1} \cup \cdots \cup V_{k_n}\right] = \varnothing$.

$U \cap V_i \subseteq U_{k_i} \cap V_{k_i} = \varnothing \forall i \in \mathbb{N} \cap [1,n]$ and result follows.

□

Lemma 1.36:

Let X be a compact topological space and Y be a Hausdorff topological space. Let $f : X \to Y$ be a continuous map. If f is a bijection, then it is a homeomorphism.

Proof:

[Claim]: $g := f^{-1} : Y \to X$ being continuous.

For all $C \subseteq X$ closed subsets, $g^{-1}(C)$ is closed in Y. But $g^{-1}(C) = f(C)$ since X is compact and C is closed hence C is compact by **Lemma 1.34**. Since f is continuous, $f(C)$ is also compact by **Lemma 1.33**. Because Y is Hausdorff, then by **Lemma 1.35** $f(C)$ is closed.

□

The Bolzano-Weierstrass theorem is a fundamental result in real analysis that guarantees the existence of convergent subsequences in bounded sequences. It is named after mathematicians Bernard Bolzano and Karl Weierstrass. The theorem plays a crucial role in understanding the behavior of sequences and is a key building block in the study of limits, continuity, and compactness.

Theorem 1.37: Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

Corollary 1.37.1:

[0,1] is compact.

Proof:

Suppose not, then there exists a collection of open subsets of $\mathbb{R}$, namely, $\{U_\alpha\}_{\alpha \in A}$, so that $[0,1] = \bigcup_{\alpha \in A} U_\alpha$ but no finite collection of U_α covers [0,1].

Then there is no $n \in \mathbb{N}$ such that for $\alpha_1, \cdots, \alpha_n \in A$ one has

$[0,1] \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ which means neither $[0,\frac{1}{2}]$ nor $[\frac{1}{2},1]$ has a finite cover constructed by U_α. Name the interval that cannot be covered by finitely many U_α, $[a_1, b_1]$, now divide $[a_1, b_1]$ into two that each of them cannot be covered by $[a_2, b_2]$, $\cdots$. Then $[0,1] \supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$. Consider $|b_n - a_n| = \frac{1}{2^n}$, $0 \le a_1 \le a_2 \le \cdots \le a_n \le \cdots \le b_n \le \cdots \le b_1$ implies that $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} b_n$ exists. Since $|b_n - a_n| \to 0$ as $n \to \infty$, one has $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = C \in \mathbb{R}$. Thus $0 \le C \le 1$, i.e. $C \in [0,1] \Rightarrow \exists \beta \in A$ such that $C \in U_\beta$. Since U_β is open there exists $\varepsilon > 0$ such that $(C - \varepsilon, C + \varepsilon) \subseteq U_\beta$ and since both a_n, b_n converges to C we conclude that there exists an N such that $(a_N, b_N) \subseteq (C - \varepsilon, C + \varepsilon) \subseteq U_\beta \Rightarrow [a_N, b_N] \subseteq U_\beta$, then this is a finite cover, hence a contradiction, so [0,1] is compact as we desired.

□

We have introduced the product topology before. Now let us build some connections between the compactness and the product. The Tube Lemma is an important result in topology, particularly in the study of topological vector spaces. It provides a powerful tool for establishing the existence of neighborhoods around subsets of a topological vector space that are contained within specified open sets.

Lemma 1.38: Tube Lemma

Let X and Y be two topological spaces with Y compact. Let $x \in X$ be a point and $U \subseteq X \times Y$ is an open subset. Consider $\{x_0\} \times Y \subseteq U$, then there exists an open neighbourhood V of x_0 in X such that $V \times Y \subseteq U$.

Proof:

$\forall y \in Y\ \exists$ an open neighbourhood V_y of x_0, W_y of y such that $V_y \times W_y \subseteq U$. Consider $\{W_y\}_{y\in Y}$ an open cover of Y since Y is compact. Then there exists an $n \in \mathbb{N}$ such that for $y_1, \cdots, y_n \in Y$ we have $Y = W_{y_1} \cup \cdots \cup W_{y_n}$. Let now $V = V_{y_1} \cup \cdots \cup V_{y_n}$. Then

$$V \times Y \subseteq V \times \bigcup_{i=1}^{n} W_{y_i} \subseteq (V_{y_1} \times W_{y_1}) \cup \cdots \cup (V_{y_n} \times W_{y_n}) \subseteq U.$$

□

The Tube Lemma is widely used in functional analysis, where it helps establish continuity properties of linear mappings and analyze the behavior of functions defined on topological vector spaces. It is also crucial in proving various results related to topological vector spaces, such as the properties of normed spaces and Banach spaces.

Corollary 1.38.1:

A product of two compact spaces is compact.

Proof:

Let X, Y be two compact spaces. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $X \times Y$ so $\forall x \in X$, $\{x\} \times Y \subseteq X \times Y$ is compact and the map defined by $Y \to X \times Y$

sending y to (x, y) is continuous. Therefore $\forall x \in X \, \exists n \in \mathbb{N}$, where $n = n(x)$ (n depends on x), such that $\alpha_1(x)\cdots, \alpha_n(x) \in A$ and $\{x\} \times Y \subseteq U_{\alpha_1(x)} \cup \cdots \cup U_{\alpha_n(x)}$. Then by **Tube Lemma**, there exists an open neighbourhood V_x of x such that $V_x \times Y \subseteq U_{\alpha_1(x)} \cup \cdots \cup U_{\alpha_n(x)}$. $\{V_x\}_{x\in X}$ is an open cover of X, $\exists x_1, \cdots, x_k \in X$ such that $X = \bigcup_{n=1}^{k} V_{x_n}$. Then $\forall i$, let $n_i = n(x_i)$, we have $V_{x_i} \times Y \subseteq \bigcup_{j=1}^{n_i} U_{\alpha_j(x_i)} \Rightarrow X \times Y = \bigcup_{i=1}^{k} V_{x_i} \times Y \subseteq \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} U_{\alpha_j(x_i)}$.

□

Corollary 1.38.2:

If $X_1, \cdots, X_n$ are compact spaces, then $X_1 \times X_2 \times \cdots \times X_n$ is also compact. In particular, $[0,1]^n$ is compact.

Proof: Follows from induction. □

Remak:

If $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}$ are such that $a_i < b_i$. Then $[a_1, b_1] \times \cdots [a_n, b_n]$ is compact because $F : [0,1]^n \to \prod_{i=1}^{n} [a_i, b_i]$ and

$$F(t_1, \cdots, t_n) = \big((a_1 + t_1(b_1 - a_1)), (a_2 + t_2(b_2 - a_2)), \cdots\big),$$

which is a continuous surjection. In fact it is a bijection, therefore it is homeomorphism. ||

Compactness in topology is also a fundamental concept that captures the essence of boundedness and finiteness in a topological space. Recall that a subset $X \subseteq \mathbb{R}^n$ is said to be bounded if $\exists R > 0$ such that $X \subseteq B_R(0)$. Note that

$$B_R(0) \subseteq (-R, R)^n \subseteq [-R, R]^n.$$

Definition: Bounded

Let X be a topological space and let $A \subseteq X$ be a subset. Then A is said to be bounded in X if there exists an open set U such that $A \subseteq U$.

This definition captures the idea that a bounded subset can be completely contained within some open set of the topological space. However, this definition is more abstract compared to the concrete notion of boundedness in metric spaces, where distances between points are involved. Keep in mind that the definition might vary depending on the specific context or type of topological space you are dealing with.

Now let us build up the connection between compactness and boundedness as we promised.

Theorem 1.39:

A subset $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ K is closed and bounded.

Proof:

"$\Rightarrow$":

Since $\mathbb{R}^n$ is Hausdorff, then any compact subset is closed. Consider $U_i = B_i(0)$, for $i = 1,2,3,\cdots$, then $\bigcup_{i=1}^{\infty} U_i = \mathbb{R}^n \Rightarrow \exists i_1, \cdots, i_k$ such that

$K \subseteq U_{i_1} \cup \cdots \cup U_{i_k}$. Let now $m := \max\{i_1, \cdots, i_k\}$. Then $K \subseteq U_m = B_m(0) \Rightarrow K$ is bounded.

"$\Leftarrow$":

Suppose that K is closed and bounded, since K is bounded then $\exists R > 0$ such that $K \subseteq [-R, R]^n \Rightarrow K$ is compact.

$\square$

Remark:

In general, compact subsets need not to be closed. ||

Corollary 1.39.1:

A compact subspace of a metric space is bounded.

Proof: Exercise.

Lemma 1.40:

Let X be a compact topological space and let $f : X \to \mathbb{R}$ be a continuous function. Then there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b) \forall x \in X$. (Such a, b might not be unique.)

Proof:

Since f is continuous, $f(X) = \mathbb{R}$ is compact, hence closed and bounded. Since it is bounded, $\inf\limits_{x\in X} f(x), \sup\limits_{x\in X} f(x)$ both exist. Since $f(X)$ is closed, there is an $a \in X$ such that $f(a) = \inf\limits_{x\in X} f(x)$, similarly, $f(b) = \sup\limits_{x\in X} f(x)$.

$\square$

Lemma 1.41:

X is compact $\Leftrightarrow \forall \{C_\alpha\}_{\alpha\in A}$ of closed subsets with $C_{\alpha_1} \cap \cdots \cap C_{\alpha_k} \neq \emptyset$ $\forall k$ such that $\alpha_1, \cdots, \alpha_k \in A$. Then $\bigcap\limits_{\alpha\in A} C_\alpha \neq \emptyset$.

Proof:

X is compact $\Leftrightarrow$ Any oepn cover $\{U_\alpha\}_{\alpha\in A}$ has finite subcover.

$\Leftrightarrow$ Any collection $\{U_\alpha\}_{\alpha\in A}$ of open sets with no finite subcover is not a cover of X.

$\Leftrightarrow$ Any collection $\{U_|alpha\}_{\alpha\in A}$ of open sets, $\forall n$, $\forall \alpha_1, \cdots, \alpha_n \in A, X \neq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$.

$\Rightarrow X \neq \bigcup\limits_{\alpha\in A} U_\alpha$.

$\Leftrightarrow$ For all collection $\{U_\alpha\}_{\alpha\in A}$ of open sets and $\forall n$ $\forall \alpha_1, \cdots, \alpha_n \in A, X \backslash (U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}) \neq \emptyset$.

$\Rightarrow X \backslash \bigcup\limits_{\alpha\in A} U_\alpha \neq \emptyset$, i.e. $(X \backslash U_{\alpha_1}) \cap \cdots \cap (X \backslash U_{\alpha_n}) \neq \emptyset$

$\Leftrightarrow \bigcup\limits_{\alpha\in A} (X \backslash U_\alpha) \neq \emptyset$.

$\Leftrightarrow$ For all collection $\{C_\alpha\}_{\alpha\in A}$ of closed sets $\forall n$, $\forall \alpha_1, \cdots, \alpha_n \in A$, $C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} \neq \emptyset$.

$$\Rightarrow \bigcap_{\alpha\in A} C_\alpha \neq \emptyset.$$

□

The finite intersection property is a fundamental concept in topology that serves as a criterion for compactness and plays a crucial role in characterizing certain properties of topological spaces. It provides a way to ensure the existence of common points among finitely many open sets in a topological space.

The motivation behind the finite intersection property lies in capturing the compactness and convergence properties of subsets within a topological space. It helps us understand how open sets can "overlap" in a way that guarantees the existence of points shared by finitely many sets. This concept becomes particularly powerful when considering compactness and sequential compactness, as well as when proving certain limit and convergence properties.

Definition: Finite Intersection Property (F.I.P.)

A collection of subsets $\{C_\alpha\}_{\alpha\in A}$ of a set X has finite intersection property if for all finite subset $\{\alpha_1, \cdots \alpha_n\} \subseteq A$, $C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} \neq \emptyset$.

Lemma 1.43:

X is compact $\Leftrightarrow$ For any collection $\{C_\alpha\}_{\alpha\in A}$ of closed subsets of X with finite intersection property, $\bigcap_{\alpha\in A} C_\alpha \neq \emptyset$.

Definition: Limit Point

Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in a space topological X, a point $p \in X$ is said to be a limit point (or cluster point, accumulation point) of this net if for all neighbourhood W of p and $\forall \lambda_0 \in \Lambda$ there exists $\lambda \in \Lambda$ such that $\lambda_0 < \lambda$ and $x_\lambda \in W$.

Example 1.17:

Let $X = \mathbb{R}$ and consider $X_n := (-1)^n$ for $n \in \mathbb{N}$. Then one can check that $p = 1$ and $p = -1$ are limit points in this case. ||

Proposition 1.44:

A point $p \in X$ be a limit point of a net $(x_\lambda)_{\lambda\in\Lambda}$ $\Leftrightarrow$ there exists a subnet $(x_{\lambda_\mu})_{\mu\in M}$ that converges to p.

Proof:

"$\Rightarrow$":

Suppose p is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$, let M be the pairs defined by

$$M := \{(\lambda, W) \mid \lambda \in \Lambda, W \text{ is neighbourhood of } p \text{ with } x_\lambda \in W\}.$$

Order M by $(\lambda, w) < (\lambda', w') \Leftrightarrow (\lambda < \lambda')$ and $(W \supseteq W')$. Define now a map $\varphi : M \to \Lambda$ by $\varphi(\lambda, W) = \lambda$. φ is order preserving.

[Claim]: φ is cofinal.

Since p is a limit point of $(x_\lambda)_{\lambda\in\Lambda}$, then $\forall \lambda_0 \in \Lambda$ and for all neighbourhood W of p, there exists $\lambda \in \Lambda$ such that $\lambda_0 < \lambda$ and $x_\lambda \in W$, i.e. $(\lambda, W) \in M$, and $\lambda_0 < \lambda = \varphi(\lambda, W)$ hence φ is cofinal. Moreover, $(x_{\varphi(\lambda,W)})_{(\lambda,W)\in M}$ is a subnet of $(x_\lambda)_{\lambda\in\Lambda}$.

Since p is a limit point of $(x_\lambda)_{\lambda\in\Lambda}$, for all neighbourhood W of p, $\exists\lambda_1 \in \Lambda$ with $x_{\lambda_1} \in W$, for any $\mu = (\lambda, U) \in M$ with $(\lambda_1, W) < (\lambda, U)$ since $\lambda_1 < \lambda$ and $x_\lambda \in U \subseteq W$. Hence $\forall\mu \in M$ with $(\lambda_1, W) < \mu$, $x_{\lambda_\mu} \in W \Rightarrow x_{\lambda_\mu} \to p$.

"$\Leftarrow$":

Suppose that $(x_{\lambda_\mu})_{\mu\in M}$ is a subnet of $(x_\lambda)_{\lambda\in\Lambda}$ and $x_{\lambda_\mu} \to p$.

[Claim]: p is the limit point of $(x_\lambda)_{\lambda\in\Lambda}$.

Let W be a neighbourhood of p, $\lambda_0 < \lambda$, we need to show that there exists a $\lambda \in \Lambda$ such that $\lambda_1 < \lambda$ and $x_\lambda \in W$. Since $x_{\lambda\mu} \to p$ there exists $\mu_0 \in M$ such that if $\mu_0 < \mu$ then $\lambda_{\lambda_\mu} \in W$. Since $(x_{\lambda_\mu})_{\mu\in M}$ is a subnet of $(x_\lambda)_{\lambda\in\Lambda}$ then $\exists\mu_1 \in W$ such that $\lambda_0 < \lambda_{mu_1}$, since λ is directed, $\exists\lambda_2 \in M$ such that $\mu_0 < \mu_2$ and $\mu_1 < \mu_2$. Then $\lambda_0 \leq \lambda_{\mu_1} \leq \lambda_{\mu_2}$ and $x_{\lambda_{\mu_2}} \in W$ since $\mu_1 < \mu_2$.

□

Proposition 1.45:

A topological space X is compact $\Leftrightarrow$ Any net $(x_\lambda)_{\lambda\in\Lambda}$ in X has a cluster point, i.e. each of them has a convergent subnet $\Leftrightarrow$ Every sequence has a convergent subsequence.

Notation:

Given a net $(x_\lambda)_{\lambda\in\Lambda}$ and $\lambda_0 \in \Lambda$, the λ_0-tail of the net $T_{\lambda_0} := \{x_\lambda \mid \lambda_0 < \lambda\}$.

In order to prove **Proposition 1.45**, we now make an observation: For any net $(x_\lambda)_{\lambda\in\Lambda}$, the set $\{T_\lambda\}_{\lambda\in\Lambda}$ of tails has Finite Intersection Property (F.I.P.). This is because $\forall\lambda_1, \cdots, \lambda_k \in \Lambda \quad \exists\mu \in \Lambda$ such that $\lambda_i < \lambda_\mu \forall i \quad \Rightarrow T_{\lambda_i} \supseteq T_{\lambda_\mu} \quad \forall i \quad \Rightarrow \bigcap_{i=1}^{\mu} T_{\lambda_i} \supseteq T_\mu \neq \varnothing$. We now proceed to the proof.

Proof of Proposition 1.45:

"$\Rightarrow$":

Suppose that $(x_\lambda)_{\lambda\in\Lambda}$ is a net in a compact space X. Consider $\{\overline{T_\lambda}\}_{\lambda\in\Lambda}$ with F.I.P. Since X is compact then $\bigcap_{\lambda\in\Lambda} \overline{T_\lambda} \neq \varnothing$, pick $p \in \bigcap_{\lambda\in\Lambda} \overline{T_\lambda}$, for any neighbourhood W of p, $\forall\lambda \in \Lambda$, $W \cap \overline{T_\lambda} \neq \varnothing$ hence $\exists\lambda \in \Lambda$ such that $\lambda < \lambda'$ and $x_{\lambda'} \in W$, i.e. p is a limit point of $(x_\lambda)_{\lambda\in\Lambda}$.

"$\Leftarrow$":

Suppose that every net in X has a cluster point, let $\mathscr{C}$ be the collection of closed subsets of X with F.I.P., we need to show that the intersection $\bigcap_{C\in\mathscr{C}} C \neq \varnothing$.

Consider that $\mathscr{G}$ is the set of finite intersections of elements of $\mathscr{C}$, i.e.

$$\mathscr{G} := \{C_1 \cap \cdots \cap C_n \mid C_1, \cdots, C_n \in \mathscr{C}, n \in \mathbb{N}\}.$$

Since $\mathscr{C}$ has F.I.P. so does $\mathscr{G}$. Direct $\mathscr{G}$ by the reverse inclusion, i.e.

$$G_1 < G_2 \Leftrightarrow G_1 \subseteq G_2.$$

$\mathscr{G}$ is now a directed set, $\forall G_1, G_2 \in \mathscr{G}$, $G_1 < G_1 \cap G_2$ and $G_2 < G_1 \cap G_2$.

$\forall G \in \mathscr{G}$ non-empty, choose $x_G \in G$, we get a net $(x_G)_{G\in\mathscr{G}}$, this net has a limit point called p. Then for any neighbourhood W of p, $\forall G \in \mathscr{G}$, $\exists G' \in \mathscr{G}$ with $G < G'$, i.e. $G \supseteq G'$, and $x_{G'} \in W$. $G \cap W \supseteq G' \cap W \ni x_{G'}$. Therefore, $G \cap W \neq \varnothing \Rightarrow p \in \overline{G} = G \Rightarrow p \in \bigcap_{G\in\mathscr{G}} G \subseteq \bigcap_{C\in\mathscr{C}} C \Rightarrow \bigcap_{C\in\mathscr{C}} C \neq \varnothing$.

□

Before we proved that the finite product of compact topological spaces is still compact, now we extend this to arbitrary product, which is the result of Tychonoff theorem, also known as the Tychonoff's product theorem, is a fundamental result in topology that characterizes the compactness of the Cartesian product of an arbitrary collection of topological spaces.

Theorem 1.46: Tychonoff's Theorem

For any collection of compact topological spaces $\{X_\alpha\}_{\alpha\in A}$ the product $\prod_{\alpha\in A} X_\alpha$ is also compact.

Remark:

The surprising fact is that this theorem is related, actually, equivalent to the axiom of choice. This equivalence is known as the Tychonoff's Theorem-Axiom of Choice Equivalence.Tychonoff's Theorem is equivalent to the axiom of choice. ||

Axiom of Choice:

Let $\mathscr{F} := \{U_\alpha\}_{\alpha\in A}$ be a collection of nonempty sets indexed by a nonempty index set A. Then there is a function $f : A \to \bigcup_{\alpha\in A} U_\alpha$ such that $\forall \alpha \in A$ one has $f(\alpha) \in U_\alpha$.

The essence of the axiom is the assertion that one can choose exactly one element from each non-empty set in a collection of sets. The function f is called a "choice function" or "selector." The Axiom of Choice has various equivalent formulations and versions, such as Zorn's Lemma and the Well-Ordering Principle. We now state another equivalent form of it, but we need some terminologies.

Definition: Comparable

Two elements $p, q \in (X, \leq)$, where X is a set and $\leq$ is a partial order (we sometimes call $(X, \leq)$ a poset) are called comparable if either $p \leq q$ or $q \leq p$. If they are not comparable then they are called incomparable.

Definition: Chain

A subset $A \subseteq X$ where X is a poset with partial order $\leq$ is called a chain if every pair of elements of A are comparable.

So in the poset $(\mathbb{R}, \leq)$, the whole partial order is a chain. In $\mathbb{N}$ with the divisibility relation, the set $\{7, 7^2, 7^3, \cdots\}$ is a chain, but the set of all odd numbers, for example, is not.

Zorn's Lemma is a fundamental result in set theory that provides a tool for establishing the existence of maximal elements in partially ordered sets (posets).

Lemma 1.47: Zorn's Lemma

Every non-empty partially ordered set (poset) in which every chain (totally ordered subset) has an upper bound contains at least one maximal element.

Now we proceed to the proof of Tychonoff's Theorem:

Proof of Theorem 1.46:

Suppose that $\mathscr{C} := \{C_i\}_{i\in I}$ is a collection of closed subsets $X = \prod_{\alpha\in A} X_\alpha$ with Finite Intersection Property.

[Claim]: $\bigcap_{i\in I} C_i \neq \varnothing$.

Let $\mathscr{F} := \left\{ B \subseteq \mathscr{P}(X) \middle| B \supseteq C \,\forall C \in \mathscr{C} \text{ and } B \text{ F.I.P.} \right\}$. Suppose that $\{B_j\}_{j\in J}$ is a chain in $\mathscr{F}$, we argue that $\bigcup_{j\in J} B_j \in \mathscr{F}$.

[Claim]: $\bigcup_{j\in J} B_j \in \mathscr{F}$.

(i) Since $C \subseteq B_j \,\forall j \in J \Rightarrow C \subseteq \bigcup_{j\in J} B_j \;\forall C \in \mathscr{C}$.

(ii) Each B_j has F.I.P., If $\{B_1, B_2, \cdots, B_k\} \subseteq \bigcup_{j\in J} B_j$, since $\{B_j\}_{j\in J}$ is a chain, then there exists $s \in J$ such that $B_{j1}, \cdots, B_{jk} \subseteq B_s$. Moreover, since B_s has F.I.P. $B_1 \cap \cdots \cap B_k \in B_s$ while $B_1 \cap \cdots \cap B_k \subseteq \bigcup_{j\in J} B_J \;\Rightarrow \bigcup_{j\in J} B_j$ has F.I.P. result follows.

Now we may apply **Zorn's Lemma**, since by our construction, $\mathscr{F}$ has a maximal element, namely, D. Then consider the projection:

$$\Pi_\beta : \prod_{\alpha\in A} X_\alpha \to X_\beta.$$

Since D has F.I.P. $\Pi_\beta(D)$ has F.I.P. where $\Pi_\beta \subseteq \mathscr{P}(X_\beta)\;\forall\beta$. Now we denote $\left\{\overline{\Pi_\beta(B)} \middle| B \in D\right\}$ to be a collection of closed sets with F.I.P., where, trivially,

$$\Pi_\beta =: \left\{\Pi_\beta(B) \middle| B \in D\right\}.$$

Since X_β is compact $\Rightarrow \bigcap_{B\in D} \overline{\Pi_\beta(B)} \neq \varnothing$. Choose now $b_\alpha \in \bigcap_{B\in D} \overline{\Pi_\alpha(B)}$ for arbitrarily chosen α. One obtains that $b = (b_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} X_\alpha = X$.

[Claim]: $b \in \bigcap_{i\in I} C_i$.

For each α, consider a neighbourhood N_α of b_α in X_α. Since $b_\alpha \in \overline{\Pi_\alpha(B)}$ $\forall B \in D$. Then $N_\alpha \cap \Pi_\alpha(B) \neq \varnothing \,\forall\alpha\,\forall\beta \Rightarrow D \cup \{\Pi_{\alpha^{-1}}(N_\alpha)\}_{\alpha\in A}$ has F.I.P. therefore $D \cup \{\Pi_\alpha^{-1}(N_\alpha)\}_{\alpha\in A} = D$ by maximality of D. Since $C_i \in D\,\forall i$ we obtain $C_i \cap \Pi_{\alpha^{-1}}(N_\alpha) \neq \varnothing \,\forall i\;\forall\alpha$. Let U be a neighbourhood of b in $\prod_{\alpha\in A} X_\alpha$, since $\left\{\Pi_{\alpha^{-1}}(U_\alpha) \middle| \alpha \in A, U_\alpha \subseteq X_\alpha \text{ open}\right\}$ is a subbasis of the

T_{prod} on X. There exists $k \in \mathbb{N}$ such that for $\alpha_1, \cdots, \alpha_k \in A$ we have $b \in \Pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \Pi_{\alpha_k}^{-1}(U_{\alpha_k}) \subseteq U \Rightarrow C_i \cap U \neq \varnothing \ \forall i \ \Rightarrow b \in \overline{C_i} \forall i$. But $\overline{C_i} = C_i$ since by our choice each C_i is being closed. Therefore $b \in C_i \forall i \ \Rightarrow b \in \bigcap_{i \in I} C_i \neq \varnothing$

Together with two claims we finally conclude that $(\Pi_{X_\alpha}, T_{\text{prod}})$ is compact as we desired.

□

This is a rather normal proof of Tychonoff Theorem using **Zorn's Lemma**. In fact, there are other approach to obtain the same result; in [10], Étienne Matheron offered two other approaches, for one of them using the categorical tools. Readers who are interested in this aspect may consult his article.

We shall close our first chapter by the discussion of convergent sequences as well as the completion of Cauchy seqeunces. First recall the definition we saw in a basic analysis course:

Definition: Cauchy sequence

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a metric space (X, d) is said to be Cauchy if it satisfies $\forall \varepsilon > 0 \, \exists N \in \mathbb{N}$ such that if $n, m \geq N$, then $d(x_n, x_m) < \varepsilon$.

Definition: Complete Metric Space

A metric space (X, d) is said to be complete if every Cauchy sequence has a limit inside it.

Example 1.18: Complete Metric Space

(i) $\mathbb{R}$ with the standard metric $d(x, y) := |x - y|$ is complete.

(ii) $(-\frac{\pi}{2}, \frac{\pi}{2})$ is homeomorphic to $\mathbb{R}$. For example, $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is homeomorphic. But $(-\frac{\pi}{2}, \frac{\pi}{2})$ is not complete.

Not that the second sub-example above states a very important feature of completeness: Homeomorphisms do not always preserve the completeness of metric spaces. A homeomorphism, by definition, preserves topological properties, including convergence properties. However, the preservation of convergence properties, such as the convergence of sequences or nets, does not necessarily imply the preservation of completeness, as completeness is a specific property related to metric spaces.

It is natural to ask, since completeness is excluded from the structure, what features or properties, in or not in the structure, the homemomorphism preserve? It is trivial to see that open sets are indeed preserved, as well as continuous functions. One anti-intuition fact is that homeomorphism does also preserve the convergence, as we specified above, this is because the convergence is a part of the structure of topologies while the completeness is a part of the structure of the metric spaces. Moreover, the topological properties such as compactness, connectedness, and separability, are also preserved by homeomoprhisms. Furthermore, as we shall see later on, the topological invariants and the topological-induced properties, are also preserved under homemomorphisms.

In topology, a subset of a metric space is said to be totally bounded if, intuitively, it can be "covered" by finitely many small balls (open balls) of a given radius. Totally bounded sets are an important concept in understanding the compactness and convergence properties of metric spaces.

Definition: Totally Bounded

A metric space (X, d) is said to be totally bounded if $\forall \varepsilon > 0\ \exists n \in \mathbb{N}$ such that for $x_1, \cdots, x_n \in X$ we have $X = \bigcup_{i=1}^{n} B_\varepsilon(x)$.

Theorem 1.47:

Let (X, d) be a metric space, then the followings are equivalent:

(i) (X, T_d) is compact.

(ii) Every sequence in X has a convergent subsequence.

(iii) X is complete and totally bounded.

Proof:

(i) $\Rightarrow$ (ii):

Suppose that (X, T_d) is compact and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence with no convergent subsequence. Then $\forall y \in X$ there exists a neighbourhood U_y of y such that $x_n \in U_y$ only for finitely many choice of n. Let $\{U_y\}_{y\in X}$ be a collection of open covers of X. Since X is compact, it has a finite subcover $\Rightarrow \{x_n\}_{n\in\mathbb{N}}$ has only finitely many terms, but this is impossible, hence contradiction.

(ii) $\Rightarrow$ (iii):

This part of the proof is divided into two parts: In the first part we prove the completeness and in the second part we prove the boundedness.

Step I: Completeness.

Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, by assumption, there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k} \to y$. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, then $x_n \to y$ as well by its definition.

Step II: Boundedness.

Suppose there exists $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls. Then $\begin{cases} \exists x_1 \text{ such that } X \backslash B_\varepsilon(x_1) \neq \varnothing \\ \exists x_2 \text{ such that } X \backslash B_\varepsilon(x_2) \neq \varnothing \end{cases}$ such that $X \backslash \left(B_\varepsilon(x_1) \cup B_\varepsilon(x_2)\right)$ is not empty. Therefore $\exists x_n \in X \backslash \bigcup_{i=1}^{n-1} B_\varepsilon(x_i)$ such that $X \backslash \bigcup_{i=1}^{n} B_\varepsilon(x_i) \neq \varnothing$. We get a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $d(x_n, x_i) \geq \varepsilon\, \forall i < n \Rightarrow \{x_n\}_{n\in\mathbb{N}}$ has no convergent subsequences, a contradiction.

(iii) $\Rightarrow$ (i):

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Suppose it has not finite subcover. Since by assumption X is totally bounded then it can be covered by finitely many balls with radius 1. Then $\exists x_0 \in X$ such that $B_1(x_0)$ cannot be covered by finitely many U_α's. There is a cover of X by finitely many balls with $r = \dfrac{1}{2}$.

Then there exists a ball $B_{\frac{1}{2}}(x_1)$ which cannot be covered by finitely many U_α's and $B_{\frac{1}{2}}(x_1) \cap B_1(x_0) \neq \emptyset$. We get a sequence of balls:

$$B_1(x_0), B_{\frac{1}{2}}(x_1), \cdots, B_{\frac{1}{2^n}}(x_n),$$

so that $B_{\frac{1}{2^n}}(x_n) \cap B_{\frac{1}{2^{n+1}}}(x_{n+1}) \neq \emptyset$ and no ball can be covered by finitely many U_α's. Moreover, $d(x_n, x_{n+1}) < \frac{1}{2^n} + \frac{1}{2^{n+1}} < \frac{2}{2^n} = \frac{1}{2^{n-1}}$, thus

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &\leq \frac{1}{2^{n-1}}(1 + \frac{1}{2} + \cdots + \frac{1}{2^k}) \\ &< \frac{2}{2^{n-1}} = \frac{1}{2^{n-2}}. \end{aligned}$$

Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is cauchy by (iii). Since X is complete by assumption, $x_n \to y$ for some $y \Rightarrow \exists \alpha_0$ such that $y \in U_{\alpha_0}$ and $\exists r > 0$ such that $B_r(y) \subseteq U_{\alpha_0}$. Since $x_n \to y$ there exist n such that $x_n \in B_{\frac{r}{2}}(y)$ and $\frac{1}{2^n} < \frac{r}{2}$ therefore $B_{\frac{1}{2^n}}(x_n) \subseteq B_r(y) \subseteq U_{\alpha_0}$ which is a finite subcover, contradiction.

□

1.10 Connectedness

Connectedness is an important concept in topology that describes the property of a topological space being "unbroken" or "not easily divided into separate pieces." A topological space is considered connected if it cannot be separated into two disjoint nonempty open sets. In other words, a space is connected if it forms a single, continuous piece without any gaps or breaks.

Definition: Connected

Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

Connectedness is obviously a topological property since it is formulated entirely in terms of the collection of open sets of X. Said differently, if X is connected so is any space homeomorphic to X since homeomorphisms are designed to preserve the topological structure. Another way of formulating the definition of connectedness is the following statement:

A space X is connected ⇔ the only subsets of X that are both open and closed in X are the empty set and X itself.

For a subspace Y of a topological space X, there is another useful way to formulate the definition of connectedness:

Lemma 1.48:

If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Since connectedness is defined to be violation of separation between disjoint non-empty open subsets. It is natural to ask, is it possible for a non-connected space having connected subspace? This question could also be viewed as "how can we coonstruct spaces that are connected?". We shall now prove several results that tell us how to form new connected spaces from the given ones.

Lemma 1.49:

Let X be a topological space and consider two non-empty open subsets $C, D \subseteq X$ such that $X = C \cup D$. If Y is a connected subspace of X then Y lies entirely within either C or D.

Proof:

Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y. These two sets are disjoint and their union is Y; if they were both non-empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in either C or D.

□

Theorem 1.50:

The union of a collection of connected subspaces of X that have a point in common is connected.

Proof:

Let $\{A_\alpha\}$ be a collection of connected subspaces of a space X; let p be a point of $\bigcap_\alpha A_\alpha$. We prove that the space $Y := \bigcup_\alpha A_\alpha$ is connected. Suppose that $Y = C \cup D$ is a separation of Y. The point p is in one of the sets C or D; without loss of generality, we may assume that $p \in C$. Since A_α's are connected, each of them must lie entirely in either C or D by **Lemma 1.49**, and it cannot lie in D since it contains the point p of C. Therefore $A_\alpha \subseteq C$ holds true for every α, so that $\bigcup_\alpha A_\alpha \subseteq C$, contradicting the fact that D is non-empty.

□

There is also a "squeeze-theorem-like" property for connectedness which is very useful when we can squeeze the set we wish to prove its connectedness by two connected sets. Note that we admit the fact that the closure of connected sets is also connected. However, note also that the interior of connected space may fail to be connected.

Theorem 1.51:

Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$ then B is also connected.

Proof:

Let A be connected and let $A \subseteq B \subseteq \overline{A}$. Suppose that $B = C \cup D$ is a separation of B. By **Lemma 1.49**, A must lie entirely in either C or D; without loss of generality, we may assume that $A \subseteq C$. Then $\overline{A} \subseteq \overline{C}$ and since C and D are disjoint, it follows that B cannot intersect D. This contradicts the fact that D is a non-empty subset of B.

□

We said before that the topological property is preserved under homeomorphisms, we now prove that continuous map also preserve the connectedness:

Theorem 1.52:

The image of a connected space under a continuous map is connected.

Proof: Exercise.

Moreover, the connectedness is preserved under finite product. In fact, this can be generalized into arbitrary product without violating the connectedness.

Theorem 1.53:

A finite Cartesian product of connected spaces is also connected.

Proof: Exercise.

1.11 IVT, MVT, and UCT

In the study of Calculus, there are three basic theorems about continuous functions, and on these theorems the rest of calculus depends. They are the followings:

(i) Intermediate Value Theorem (IVT): If $f : [a, b] \to \mathbb{R}$ is continuous and if r is a real number between $f(a)$ and $f(b)$, then there exists an element $c \in [a, b]$ such that $f(c) = r$.

(ii) Maximum Value Theorem (MVT): If $f : [a, b] \to \mathbb{R}$ is continuous then there exists an element $c \in [a, b]$ such that $f(x) \leq f(c) \forall x \in [a, b]$.

(iii) Uniform Continuity Theorem (UCT): If $f : [a, b] \to \mathbb{R}$ is continuous, then given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for every pair of number $x_1, x_2 \in [a, b]$ for which $|x_1 - x_2| < \delta$.

These theorems are used in a number of places. The IVT is used for instance in constructing inverse functions such as $\sqrt[3]{x}$ and $\arcsin x$; and the MVT is used for proving the IVT for derivatives, upon which the two fundamental theorems of calculus depend. The UCT is used, among other things, for proving that every continuous functions is integrable.

We have spoken of these three theorems as theorems about continuous functions. But they can also be considered as theorems about the closed intervals $[a, b]$ of real numbers. The theorems depend not only on the continuity of the function f but also on properties of the topological spaces $[a, b]$.

The property of the space $[a, b]$ on which the IVT depends on the connectedness, and the property on which the other two depend on is the compactedness. We have so far introduced all of them so let us now talk about their applications to these fundamental theorems.

We now introduce a fact that the intervals and rays in $\mathbb{R}$ are connected and this concept should be already for familiar for those who are confortable with analysis. We prove it here again, in generalized form. It turns out that this fact does not depend on the algebraic properties of $\mathbb{R}$ but only on its order properties. To make this clear, we shall prove the theorem for an arbitrary ordered set that has the order properties of $\mathbb{R}$. Such a set is called linear continuum.

Definition: Linear Continuum

A simply ordered set L having more than one element is called a linear continuum if:

(i) L has the least upper bound property.

(ii) If $x < y$ there exists z such that $x < z < y$.

The terms "linear continuum" and "continuum" are closely related, but they refer to slightly different concepts, particularly in the context of topology and real analysis. Let's explore the differences between these terms:

Linear Continuum:

A "linear continuum" refers to a specific type of topological space that is ordered, connected, and densely ordered. In other words, it's a linearly ordered set (often the real numbers $\mathbb{R}$) that forms a connected space, and between any two elements, there is another element.

The properties of a linear continuum include:

Ordered Set:

The elements of a linear continuum can be arranged in a linear order (usually denoted by $\leq$) that is reflexive, transitive, and connected.

Connectedness:

A linear continuum is connected as a topological space. This means that there are no disjoint open sets that partition the space.

Dense Ordering:

Between any two distinct elements of a linear continuum, there is another element. In other words, the space is densely ordered.

The classical example of a linear continuum is the set of real numbers $\mathbb{R}$ equipped with the usual order and topology. As for continuum:

Continuum:

In a more general sense, "continuum" refers to a connected, unbroken space that does not have any gaps or jumps. It emphasizes the idea of a smooth, uninterrupted flow of points.

The properties of a continuum include:

Connectedness:

A continuum is connected as a topological space, meaning that it cannot be partitioned into two disjoint nonempty open sets.

Unbroken Flow:

A continuum is a space where there are no "holes" or "gaps." It can be thought of as a space that is continuously connected without interruptions.

Remark:

In this broader sense, a linear continuum is a specific type of continuum that possesses additional properties related to linear ordering and density. ||

Now we prove an important result of the linear continuum:

Theorem 1.54:

If L is a linear continuum in the order topology, then L is connected, and so are the intervals and rays in L.

Proof:

Recall that a subspace Y of L is said to be convex if for every pair of points $a, b \in Y$ with $a < b$, the entire interval $[a, b]$ of points of L lies in Y. We prove that if Y is a convex subspace of L, then Y is connected.

So suppose that Y is the union of the disjoint nonempty sets A and B, each of which is open in Y. Choose $a \in A$ and $b \in B$; suppose for convenience that $a < b$. The interval $[a, b]$ of points of L is connected in Y. Hence $[a, b]$ is the union of the disjoint sets

$$A_0 = A \cap [a, b] \text{ and } B_0 = B \cap [a, b],$$

each of which is open in $[a, b]$ in the sense of subspace topology, which is the same as the order topology (see **Remark** below). The sets A_0 and B_0 are nonempty because $a \in A_0$ and $b \in B_0$. Thus, A_0 and B_0 constitute a separation of $[a, b]$.

Let now $c := \sup A_0$. We show that c belongs to neither A_0 nor B_0, which contradicts the fact that $[a, b]$ is the union of A_0 and B_0.

Case I: $c \in B_0$.

Suppose $c \in B_0$. Then $c \neq a$ so either $c = b$ or $a < c < b$. In either case, it follows from the fact that B_0 is open in $[a, b]$ that there is some interval of the form $(d, c]$ contained in B_0: If $c = b$, we have a contradiction at once, for f is a smaller upper bound on A_0 than c. If $c < b$, we note that $(c, b]$ does not intersect A_0 since c is an upper bound of A_0. Therefore $(d, b] = (d, c] \cup (c, b]$ does not intersect A_0. Again, since d is a smaller upper bound on A_0 than c, contradiction.

Case II: $c \in A_0$.

Suppose now $c \in A_0$. Then $c \neq b$ so either $c = a$ or $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. According to the definition of the linear continuum, we can choose a point $z \in L$ satisfying $c < z < e$, still, a contradiction since this implies $z \in A_0$.

□

Remark:

A key result we used in the proof is that the order topology and the subspace topology coincide when the subset under consideration is itself a convex subset of the ordered space. ||

Corollary 1.54.1:

The real line $\mathbb{R}$ is connected and so are the intervals and rays in $\mathbb{R}$.

As an application, we shall prove the intermediate value theorem (IVT) as we promised.

Theorem 1.55: Intermediate Value Theorem (IVT)

Let $f : X \to Y$ be a continuous map where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Proof:

Assume this is true. The sets $A := f(X) \cap (-\infty, r)$ and $B := f(X) \cap (r, +\infty)$ are obviously disjoint and they are nonempty since one contains $f(a)$ while the other having $f(b)$ inside. Each is open in $f(X)$, being the intersection of an open ray in Y with $f(X)$. If there were no point c of X such that $f(c) = r$, then

$f(X)$ would be the union of the sets A and B. Then A and B would constitute a separation of $f(X)$ and lead us to the contradiction since the connectedness is closed under the continuous mapping.

□

Connectedness of intervals in $\mathbb{R}$ gives rise to an especially useful criterion for showing that a space is connected; namely, the condition that every pair of points of X can be joined by a path in X. The following discussion of path connectedness and the relations between it and the connectedness follows from [22] and [23].

We now turn to a closely related conception: the path connectedness. It is more intuitive, and, as we will see soon, can be extended to define "higher level connectedness" which is described by computable algebraic objects.

Definition: Path and Loop

Let X be a topological space and let $x, y \in X$ be two points.

(i) A **path** from x to y is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

(ii) In the case $x = y$, we will call the path a loop with base point x.

(iii) There is a special path/loop from x to x: the **constant path** γ_x defined by $\gamma_x(t) = x \ \forall t \in [0,1]$.

Remark:

So path is a continuous map, not just a "geometric curve". Different parameterizations of the same "geometric pictures" will be regarded as different paths. ||

Path-connectedness is a fundamental concept in topology that describes the degree to which points in a topological space can be connected by continuous curves or paths. A space is path-connected if you can find a continuous path between any two points in the space. Path-connectedness is a stronger notion than simple connectedness, as it not only ensures that the space is connected as a whole but also allows for a "path" between any two points.

Definition: Path Connceted

We say a topological space X is path-connected if any two points in X can be connected by a path.

It is easy to prove that path-conncetedness is stronger than connectedness:

Proposition 1.56:

If X is path-connected then X is connected.

Proof:

Suppose that X is path connected but not connected. Assume that there exist nonempty disjoint open sets A and B such that $X = A \cup B$. Take a point $x \in A$ and a point $y \in B$ and a path γ from x to y. Then $[0,1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$, which makes the union of non-empty disjoint open sets, which contradicts with the connectedness of $[0,1]$.

□

We now introduce some results from the compactness and derive the desired Extreme Value Theorem (EVT) as well as the Uniform Continuity Theorem (UCT) and then talk a bit of the Lebesgue number lemma.

Results:

(i) Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact. ([1], Theorem 27.1)

(ii) Every closed interval in $\mathbb{R}$ is compact. ([1], Corollary 27.2)

Now we prove the extreme value theorem of calculus, in suitably generalized form. One should note that the EVT is the special case of the following generalization occurs even when X is closed interval in $\mathbb{R}$ and $Y = \mathbb{R}$.

Theorem 1.57: Extreme Value Theorem (EVT)

Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then $\exists\, c, d \in X$ such that $f(c) \leq f(x) \leq f(d)\ \forall x \in X$.

Proof:

Since f is continuous and X is compact, the set $A := f(X)$ is compact. We show that A has a largest element M and a smallest element m. Then since m and M belong to A, we must have $m = f(c)$ while $M = f(d)$.

If A has no largest element then the collection $\{(-\infty, a) \mid a \in A\}$ forms an open covering of A. Since A is compact, some finite subcollection $\{(-\infty, a_1), \cdots, (-\infty, a_n)\}$ covers A. If a_i is the largest of the elements $a_1, \cdots, a_n$ then a_i belongs to none of these sets, contradiction to the fact that they cover A. A similar argument applies to the smallest elements, result follows.

□

Now we prove the uniform continuity theorem of Calculus. In the process, we are led to introduce a new notion that will prove to be surprisingly useful, that of a Lebesgue number for an open covering of a metric space.

Definition: Diameter

Let (X, d) be a metric space, let $Y \subseteq X$ be a subset. The diameter of Y is defined to be $\operatorname{diam} Y := \sup\{d(y_1, y_2) \mid y_1, y_2 \in Y\}$, where $\operatorname{diam} Y$ could be infinite.

Lemma 1.58: Number Lemma

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of a compact metric space (X, d). Then for some $\delta > 0$ so that for all subset $Y \subseteq X$ with $\operatorname{diam} Y < \delta$ there exists an $\alpha \in A$ such that $Y \subseteq U_\alpha$.

Proof:

Pick $x \in X$ then $x \in U_\alpha$ for some $\alpha = \alpha(x) \in A$. Then there exists $\varepsilon(x) > 0$ such that $B_{2\varepsilon(x)}(x) \subseteq U_{\alpha(x)}$, so we get an open cover $\{B_{\varepsilon(x)}(x)\}_{x \in X}$, which is an open cover of X.

Since X is compact, there exists $k \in \mathbb{N}$ such that for $x_1, \cdots, x_k \in X$ one has $X = B_{\varepsilon(x_1)}(x_1) \cup \cdots \cup B_{\varepsilon(x_k)}(x_k)$. Let now $\delta := \min\{\varepsilon(x_k), \cdots, \varepsilon(x_k)\}$. If $Y \subseteq X$ with $\operatorname{diam} Y < \delta$, then $Y \cap B_{\varepsilon(x_i)}(x_i) \neq \varnothing$ for some $1 \leq i \leq k$.

[Claim]: $Y \subseteq U_\alpha(x_i)$.

Suppose $y_0 \in Y \cap B_{\varepsilon(x_i)}(x_i)$, then $\forall y \in Y$,

$$d(y, x_i) \leq d(y, y_0) + d(y_0, x_i)$$

$$< \delta + \varepsilon(x_i) \qquad \text{(since } d(y, y_0) < \delta \text{ and } d(y_0, x_i) < \varepsilon(x_i))$$
$$\leq 2\varepsilon(x_i) \qquad \text{(By the choice of } \delta)$$

Therefore $Y \subseteq B_{\varepsilon(x_i)}(x_i) \subseteq U_\alpha(x_i)$ as we claim.

□

The Lebesgue Number Lemma is a fundamental result in topology that provides a useful tool for understanding the relationship between an open cover of a compact metric space and the existence of a "Lebesgue number," which is a positive number that ensures that any subset of the space with diameter less than the Lebesgue number can be fully contained in one of the open sets of the cover.

Note that in different literature we may see different treatments. For example, the version we adapt here comes from Eugene Lerman (see [2]). There are also other approachses, for instance, in [1], the approach is done by

$$d(x, A) := \inf\{d(x, a) \,|\, a \in A\},$$

where $d(x, A)$ is called the distance from x to A. See also the treatment given by [24].

Recall in basic analysis we knew the difference between being continuous and being uniform continuous.

Definition: Uniform Contiuity

A function $f : (X, d_X) \to (Y, d_Y)$ is said to be uniformly continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall x_0, x_1 \in X$ one has:

$$d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 1.59: Uniform Continuity Theorem

Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous map where (X, d_X) is compact. Then f is uniformly continuous.

Proof:

Given $\varepsilon > 0$, take the open covering of Y by balls $B_{\frac{\varepsilon}{2}}(y)$ of radius $\dfrac{\varepsilon}{2}$. Let $\mathscr{A}$ be an open covering of X by the inverse images of these balls under f. Choose δ to be the Lebesgue number for the covering $\mathscr{A}$. Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$ the two-point set $\{x_1, x_2\}$ has diameter less than δ so that the image $\{f(x_1), f(x_2)\}$ lies in some open ball $B_{\frac{\varepsilon}{2}}(y)$. Then $d_Y(f(x_1), f(x_2)) < \varepsilon$ as we desired.

□

1.12 Local Connectedness and Limit Point Compactness

We now generaliz the results we have proved so far. Both connectedness and compactness are topological properties hence they are closed under continuos mappings. We also know that according to Tychonoff's Theorem arbitrary product of compact spaces is also compact, since compactness implies connectedness, this means that the arbitrary product of connected spaces is connected, not disconnected. We used these terminologies to prove the IVT, EVT, and UCT and we saw how powerful these tools are. It is natural to ask what should we do when we try to prove IVT, for example, when we do not have the connectedness involved? This is where the locally connectedness, as well as the locally compactness come to our sight.

Locally connectedness is a property in topology that characterizes the "closeness" of points within a topological space. A space is locally connected if, intuitively, every point has a neighborhood that is connected. This property provides information about how the space is connected on a small scale, even if it might not be globally connected.

The idea to derive the local connectedness is: given an arbitrary space X, there is anatural way to break it up into pieces that are connected (or path-connected).

Definition: Components

Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called componenets (or the connected components) of X.

Property: (see [1], Theorem 25.1)

The components of X are connected disjoint subspaces of X whose union is X such that each nonempty connected subspace of X intersects only one of them.

Definition: Path Components

We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X.

Property: (see [1], Theorem 25.2)

The path components of X are path-connected disjoint subspaces of X whose union is X such that each nonempty path-connected subspace of X intersects only one of them.

Definition: Locally Connected

A topological space X is said to be locally connected if $\forall x \in X$ and for all open subsets $U \subseteq X$ containing x, there exists an open, connected set V such that $x \in V \subseteq U$.

Definition: Locally Path Connected

We say a topological space X is:

(i) Locally Path Connected at $x \in X$ if for any open neighbourhood U of x there exists an open neighbourhood V of x inside U which is path connected.

(ii) Locally Path Connected if it is locally path connected at any point.

Theorem 1.60: Criterion for Locally Connected (see [1], Theorem 25.3)

A topological space X is locally connected $\Leftrightarrow$ for every open set $U \subseteq X$, each component of U is open in X.

Theorem 1.60: Criterion for Locally Path Connected (see [1], Theorem 25.4)

A topological space X is locally path connected $\Leftrightarrow$ for every open set $U \subseteq X$, each path component of U is open in X.

Theorem 1.61: Relations (see [1], Theorem 25.5)

If X is a topological space, each path components of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Proposition 1.62: Connected + Locally Path Connected $\Rightarrow$ Path Connected

If X is a connected and locally path connected topological space. Then X is also path connected.

Proof:

Fix a point $x \in X$. Consider the set

$$A = \{y \in X \,|\, y \text{ can be connected by path to } x\}.$$

By locally path connectedness, we know if a point is in $A \Rightarrow$ a neighbourhood of this point is in A. If a point is in A^c then a neighbourhood of this point is in A^c. Therefore, $A \neq \emptyset$ is both open and closed. Since X is connected then $X = A$.

□

From the above results, one can see that connectedness does not necessarily imply the locally connectedness, similar result applies to path connectedness and locally path connectedness. The reason is that they are different topological properties.

Instead of introducing the local compactness, which involve concrete background of the Hausdorff space, we now introduce the limit point compactness. The key difference between compactness and limit point compactness lies in the way they handle infinite sets. Compactness deals with open covers and their finite subcovers, while limit point compactness concerns the existence of limit points for infinite subsets.

Definition: Limit Point Compact

A topological space X is said to be limit point compact if every infinite subset of X has a limit point in X

Theorem 1.63: (see [1], Theorem 28.1)

Compactness implies limit point compactness, but not conversely.

Now we introduce another version of compactness called sequential compactness which deals specifically with sequences of points in a space. A space is sequentially compact if every sequence of points in the space has a convergent subsequence whose limit lies within the space. Sequential compactness is a useful property in spaces where sequences play a significant role.

Definition: Sequentially Compact

A topological space X is said to be sequentially compact if every seqeunce of points in X has a convergent subsequence.

Theorem 1.64: (see [1], Theorem 28.2)

Let X be a metrizable space. Then the followings are equivalent:

(i) X is compact.

(ii) X is limit point compact.

(iii) X is sequentially compact.

The term "metrizable" is quite unfamiliar at this time since we have not yet introd-uced it. The treatment of it will be given in the third chapter, where we zoom in to the discussion of the metric space and metric topology.

To close this chapter, we shall talk a bit of the local-to-global lemma. In the world of advanced mathematics, we are often interested in comparing the local properties of a space to its global properties. Connectedness is one of our most important tools in doing this. Often, it is easier to prove that a property holds in a neighborhood of each point than to prove that it holds for the entire space. If the space is connected, then using the following lemma, we can sometimes deduce that a property holds globally, simply from the fact that it holds in a neighborhood of every point.

Lemma 1.65: Local-To-Global Lemma

Let X be a connected topological space. Suppose that we have an equivalence relation $\sim$ on X such that every point has a neighborhood of equivalent points. Then there is only one equivalence class, i.e., all points of X are equivalent.

Proof:

Let $X = \bigcup_{\alpha} X_\alpha$ be the partition of X into the equivalent classes X_α. Thus, X is the disjoint union of the X_α, and if $x \in X_\alpha$ then $(x \in X_\alpha \Rightarrow y \in X_\alpha) \Leftrightarrow (x \sim y)$. By assumption, $\forall x \in X$ there exists an open set U such that $x \in U$ and every element of U is equivalent to x; i.e. $x \in U \subseteq X_\alpha$. Thus, every equivalence class X_α is open.

Suppose now that there there is more than one equivalent class. Let $U = X_\alpha$ be an equivalence class, and $V = \bigcup_{\beta \neq \alpha} X_\beta$. Then U, V form a separation, contradicting with the connectedness of X. Hence the uniqueness holds.

□

Comment:

The "local-to-global" principle is a general approach used in topology where a global property of a topological space is deduced from the local properties of its points. While the local-to-global principle can be applied to many local-to-global properties, it might not apply to all of them. It depends on the specific property and the nature of the space being considered.

The local-to-global principle can be applied to many properties that have a local character and are preserved under open sets. Examples of local-to-global properties include connectedness, path connectedness, and locally connectedness.

However, there are properties for which the local-to-global principle might not apply:

Compactness and Limit Point Compactness:

Compactness and limit point compactness are not local-to-global properties. A locally compact space does not necessarily imply global compactness, and a space where every point has a limit point does not necessarily imply limit point compactness.

Separation Axioms:

Some separation axioms, such as regularity and normality have local-to-global properties, but others like T_0 and T_1 do not necessarily follow this principle.

Completeness:

Completeness, as seen in metric spaces, is a global property that depends on the entire metric space and not just local neighborhoods.

2.1 Separation Axioms

We have introduced compatibility in the previous chapter. In this chapter we shall deal with other topological properties such as connectedness and separability. The order of their presence is only due to convenience. We shall now consider the separability.

A topological space is said to be separable if it contains a countable dense subset. A dense subset is one that is "closely packed" in the sense that every point in the space is either a part of the subset or a limit point of the subset. If a space is separable, it means that you can find a countable set that is "everywhere dense" in the space. The concept of separability is important because it measures how "rich" the space is in terms of having points that are "close" to each other. The overview over separability is mainly from [14]:

Definition: Dense

Let X be a metric space. For $A \subseteq X$ a subset we say A is dense in X if $\overline{A} = X$.

Remark:

Recall that $\overline{A} = A \cup A'$ where A' is the set containing all the limit points of A, therefore, according to this definition, it follows that A is dense in $X \Leftrightarrow$ every open ball in X contains a point in A. Moreover, by the sequential characterization of the closure, we can say that A is dense in $X \Leftrightarrow \forall a \in X$ there exists a sequence $(x_n)_{n\in N}$ in A such that $x_n \to a$ in X. ||

Definition: Separable

Let X be a metric space (or a topological space). We say that X is separable when it has a finite or countable dense subset.

Theorem 2.1:

Let X be a metric space. Then the followings hold:

(i) If X is separable then there is a finite or countable basis for the metric topology on X.

(ii) If every infinite subset of X has a limit point then X is separable.

(iii) If X is separable then every subspace of X is separable.

Proof: Consult [14].

In order not to deviate from our main goal, which is the discussion of separability axioms, we shall close the discussion for now, with a statement on its stability under homeomorphisms. Before that we introduce a competitive definition for being dense:

Definition: Dense (comparable definition)

Let (X, T) be a topological space. The subset $A \subseteq X$ is said to be dense in X if the intersection of every nonempty open set with A is nonempty, i.e. $A \cap U \neq \varnothing$ holds true $\forall U \in T \backslash \varnothing$.

Theorem 2.2: Separability is closed under Homeomorphisms

Let (X, T_X) and (Y, T_Y) be two topological spaces and let $f : X \to Y$ be a homeomorphism. If X is separable then so is Y.

Proof:

Since X is separable there exists a subset $A \subseteq X$ that is both countable and dense. That is, for all open sets $U \subseteq X$ we have that $A \cap U \neq \varnothing$.

[Claim]: $f(A)$ is a countable and dense subset of Y.

Since f is surjective and A is a countable subset of X we see that $f(A)$ is a countable subset of Y. Now let V be an open subset of Y, since f is continuous we have that $f^{-1}(V)$ is open in X and since A is dense in X. Therefore we have $A \cap f^{-1}(V) \neq \emptyset$. Hence,

$$f\big(A \cap f^{-1}(V)\big) \neq \emptyset, f(A) \cap f(f^{-1}(V)) \neq \emptyset, f(A) \cap V \neq \emptyset.$$

Since V is chosen arbitrarily, $f(A)$ is a countable and dense subset of Y.

□

Separation axioms, also known as separation properties, are fundamental concepts in topology that describe how well distinct points and closed sets can be separated from each other within a topological space. These axioms characterize the “separateness" and "closeness" of points and sets, which is crucial for understanding the structure of topological spaces.

It may be bit confusing to claim the fact that the separability and the separation axioms indeed are not the same: They refer to different aspects of the properties of topological spaces:

Separability:

Separability is a property of a topological space that relates to the existence of a dense subset with a specific cardinality. A topological space X is said to be separable if there exists a countable dense subset in X. This means that there is a countable set of points that are "dense" in the sense that every point in the space is either in this set or is a limit point of this set. In other words, one can approximate any point in the space arbitrarily closely using elements from the countable dense subset.

Separation Axioms:

Separation axioms, on the other hand, are a set of properties that define how well-behaved the open sets of a topological space are in relation to each other and to the points in the space. These axioms provide information about how "separated" or "disconnected" different parts of the space are from each other.

The materials we use to cover the separation axioms are from [9], [10], [11], [12], [13], and mainly results from [14].

Definition: T_0 (Kolmogorov)

For any two distinct points, there exists an open set containing one of the points but not the other. That is to say, a topological space X is T_0 if $\forall x, y \in X$ with $x \neq y$, there is an open set containing one and only one of x or y.

Example 2.1: T_0

Let $X = \{a, b, c\}$ and consider the topology $T = \{\emptyset, X, \{b\}, \{a, b\}\}$ on X. ||

We make a comment taken from [17], where a relationship between topological spaces and their Kolmogorov quotients are concerned.

Comment: [17]

Every topological space has a Kolmogorov quotient that is obtained by identifying topologically indistinguishable points, that is, points that are contained in exactly the same open sets. This means that there is no sequence of operations on an open sets that would give a set A such that $x \in A$ but $y \notin A$ for $x \neq y$. Nothing topologically important to the space X is lost in identifying these points.

The resulting space is T_0-space: a space where all points are topologically distinguishable. Most topological spaces we concern are T_0. In a T_0 space, every point serves a purpose!

However, there are situations where it is inconvenient if a space is T_0. Such a situation occurs when one is interested in refinements of the topology: the more points there are in X, the more choices there are for refinements. The same is true for subspaces, though the loss here is not so dramatic, still, if one is interested in the specific points of the space, one might not wish to clump them together in equivalence classes.

Definition: T_1 (Fréchet)

For any two distinct points, there exist disjoint open sets containing each of the points. That is to say, a topological space is T_1 if $\forall x, y \in X$ such that $x \neq y$, there exist open neighbourhoods U_x of x and U_y of y such that $y \notin U_x$ and $x \notin U_y$.

Example 2.2: T_1

Let X be a set and let T be a cofinite topology (where singletons are closed). Cofinite topology is defined by declaring a subset of X to be open if and only if its complement in X is either finite or the whole set X. If $x \neq y$, consider the neighbourhoods $U_x := x \backslash \{y\}$ and $U_y := y \backslash \{x\}$. ||

Proposition 2.3: Criterion for T_1

A topological space X is T_1 $\Leftrightarrow \forall x \in X$, $\{x\}$ is closed.

Proof:

"$\Rightarrow$":

Suppose that X is T_1. Then for $x \in X$ and $\forall y \in X$ such that $x \neq y$ there exists an open neighbourhood U_y of y such that $x \notin U_y$. $\{x\} = X \backslash \bigcup_{y \neq x} U_y$ therefoer $\{x\}$ is closed.

"$\Leftarrow$":

Suppose singletons are closed, for $x, y \in X$ such that $x \neq y$ and let $U_x = X \backslash \{y\}$, easily see $U_y = X \backslash \{x\}$.

□

We now give some comments on T_1 space, which are from [18], [19], [20]. For readers interested in this topic we highly recommend [20] for further readings. Moreover, there are many open problems listed in [18].

Comment:

In topology, the T_1 separation axiom (also known as the Fréchet-Urysohn property) is a fundamental property that characterizes the degree of separation between points in a topological space. A topological space is said to satisfy the T_1 axiom if, for any two distinct points in the space, there exist open sets containing each point but not the other. In other words, the points in the space can be separated by disjoint open sets.

A T_1 space is stronger than a T_0 space (a space satisfying the Kolmogorov property). In a T_1 space, not only can distinct points be distinguished by open sets, but they can also be separated by disjoint open sets. Many common topological spaces, including the Euclidean topology on $\mathbb{R}^n$, metric spaces, and discrete spaces, satisfy the T_1 axiom.

The T_1 axiom is often used as a basic requirement when proving various properties in topology. For instance, the uniqueness of limits in topological spaces is a consequence of the T_1 property. Moreover, as we proved in **Proposition 2.3**, The T_1property is preserved under homeomorphisms. If two spaces are homeomorphic and one of them satisfies the T_1 axiom, the other will also satisfy it.

A product of T_1 spaces is again a T_1 space. This is an important property when dealing with products of topological spaces. However, this may fail to be true when the amount of products we consider is infinity. In the case of infinite products, specifically uncountably infinite products, some additional conditions or restrictions might be needed to ensure that the product remains Fréchet. This is due to the potential subtleties that arise in dealing with uncountable collections of open sets and closed sets in the product topology. In some cases, extra assumptions or restrictions, such as the **Axiom of Choice** or specific properties of the spaces involved, might be required to ensure the validity of the statement for infinite products.

An equivalent statement for a space X to be Fréchet (or Fréchet-Urysohn) if whenever x is in the closure of a set A, there is a sequence of points a_n in A which converge to x. In a letter written to Gary Gruenhage, F. Galvin asked the follow-ing qeustion: If $X_0, X_1, X_2, \cdots$ are such that $\prod_{i \leq n} X_i$ is Fréchet for all $n \in \omega$, must $\prod_{i \in \omega} X_i$ be Fréchet? Y. Tanaka has asked the same question. Gary Gruenhage's paper [21] offers a construction, assuming Martin's Axiom (MA), a Fréchet space X such that X^n is Fréchet for all $n \in \omega$ but X^ω is not Fréchet, where the space X is countable and has only one non-isolated point.

Definition: T_2 (Hausdorff)

For any two distinct points, there exist disjoint open sets containing each of the points. That is to say, a topological space X is T_2 if $\forall x, y \in X$ such that $x \neq y$, there exists an open neighbourhood U_x of x, U_y of y, such that $U_x \cap U_y = \varnothing$.

It is natural to ask that why is T_0, T_1, and Hausdorff space are often mentioned, with Hausdorff seems to be the only "famous" one? One possible reason is that the Hausdorff separation axiom aligns well with our geometric and intuitive understanding of "closeness." When points can be separated by disjoint open sets, it reflects the idea that distinct points can be "strictly distinguished" from each other based on open neighborhoods.

Remark: ([1] **Theorem 31.2**)

A subspace of a Hausdorff space is Hausdorff and a product of Hausdorff space is Hausdorff. ||

Definition: T_3 (regular)

Given a closed set and a point not in that set, there exist disjoint open sets containing the closed set and the point. That is to say, a topological space X

is regular if for all points $x \in X$ and for all closed subsets $C \subseteq X$, there exist open sets U, V with $U \cap V = \varnothing$ such that $x \in U$ and $C \subseteq V$.

Propotion 2.4: Criterion for T_3

A topological space X is $T_3 \Leftrightarrow X$ is T_1 and open sets separate points and closed sets.

As **Proposition 2.4** implies, there are implications between all $T_0, T_1, T_2, T_3,$ and T_4, which we shall introduce later. Our treatment with separation axioms concern with the most common ones, T_5, T_6, and some others will not be considered throughout these notes. However, the as there are a rational number between 3 and 4, there is a $T_{3\frac{1}{2}}$ between T_3 and T_4 as an intermediate, which we shall deal with later.

Remark: ([1] **Theorem 31.2**)

A subspace of a regular space is regular and a product of regular spaces is regular. ||

Comment:

The T_3 separation axiom is a property in topology that characterizes a certain level of "closeness" between points and closed sets in a topological space. The T_3axiom is a stronger separation property than both the T_0 and T_1 axioms. A T_3 space is also T_2 (Hausdorff), T_1 (Fréchet), and T_0. A T_3 space generalizes the T_2 (Hausdorff) property by ensuring that points and closed sets can be separated, rather than just distinct points.

Moreover, The T_3 property captures an important aspect of continuity. In a T_3 space, you can separate a point from a closed set using disjoint open neighborhoods, which is a fundamental requirement for various continuity-related concepts. Furthermore, The T_3 property is preserved under continuous maps. If X is T_3 and $f : X \to Y$ is a continuous map, where Y is another topological space, then Y is also T_3. Notwithstanding, the product of two T_3 spaces is T_3, this can be extended to finite product, however, the arbitrary product of T_3 spaces is not necessarily a T_3 space. While finite products of T_3 spaces are guaranteed to be T_3, the situation changes when dealing with an arbitrary (possibly uncountable) product of T_3 spaces.

The counterexample lies in the realm of set theory and the cardinality of the product. When dealing with an arbitrary product, issues related to the size of the product index set can arise. In particular, if the index set is too large (uncountably large), the product might not satisfy the T_3 separation axiom.

Example 2.3: Arbitrary product of T_3 may not be T_3.

This can be illustrated using the product topology and considering an uncountable product of the Sorgenfrey line, which is a well-known example of a T_3 space. In this case, when you take an uncountable product (for instance, the product over the real numbers), issues related to the size of the index set can lead to counterexamples where the product is not T_3. ||

It's worth noting that set-theoretic issues and cardinality considerations play a role in such counterexamples. When dealing with arbitrary products of spaces, especially when the index set is uncountable, extra assumptions or conditions might be needed to ensure that the product retains certain topological properties.

Definition: T_4 (Normal)

Given two disjoint closed sets, there exist disjoint open sets containing each of the closed sets. That is to say, for any two closed subsets $C_1, C_2 \subseteq X$ with $C_1 \cap C_2 = \emptyset$, there exist open sets U and V with $U \cap V = \emptyset$ such that $C_1 \subseteq U$ and $C_2 \subseteq V$.

Proposition 2.5: Criterion for T_4

A topological space X is $T_4 \Leftrightarrow X$ is T_1 and any two disjoint closed sets can be separated by open sets.

Comment:

The T_4 separation axiom, also known as the normal space axiom, is a property in topology that characterizes a higher level of separation between disjoint closed sets in a topological space. The T_4 axiom is stronger than both the T_3 and T_2 (Hausdorff) axioms. A T_4 space is also T_3, T_2, T_1, and T_0. Moreover, the T_4 property is important for applications in functional analysis, measure theory, and other areas of mathematics that involve working with spaces with "enough separation." Furthermore, the T_4 property is preserved under continuous maps.

The arbitrary (finite or infinite) product of T_4 spaces is still a T_4 space. Unlike the situation with T_3 spaces, where infinite products might not satisfy the T_3 property, the T_4 property is well-behaved with respect to products. This is a significant result known as the Tychonoff theorem, which we shall introduce later.

2.2 Relations between Separation Axioms

We have offered criterions for T_1, T_3, and T_4 with the corr-esponding description in **Proposition 2.3**, **Proposition 2.4**, and **Proposition 2.5**, res-pectively. Now we present a criterion for the missing T_2.

Proposition 2.6: Criterion for T_2 (Hausdorff)

Let (X, T_X) be a topological space.

Then (X, T_X) is T_2 (Hausdorrf) $\Leftrightarrow$The diagonal $\Delta := \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Proof:

"$\Rightarrow$":

$\forall x \neq y$, T_2 implies that there exist open sets U and V such that $x \in U, y \in V$, while $U \cap V = \emptyset$. This can be modified into the form: there exists $U_x \times U_y$ in $X \times X$ such that

$$(x, y) \in U_x \times V_y \text{ and } \Delta \cap (U_x \times V_y) = \emptyset.$$

This means that every point in Δ^c has an open neighborhood entirely contained in the complement. Therefore, Δ^c is open and by the open-closed duality Δ is open.

"$\Leftarrow$":

Assume now that Δ is closed in $X \times X$, we wish to show (X, T_X) is Hausdorff.

Let $x, y \in X$ be such that $x \neq y$. Consider the open set $U := X \backslash \{y\}$ in X. Obviously $(x, y) \notin \Delta$. Since Δ is closed, there exist open sets V_x and V_y such that $(x, y) \in V_x \times V_y$ and $V_x \times V_y \cap \Delta = \emptyset$. If there exists $z \in V_x \cap V_y$, then $(z, z) \in (V_x \times V_y) \cap \Delta$, contradiction. Therefore (X, T_X) is Hausdorff.

□

Relations between Separation Axioms: [11]

We can also study the relations between these axioms. Obviously we have:

(i) $T_2 \Rightarrow T_1$.

(ii) $T_1 + T_3 \Rightarrow T_2$, $T_1 + T_4 \Rightarrow T_2$, $T_1 + T_4 \Rightarrow T_3$.

Note that we also have

(iii) $T_1 \not\Rightarrow T_2$, $T_1 \not\Rightarrow T_3$, $T_1 \not\Rightarrow T_4$ with counterexample $(\mathbb{R}, T_{\text{cofinite}})$.

(iv) $T_4 \not\Rightarrow T_3$, $T_4 \not\Rightarrow T_2$, $T_4 \not\Rightarrow T_1$ with counterexample $(\mathbb{R}, T)$ with $T := \{(-\infty, a) \,|\, a \in \mathbb{R}\}$.

(v) $T_3 \not\Rightarrow T_2$, $T_3 \not\Rightarrow T_1$ with counterexample $(\mathbb{R}, T)$ where T is generated by the basis $\mathscr{B} := \{[n, n+1) \,|\, n \in \mathbb{Z}\}$. In this topology, closed subsets and open subsets are the same.

(vi) $T_2 \not\Rightarrow T_3$, $T_2 \not\Rightarrow T_4$ with counterexample $(\mathbb{R}, T)$ where T is generated by the subbasis $\mathscr{S} := \{(a, b) \,|\, a, b \in \mathbb{Q}\} \cup \mathbb{Q}$. $\mathbb{Q}^c$ is closed but it cannot be separated from $\{0\}$.

(v) $T_3 \not\Rightarrow T_4$ with counterexample being the Sorgenfrey plane $(\mathbb{R}, T_{\text{sorgenfrey}}) \times (\mathbb{R}, T_{\text{sorgenfrey}})$, one may consult [1] in Section 31, Page 152.

Proposition 2.7: T_2 + Criterion $\Rightarrow T_3$

A T_2 (Hausdorff) space X is T_3 (Regular) $\Leftrightarrow \forall x \in X$, there exists an open neighbourhood N of x containing a closed neighbourhood, i.e., there exists an open set V such that $x \in V \subseteq \overline{V} \subseteq N$.

Proof:

"$\Rightarrow$":

Suppose now X is regular. Take $x \in X$, let N be a neighbourhood of x. Then there exists an open set V such that $x \in V \subseteq N$. Since V is open, then $C := X \backslash V$ is closed and $x \notin C$. Since X is regular, there exist open neighbourhoods W of C, U of x, such that $U \cap W = \emptyset$. Consequently $U \subseteq X \backslash W$, since $C \subseteq W$, one has $U \subseteq X \backslash W \subseteq X \backslash C = V \subseteq N$. Therefore, $X \backslash W$ is the desired closed neighbourhood of x.

"$\Leftarrow$":

Suppose now $x \in X$, $C \subseteq X$ is a closed subset and $x \notin C$, then $X \backslash C$ is an open neighobourhood of x. By assumption there exists a closed neighbourhood N of x with $N \subseteq X \backslash C$. Since N is an neighbourhood of x, there exists an open neighbourhood U of x with $U \subseteq N$. Set $V := X \backslash N$ to be an openset, then one has $V = X \backslash N \supseteq X \backslash (X \backslash C) = C$. Since $U \subseteq N$ and $V = X \backslash N$, one has $U \cap V = \emptyset$.

□

Lemma 2.8: T_2+Compact $\Rightarrow T_3$

Compact Hausdorff space is regular.

Proof:

Let X be a compact Hausdorff space and consider a point $x \in X$, a closed subset $C \subseteq X$ such that $x \notin C$. $\forall c \in C$ there exists an open neighbourhood U_x of x and U_c of c such that $U_x \cap U_c = \varnothing$ by Hausdorff. Since X is compact, $\{U_c\}_{c \in C}$ is an open cover of C. Moreover, since C is closed it is compact since it has bounded neighbourhoods. Therefore there exists $n \in \mathbb{N}$ such that for the points $c_1, \cdots, c_n$ satisfying $C \subseteq U_{c_1} \cup \cdots \cup U_{c_n} =: U$, let $V := U_{c_1} \cap \cdots \cap U_{c_n}$ then $U \cap V = \varnothing$ with $x \in V$ and $C \subseteq U$.

$\square$

Theorem 2.9: T_2+Compact $\Rightarrow T_4$

Compact Hausdorff space is normal.

Proof:

Suppose that X is compact Hausdorff and let $C_1, C_2 \subseteq X$ be closed subsets such that $C_1 \cap C_2 = \varnothing$. By **Lemma 2.8** $\forall c \in C_1$ there exist open sets $U_c \ni c$, $V_c \supseteq C_2$ such that $U_c \cap V_c = \varnothing$. Then $\{U_c\}_{c \in C_1}$ is an open cover of C_1. Therefore there exists an $n \in \mathbb{N}$ with $c_1, \cdots, c_n \in C_1$ such that $C_1 \subseteq U_{c_1} \cup \cdots \cup U_{c_n} =: U$. Let now $V := V_{c_1} \cap \cdots V_{c_n} \supseteq C_2$, then $U \cap V = \varnothing$ with $C_1 \subseteq U$ while $C_2 \subseteq V$.

$\square$

We talk about the properties of the normal spaces to close this subsection. Normality, may not behave well as its name suggested. However, most of the spaces we are familiar with are T_4. It importance comes from the fact that the results one can prove under the hypothesis of normality are central to much of topology. The Urysohn metrization theorem and the Tietze extension theorem are two such results, which we shall deal with later.

Properties: of Normal

(i) Every regular space with a countable basis is normal. ([1], **Theorem 32.1**)

(ii) Every metrizable space is normal. ([1], **Theorem 32.2**)

(iii) Every compact Hausdorff space is normal. (**Theorem 2.9**)

(iv) Every well-ordered set X is normal in the order topology. ([1], **Theorem 32.4**)

2.3 Countability Axioms

In order to obtain more relationships between separation axioms, we need to introduce more materials such as the countability axioms. One may see an elegant introduction in [29], there are some very good lecture notes corresponding to this topic, one may also consult [27] and [28]. For a thouroughly treatment, one may also review the 31st section of [1]. First recall the definition of first countable we offered in **1.8**.

Definition: First Countable (A_1)

A topological space X is said to be first countable if every point $x \in X$ has a countable neighbourhood basis.

Remark:

If (X, T_X) is first countable, then for each point one can choose a countable neighbourhood base $\{U_n\}$ satisfying $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$, since if one has a countable neighbourhood basis $V_1, V_2, \cdots$ at x, then one can take

$$U_1 = V_1, U_2 = V_1 \cap V_2, U_3 = V_1 \cap V_2 \cap V_3, \cdots. \qquad \|$$

A first-countable topological space is a space in which every point has a countable neighborhood basis. This means that for each point in the space, there exists a countable collection of open sets that form a basis for the neighborhoods of that point. The concept of a first-countable space is important in topology as it leads to some convenient properties related to sequences, limits, and continuity.

We shall introduce four basic countability properties, they are (1) The first countable axiom, (2) The second countable axiom, and (3) The Lindelöf condition. They are denoted as A_1 space, A_2 space, and A_3 space, respectively. We first introduce some results of A_1.

Theorem 2.10: A_1 and Convergence

Let X be a topological space, then

(i) Let $A \subseteq X$ be a subset. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is A_1.

(ii) Let $f : X \to Y$. If f is continuous, then for every convergent sequences $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$. The converse holds if X is A_1.

Proposition 2.11: A_1 and Compactness

Suppose that the topological space X is A_1. If X is also Hausdorff, then a subset $A \subseteq X$ is limit point compact $\Leftrightarrow$ it is sequentially compact.

Example 2.4: A_1-space

(i) Any metric space is A_1 since one can take $\{U_n\} := B_{\frac{1}{n}}(x)$.

(ii) The set of real numbers $\mathbb{R}$ equipped with the standard Euclidean topology is a first-countable space. $\forall x \in \mathbb{R}$ the collection of open intervals with rational endpoints centered at x forms a countable neighborhood basis.

(iii) The discrete topology on any set X is first-countable. Each singleton set $\{x\}$ is an open set, and these singleton sets form a countable neighborhood basis at each point.

First-countable spaces have a nice property when it comes to sequences and their limits. If a space is first-countable, then every limit of a sequence can be described using the countable neighborhood basis of the limit point. This is particularly useful in metric spaces where sequences play a crucial role. As we see above, a sequentially compact space, where every sequence has a convergent subsequence, is always first-countable. However, the reverse is not necessarily true; a first-countable space is not necessarily sequentially compact. Moreoever, in a first-countable space, a function is continuous at a point if and only if the limit of the function at that point coincides with the value of the function at that point. Furthermore, all metric spaces are first-countable. This is because, in metric spaces, the open balls centered at a point with rational radii form a countable neighborhood basis.

Now we move to the discussion on the second countable axiom, namely A_2. A second countable topological space is a space that possesses a countable basis for its topology. This means that the space has a collection of open sets that is both sufficient to generate the entire topology and countable in size. The concept of second countability is important in topology as it leads to some useful properties related to compactness, separability, and metrizability.

Definition: Second Countable (A_2)

If a topological space X is said to be second countable if it has a countable basis.

Remark:

Obviously any second contable space is also first countable. But the converse is not true, for example, the discrete topology is first countable but not second countable. Moreover, one need to note that the second countable axiom, being stronger than the first countable axiom, sometimes is so strong that not even every metric space satisfies this property. ||

Second-countable spaces have some convenient properties related to compactness. For instance, every compact subset of a second-countable space is itself second-countable. A second-countable space is always separable, meaning that it contains a countable dense subset. This is because the countable basis can be used to construct a countable dense subset. As we shall introduce later, all metric spaces are second-countable. The collection of open balls with rational radii centered at all points of the space forms a countable basis.

Recall the definition of total boundedness:

Definition: Totally Bounded

A metric space (X, d) is said to be totally bounded if $\forall \varepsilon > 0\ \exists n \in \mathbb{N}$ such that for $x_1, \cdots, x_n \in X$ we have $X = \bigcup_{i=1}^{n} B_\varepsilon(x)$.

Proposition 2.12:

Any totally bounded metric space is A_2.

Proof:

Suppose that (X, d) is a totally bounded metric space. By definition, for any $n \in \mathbb{N}$, one can form a finite $\dfrac{1}{n}$ - net, i.e. there exists finitely many points, namely $x_{n,1}, x_{n,2}, \cdots, x_{n,k(n)} \in X$ such that $X = \bigcup_{i=1}^{k(n)} B_{\frac{1}{n}}(x_i)$.

[Claim]: $\mathscr{B} := \left\{ B_{\frac{1}{n}}(x_{n,i}) \,\middle|\, n \in \mathbb{N}, 1 \leq i \leq k(n) \right\}$ is a countable basis.

Take any open subset U and an arbitrary point $x \in U$. Then there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$ by openness. Now choose $n \in \mathbb{N}$ and $1 \leq i \leq k(n)$ such that

$$\frac{1}{n} < \frac{\varepsilon}{2} \text{ and } d(x, x_{n,i}) < \frac{1}{n}.$$

It follows immediately that

$$B_{\frac{1}{n}}(x_{n,i}) \subseteq B_{\frac{2}{n}}(x) \subseteq B_{\varepsilon}(x) \subseteq U,$$

therefore the countable family $\mathscr{B}$ is a basis.

□

Corollary 2.12.1:

Any compact metric space is A_2.

Theorem 2.13:

Let a topological space X be A_2. Then X is separable and first countable.

Before we proceed to the proof of **Theorem 2.13**, we need a property for the dense subsets.

Lemma 2.14:

Let $D \subseteq X$ be a subset and let $\mathscr{B}$ be a basis of open sets not containing $\varnothing$. Then D is dense $\Leftrightarrow \forall B \in \mathscr{B}, B \cap D \neq \varnothing$.

Proof:

"$\Rightarrow$":

Suppose that $B \in \mathscr{B}$ but $B \cap D = \varnothing$. Then D is contained in the closed set $X \backslash B$, which implies that $\overline{D} \neq X$, hence not dense, contradiction.

"$\Leftarrow$":

Let $x \in X$ be an arbitrary point and let N be a neighbourhood of x in X. Then there is a $B \in \mathscr{B}$ such that $x \in B \subseteq N$. Since $B \cap D \neq \varnothing$, it follows that $N \cap D \neq \varnothing$, therefore $x \in \overline{D}$ which implies $X = \overline{D}$. Result follows.

□

Proof of Theorem 2.13:

Let $\mathscr{B}$ be a countable basis. We may assume that $\varnothing \notin \mathscr{B}$. For each $B \in \mathscr{B}$, choose $x_B \in B$. Then by **Lemma 2.14**, $\{x_B \mid B \in \mathscr{B}\}$ is a countable dense subset of X, so X is separable.

For each $x \in X$, the family $\{B \in \mathscr{B} \mid x \in B\}$ is a countable neighbourhood basis at x. Therefore the first countable axiom follows.

□

Remark:

Note that the reverse may not always be valid. A famous example is done by the Sorgenfrey line, this is a concept we try to aviod in the first chapter, in order for a detailed description we shall offer below. ||

The Sorgenfrey line and the Sorgenfrey space are topological spaces that have unique properties and are often used as counterexamples in topology to illustrate concepts that might not hold in more familiar spaces like the real numbers with the standard Euclidean topology.

The lower limit topology is the topology generated by the set of all open intervals in the real numbers. The lower limit topology is denoted by R_{ℓ}. The standard example of the lower limit topology is the real line, which is called the Sorgenfrey line.

Sorgenfrey Line:

The Sorgenfrey line, denoted by $\mathbb{R}_\ell$, is a topological space where the open sets are generated by half-open intervals of the form $[a, b)$ for $a, b \in \mathbb{R}$. In other words, the basis for the topology consists of half-open intervals. This topology is also known as the lower-limit topology.

Properties:

The Sorgenfrey line has some interesting properties:

(i) It is separable and second-countable.

(ii) It is not a normal space.

(iii) The space is connected, path-connected, and locally connected.

Sorgenfrey Space:

The Sorgenfrey space, denoted by $\mathbb{R}_\ell^2$, is a two-dimensional analog of the Sorgenfrey line. It is defined by taking the product of two copies of the Sorgenfrey line, with the basis for the topology consisting of sets of the form $[a, b) \times [c, d)$ with $a, b, c, d \in \mathbb{R}$. Sorgenfrey space share the same properties (i), (ii), and (iii) of Sorgenfrey line, but it does not have all the properties of Sorgenfrey line.

Let us now return to the remark of **Theorem 2.13**, we claimed that the reverse, i.e. a separable A_1 space may not be A_2, we now offer a counterexample.

Example 2.5: Separable A_1 space may not be A_2 (see [30])

The Sorgenfrey line, also know as $\mathbb{R}$ equipped with lower limit topology. In $\mathbb{R}_\ell$, $\mathbb{Q}$ is still a dense set, and each point x has a countable local base of sets of the form $[x, x + \frac{1}{n})$ for $n \in \mathbb{Z}^+$, but the space is not second countable.

In the Sorgenfrey space, namely, $\mathbb{R}_\ell^2$, the situation is even worse: $\mathbb{Q} \times \mathbb{Q}$ is a dense subset, so it's separable, and as a product of two first countable spaces it is certainly first countable, but the reverse diagonal, $\Delta := \{(-x, x) \mid x \in \mathbb{R}\}$is an uncountable closed discrete set. It's easy to see that no space with an uncountable closed discrete subset can be second countable. ||

We now state an important behaviour of A_1 and A_2 spaces, which is very useful wh-en we consider the subspace or the product of the A_1 (resp. A_2) spaces.

Theorem 2.15:

(i) A subspace of A_1 space is A_1, a countable product of A_1 space is A_1.

(ii) A subspace of A_2 space is A_2, a countable product of A_2 space is A_2.

Proof: (see [1], **Theorem 30.2**).

Corollary 2.15.1:

A subspace of a separable space is separable, a countable product of a separable space is also separable.

Proof: (see [30], **Theorem 3.4**).

Moreover, we state a fact that any second countable topological space is separable. This could be easily derived from the following proposition:

Proposition 2.16:

Any second countable topological space has a countable dense subset.

Proof:

Let $\{U_n\}$ be a countable basis of the topology (X, T_X). For each $n \in \mathbb{N}$, choose

a point $x_n \in U_n$ and let $A = \{x_n\}$. Then A is a countable subset in X.
[Claim]: $\overline{A} = X$.

In fact, $\forall x \in X$ and any open neighbourhood U of x, there exists an n such that $x \in U_n \subseteq U$. In particular, $U \cap A \neq \varnothing$ therefore $\overline{A} = X$.

□

Remark:

$A_2 \Rightarrow$ separable but separable $\nRightarrow A_2$. ||

Example 2.6: separable but not A_2 space

Again, this counterexample is done by the Sorgenfrey line, the separability of $(\mathbb{R}, \mathbb{R}_\ell)$ follows from the fact that $\overline{\mathbb{Q}} = \mathbb{R}$. To see that $(\mathbb{R}, \mathbb{R}_\ell)$ is not A_2, we let $\mathscr{B}$ be any basis of $\mathbb{R}_\ell$. Then $\forall x \in \mathbb{R}$ there exists an open set $B_x \in \mathscr{B}$ such that

$$x \in B_x \subseteq [x, x+1),$$

which implies that $x = \inf B_x$. As a consequence, for any $x \neq y$, we have $B_x \neq B_y$. So $\mathscr{B}$ is not a countable family. ||

However, in some cases, the converse is true:

Proposition 2.17:

A metric space is $A_2 \Leftrightarrow$ it is separable.

Proof: (see [27], **Proposition 1.13**).

Remark:

Separability is a very useful concept in functional analysis. It is used to prove certain compactness results. Another well-known result is:

A Hilbert space $\mathscr{H}$ is separable $\Leftrightarrow$ it has a countable orthogonal basis.

From this fact one can easily construct a non-separable Hilbert spaces. ||

Theorem 2.18:

Suppose that X has a countable basis. Then the following statements are valid:

(i) Every open covering of X contains a countable subcollection covering X.

(ii) There exists a countable subset of X that is dense in X.

Proof:

Let $\{B_n\}$ be a countable basis for X.

(i):

Let $\mathscr{A}$ be an open covering of X. For each positive integer n for which it is posible, choose an element A_n of $\mathscr{A}$ containing the basis element B_n. The collection $\mathscr{A}'_n$ of the sets A_n is countable, since it is indexed with a subset J of the positive integers. Furthermore, it covers X. Given a point $x \in X$, we can choose an element $A \in \mathscr{A}$ containing x. Since A is open there exists a basis element B_n such that $x \in B_n \subseteq A$. Because B_n lies in an element of $\mathscr{A}$, the index n belongs to the set J, so A_n is defined; since A_n contains B_n, it contains x. Thus $\mathscr{A}'$ is a the desired countable subcollection of $\mathscr{A}$.

(ii):

From each nonempty basis element B_n, choose a point x_n. Let D be the set consisting of the points x_n. Then D is dense in X: Given any point x of X, every basis element containing x intersects D, therefore $x \in \overline{D}$, denseness follows.

□

Two properties listed in the above theorem are sometimes taken as an alternative countability axioms.

Definition: Lindelöf (A_3)

A topological space X is said to be A_3 if every open cover of X has a countable subcover. In other words, for any collection of open sets that covers the space, you can select a countable subset of those sets that still covers the entire space.

Weaker in general than A_2, each of these properties is equivalent to A_2 when the space is metrizable (we will see this later). They are less important than the second countability axiom but one should be aware of its existance for it is sometimes useful. For example, to show that a space X has a countable dense subset than it is to show that X has a countable basis.

The Lindelöf property is a topological property that ensures a certain level of "compactness" or "coverage" by open sets in a space. It is a countability axiom that imposes a limitation on the open covers of a space. It ensures that no matter how "large" or "uncountable" the open cover might be, you can still find a countable subcollection that covers the entire space. Moreover, compact spaces are Lindelöf, but the reverse is not necessarily true. Every compact space has a finite subcover, which is also a countable subcover. Therefore, compact spaces satisfy the Lindelöf property. However, there exist Lindelöf spaces that are not compact. Furthermore, every sequentially compact space is Lindelöf, but the reverse is not necessarily true. Sequential compactness is a stronger condition than the Lindelöf property.

2.4 Urysohn's Lemma

The Urysohn Lemma is a fundamental result in topology that provides a powerful tool for constructing continuous functions that separate points and sets in topological spaces. It is a key ingredient in proving various properties of topological spaces, especially in the context of normal spaces and in establishing metrization theorems.

Definition: Completely Regular

A topological space is said to be completely regular if $\forall x \in X$ and for all closed subsets $C \subseteq X$ with $x \notin C$, there exists a real-valued continuous function $f : X \to [0,1]$ such that $f(0) = 0$ and $f|_C = 1$, i.e. $f(c) = 1 \, \forall c \in C$.

A completely regular topological space is a type of topological space that extends the notion of regularity by allowing the separation of points from closed sets by continuous functions.

Theorem 2.19:

A subspace of a completely regular space is completely regular. An arbitrary product of completely regular spaces is completely regular.

Proof:

Let X be a completely regular space and let Y be a subspace of X. Let $x_0 \in Y$ and let $A \subseteq Y$ be a closed subset such that $x_0 \notin A$. Now let $A = \overline{A} \cap Y$, then $x_0 \notin \overline{A}$. Since X is completely regular it follows that one can choose a continuous function, namely, $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(\overline{A}) = \{0\}$. The restriction of $f|_Y$ is the desired continuous function on Y.

Let now $X := \prod_{\alpha \in A} X_\alpha$ be a product of completely regular spaces where A is an arbitrary index set (i.e. countable or not, finite or not). Let $b := (b_\alpha)$ be a point of X and let A be a closed set of X disjoint from b. Choose a basis element $\prod U_\alpha$ containing b that does not intersect A; then $U_\alpha = X_\alpha$ except for finitely many α, say $\alpha = \alpha_1, \cdots, \alpha_n$. Given $i = 1, \cdots, n$, choose a continuous function

$$f_i : X_{\alpha_i} \to [0,1]$$

be such that $f_i(b_{\alpha_i}) = 1$ and $f_i(X \backslash U_{\alpha_i}) = \{0\}$. Let now $\varphi_i(x) = f_i(\pi_{\alpha_i}(x))$, where π_{α_i} denotes the projection. Then φ_i maps X continuously into $\mathbb{R}$ and vanishes outsied $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$. Then the product

$$f(x) = \varphi_1(x) \cdot \varphi_2(x) \cdot \cdots \cdot \varphi_n(x))$$

is the desired continuous function on X, for it equals to 1 at b and vanishes outside $\prod U_\alpha$.

□

Remark:

Every completely regular space is regular, but not all regular spaces are completely regular. The completely regular property is stronger than regularity. However, the completely regular space is not necessarily normal, a counterexample could be viewed in [31]. ||

Definition: $T_{3\frac{1}{2}}$ space (Tychonoff)

A topological space X is said to be Tychonoff ($T_{3\frac{1}{2}}$) if it is Hausdorff and completely regular.

Tychonoff spaces play a significant role in continuity and function theory. The Urysohn Lemma, which allows the construction of continuous functions that separate points and sets, holds in Tychonoff spaces.

We now prove two claims corresponding to the relationship between $T_{3\frac{1}{2}}$ and T_3:

Theorem 2.20:

A Tychonoff space is regular, i.e. $T_{3\frac{1}{2}} \Rightarrow T_3$.

Proof:

Given $x \in X$ and a closed subset $C \subseteq X$ with $x \notin C$. There exists a function $f : X \to [0,1]$ such that $f|_C = 1$ and $f(x) = 0$. Let $U := f^{-1}([0, \frac{1}{2}]) \ni x$ and $V := f^{-1}((\frac{1}{2}, 1]) \supseteq C$, result follows.

□

Lemma 2.21: Urysohn's Lemma ($T_4 \Rightarrow T_{3\frac{1}{2}}$)

Let X be a normal space and let $A, B \subseteq X$ be two closed subsets such that $A \cap B = \varnothing$. Then there exists a continuous function $f : x \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$. That is to say, every normal space is Tychonoff.

Lemma 2.22:

Suppose X is normal. Let A be a closed subset of X and U be an open subset of X such that $A \subseteq U$. Then there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Proof:

Since U is open then $X \backslash U$ is closed. Since $A \subseteq U$ then $(X \backslash U) \cap A = \varnothing$. Assume X is normal, then there exist open sets $V, V' \subseteq X$ with $A \subseteq V$ and $X \backslash U \subseteq V'$ such that $V \cap V' = \varnothing$. Then $\overline{V} \subseteq X \backslash V' \subseteq X \backslash (X \backslash U) = U$.

□

Proof of Lemma 2.21:

The proof of Urysohn's lemma is divided into three steps:

Step I:

Inductively construct open sets $U_r \subseteq X$ where $r \in \mathbb{Q} \cap [0,1]$ so that

$$r < s < 1 \Rightarrow A \subseteq U_r \subseteq \overline{U_r} \subseteq U_s \subseteq \overline{U_s} \subseteq X \backslash B.$$

Since $A \subseteq X \backslash B =: U_1$, then by **Lemma 2.22**, there exists an open set U_0 such that

$$A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1 \subseteq X \backslash B. \qquad \text{(Base case)}$$

Since $\mathbb{Q} \cap [0,1] \in \aleph_0$[2]. Then there exists a bijection

$$r : \{0,1,2,\cdots\} \to \mathbb{Q} \cap [0,1]$$
$$n \mapsto r_n$$

such that $r_0 = 0$ and $r_1 = 1$.

Now we proceed to the inductive step:

Suppose that $U_0 := U_{r_0}, U_{r_1}, \cdots, U_{r_n}$ have been defined. We now construct $U_{r_{n+1}}$. Let $r_l := \min\{r_i\}_{i \in [0,n]}$ such that $r_l < r_{n+1}$ and let $r_m := \max\{r_i\}_{i \in [0,1]}$ such that $r_m > r_{n+1}$.

(i) Since $0 = r_0 < \cdots < r_l < r_{n+1} < r_m < \cdots r_1 = 1$, we have $\overline{U_{r_l}} \subseteq U_{r_m}$.

(ii) By **Lemma 2.22**, there exists an open set $U_{r_{n+1}}$ such that $\overline{U_{r_l}} \subseteq U_{r_{n+1}} \subseteq \overline{U_{r_{n+1}}} \subseteq U_{r_m}$.

Step II:

Define the function $f : X \to [0,1]$ piecewisely

$$f(x) := \begin{cases} 1, \text{ if } x \in X \backslash U_1 =: B \\ \inf \left\{ r \in \mathbb{Q} \cap [0,1] \middle| x \in U_r \right\}, \text{ if } x \in U_1 \end{cases}.$$

By our construction, $f(x) \in [0,1] \forall x$ and $f(x) = 1$ if $x \in B$. We wish to show that f vanishes everywhere ouside B and f is continuous. For any subset $Z \subseteq \mathbb{R}$ such that $\inf Z$ exists and $\forall a, b \in \mathbb{R}$,

$$\inf Z < a \Leftrightarrow \exists r \in \mathbb{Z} \text{ such that } r < a,$$
$$b < \inf Z \Leftrightarrow \exists r' \notin Z \text{ such that } b < r'.$$

Therefore,

---

[2] The symbol $\aleph_0$ (read as "aleph-null") represents the cardinality of the set of natural numbers $\mathbb{N}$, which is the smallest infinity in the hierarchy of infinite cardinal numbers introduced by the mathematician Georg Cantor.

(i) $f(x) = \inf\{r \mid x \in U_r\} < a \Leftrightarrow \exists r$ such that $x \in U_r, r < a$. (2.1)
(ii) $f(x) = \inf\{r \mid x \in U_r\} > b \Leftrightarrow \exists r'$ such that $x \notin U_{r'}, r' > b$. (2.2)
Since $A \subseteq U_0, f(x) = 0 \forall x \in A$, now it left us to prove the continuity of f.
Step III:
Note that $\mathcal{S} := \{[0,a) \,|\, a \in [0,1]\} \cup \{(b,1] \,|\, b \in [0,1]\}$ is a subbasis for a topology on [0,1]. To show that f is continuous is to show that the preimage of the elements of the subbasis are open, i.e. $f^{-1}([0,a))$ and $f^{-1}((b,1])$ are open.
[Claim]: $f^{-1}([0,a))$ is open.

Pick $x \in f^{-1}([0,a)) \Leftrightarrow f(x) < a \Leftrightarrow \exists r$ such that $r < a$ and $x \in U_r$ (2.1)
$\Leftrightarrow x \in \bigcup_{r<a} U_r$ which is open.

Thus, $\bigcup_{r<a} U_r = f^{-1}([0,a))$ is open.

[Claim]: $f^{-1}((b,1])$ is open.

Similarly, $x \in f^{-1}((b,1]) \Leftrightarrow b < f(x)$
$\Leftrightarrow \exists r'$ with $b < r'$ and $x \notin U_{r'}$ (2.2)
$\Leftrightarrow \exists s > b$ such that $x \notin \overline{U_s}$
$\Leftrightarrow x \in \bigcup_{s>b} (X \setminus \overline{U_s})$
$\Rightarrow f^{-1}((b,1]) = \bigcup_{s>b} (X \setminus \overline{U_s})$ which is open.

Continuity follows from two claims, result follows thereafter.

□

2.5 Urysohn's Metrization Theorem

Our goal of this subsection is to arrive at the following result:

$$A_2 + T_1 + \text{Completely Regular} \Rightarrow \text{Metrizable}. \quad (2.3)$$

This is the famous result called Urysohn's Metrization Theorem, which is a fundamental result in topology that provides a condition under which a topological space can be metrized, meaning that its topology can be induced by a metric (distance function). The theorem is named after the Russian mathematician Pavel Urysohn and is a significant contribution to the study of topological spaces and their properties.

Definition: Metrizable

A topological space (X, T_X) is said to be metrizable if there exists a metric d on X such that $T_d = T_X$.

In topology, an embedding is a way to represent one topological space within another in a manner that preserves certain properties and relationships. An embedding essentially allows us to view a space as a subset of another space while maintaining its topological structure. This concept is fundamental in understanding the relationships and properties between different spaces.

Definition: Embedding

A continuous map $f : X \to Y$ is called an embedding if $f : X \to f(X)$ is a homeomorphism where $f(X)$ is given a subspace topology.

Remark:

(i) An embedding is a way to map one space into another while preserving the topological properties. The topology on the embedded space is the same as the subspace topology induced by the larger space.

(ii) The simplest form of embedding is the inclusion map. ||

Example 2.6: Embedding

(i) The continuous mapping $f : \mathbb{R} \to \mathbb{R}^2$ by sending x to $(x,0)$ is an embedding.

(ii) The continuous mapping $g : [0,2\pi) \to \mathbb{C}$ by sending θ to $e^{i\theta}$ is NOT an embedding. ||

Lemma 2.23:

The space $[0,1]^{\mathbb{N}}$ with product topology is metrizable.

Proof:

Since $[0,1]^{\mathbb{N}} := \left\{(x_n)_{n\in\mathbb{N}} \,\middle|\, x_n \in [0,1]\right\}$. Given tw sequences $(x_n), (y_n)$ in $[0,1]^{\mathbb{N}}$, define a function

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n}\left|x_n - y_n\right|.$$

It is not hard to see that such a d is a metric. Let now T_d denote the corresponding topology generated from the metric d.

[Claim]: $T_d = T_{\text{prod}}$.

That is, $Id : \left([0,1]^{\mathbb{N}}, T_d\right) \to \left([0,1]^{\mathbb{N}}, T_{\text{prod}}\right)$ is a homeomorphism.

"$T_{\text{prod}} \subseteq T_d$":

Consider the projections $p_j : [0,1]^{\mathbb{N}} \to [0,1]$ such that $p_j(x) = x_j$.

Fix j and $\forall \varepsilon > 0$, if $d(x,y) < \frac{1}{2^j} \cdot \varepsilon$ then

$$\begin{aligned} |p_j(x) - p_j(y)| &= |x_j - y_j| \\ &= 2^j \cdot \frac{1}{2^j} \cdot |x_j - y_j| \\ &\le 2^j \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n - y_n| \\ &= 2^j \cdot d(x,y) < 2^j \cdot \frac{1}{2^j} \cdot \varepsilon = \varepsilon. \end{aligned}$$

Therefore such a function p_j is continuous. An immediate consequence is that $Id : \left([0,1]^{\mathbb{N}}, T_d\right) \to \left([0,1]^{\mathbb{N}}, T_{\text{prod}}\right)$ is continuous. That is to say, $\forall V \in T_{\text{prod}}, V = Id^{-1}(V) \in T_d \Rightarrow T_{\text{prod}} \subseteq T_d$.

"$T_d \subseteq T_{\text{prod}}$":

Suppose that $V \in T_d$ is an open set. Then $\forall x \in V$, where x is a sequence then there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq V$. There exists $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{\varepsilon}{2}$. Consider

$$U := \left\{ y \in [0,1]^{\mathbb{N}} \,\middle|\, |x_j - y_j| < \frac{\varepsilon}{2} \text{ for } j = 1, \cdots, N \right\}.$$

$$\begin{aligned}\forall y \in U, d(x,y) &= \sum_{n=1}^{N} \frac{1}{2^n} |x_n - y_n| + \sum_{n=N+1}^{\infty} \frac{1}{2^n} |x_n - y_n| \\ &\leq \frac{\varepsilon}{2} \cdot \sum_{n=1}^{N} \frac{1}{2^n} + \frac{1}{2^N} \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &< \frac{\varepsilon}{2} \cdot 1 + \frac{\varepsilon}{2} \cdot 1 = \varepsilon.\end{aligned}$$

Thus, $U \subseteq B_\varepsilon(x) \subseteq V$ therefore V is open in $T_{\text{prod}} \Rightarrow T_d \subseteq T_{\text{prod}}$.

□

Lemma 2.24:

Suppose that X is T_1, and $\{f_\alpha : X \to [0,1]\}_{\alpha \in A}$ is a collection of continuous functions that separate points and closed sets, i.e. if $x \in X$ and $C \subseteq X$ closed subsets with such that $x \notin C$, there exists an α suc that $f_\alpha(x) = 1$ and $f_\alpha|_C \equiv 0$. Then there is a function $F : X \to [0,1]^A$ such that $F(x) = (f_\alpha(x))_{\alpha \in A} \in [0,1]^A$ is an embedding.

Proof:

Recall in the proof of **Lemma 2.23** we built projections $p_\alpha : [0,1]^A \to [0,1]$ such that $p_\alpha((x_\beta)_{\beta \in A}) = x_\alpha$, which are continuous and for all Y and for all G the mapping $Y \to [0,1]^A$ is continuous $\Leftrightarrow \forall \alpha, p_\alpha \circ G : Y \to [0,1]$ are continuous.

Now, setting $p_\alpha \circ F = f_\alpha \forall \alpha$ implies that $F : X \to [0,1]^A$ is continuous.

Suppose that $x, y \in X$ with $x \neq y$ since $\{y\}$ is closed and $x \notin \{y\}$, there then exists an α such that $f_\alpha(y) = 0$ and $f_\alpha(x) = 1$. (In T_1 singletons are closed.)

Therefore we have $F(x) \neq F(y)$ which means F is injective.

[Claim]: $F : X \to F(X)$ is closed.

Let $C \subseteq X$ be a closed subset, $z \in F(x)$ be a point such that z is a limit point of $F(C)$. Then $z = f^{-1}(y)$ for some $y \in X$ and there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in C such that $F(x_\lambda) \to z = F(y)$. If $y \notin C$, then there is an $\alpha \in A$ such that $f_\alpha|_C \equiv 0$ and $f_\alpha(y) = 1$.

$\Rightarrow f_\alpha(x_\lambda) = 0 \forall \lambda \in \Lambda \Rightarrow f_\alpha(x_\lambda) \nrightarrow f_\alpha(y) \Rightarrow F(x_\lambda) \nrightarrow F(y) = z$, contradcition. Therefore $y \in C$ and $z = F(y) \in F(C)$ which means $F(C)$ is closed and then result follows.

□

Lemma 2.25:

Suppose X is a topological space which is completely regular and A_2. Then there exists a countable family of functions $\{f_\alpha : X \to [0,1]\}_{\alpha \in A}$ that separates points and closed sets.

Proof:

Suppose $C \subseteq X$ is a closed subset and $x \in X$ is a point such that $x \notin C$. Then $f : X \to [0,1]$ is a continuous function such that $f(x) = 1$ and $f|_C = 0$. We may

assume that $f \equiv 1$ on a neighbourhood of x.
Since X is A_2, there exists a countable basis, namely $\mathscr{S}$, for the topology on X. Suppose $U, V \in \mathscr{S}$ with $\overline{U} \subseteq V$, consider the following set:

$$\left\{f : X \to [0,1] \text{ continuous} \,\middle|\, f|_U = 1, f|_{X\setminus V} = 0\right\}. \tag{2.4}$$

If this set is nonempty, choose one and exactly one function from the set. We get a countable set of continuous function and we shall denote this set as $\{f_\alpha : X \to [0,1]\}_{\alpha\in A}$ where $A \subseteq \mathscr{S} \times \mathscr{S}$ so that it is countable.
[Claim]:The collection $\{f_\alpha\}_{\alpha\in A}$ separate points and closed sets.

Suppose $C \subseteq X$ is a closed subset and $x \in X\setminus C$ is a point. Since $\mathscr{S}$ is a basis and $X\setminus C$ is open, there exists a $V \in \mathscr{S}$ such that $x \in V \subseteq X\setminus C$. Since X is assumed to be completely regular, it follows that there is a function $g : X \to [0,1]$ such that $g(x) = 1$ and $g|_{X\setminus V} = 0$. We may assume $g \equiv 1$ on a neighbourhood of x. There exists $U \in \mathscr{S}$ such that $x \in U$ and $g|_U \equiv 1$. Since g is continuous, $g|_{\overline{U}} \equiv 1$, since $g|_{X\setminus V} = 0$ then $\overline{U} \subseteq V$. Therefore we have (2.4) $\neq \varnothing$. Hence $\exists \alpha \in A$ and the corresponding f_α such that $f_\alpha|_U \equiv 1$ and $f_\alpha|_{X\setminus V} \equiv 0$. Then it follows that $f_\alpha(x) = 1$ and $f_\alpha|_C \equiv 0$, as we desired.

□

Theorem 2.26: Urysohn's Metrization Theorem

Let X be a T_1, A_2, and completely regular space, then X is metrizable.

Proof:

According to **Lemma 2.25**, there exists a countable collection $\{f_\alpha : X \to [0,1]\}_{\alpha\in A}$ of continuous functions that separate points and closed sets. We may assume that $A = \mathbb{N}$, then by **Lemma 2.24**,

$$F : X \to [0,1]^{\mathbb{N}}, F(x) = (f_\alpha(x))_{\alpha\in A}$$

is an embedding and since $[0,1]^{\mathbb{N}}$ is metrizable by **Lemma 2.23**, and F is an embedding therefore X is metrizable.

□

2.6 Tietze Extension Theorem

Before our discussion of Tietze Extension, we prove some results derived from Urysohn's Metrization Theorem.

Theorem 2.27:

A_2 T_3 space is metrizable.

Recall that according to Urysohn's Lemma, being normal implies completely regularity; therefore, to prove **Theorem 2.27** is equivalent to prove the following:

Theorem 2.28:

A_2 T_3 space is T_4.

Before the proof, recall the definition of Lindelöf space: A topological space is said to be Lindelöf if every open cover has a countable subcover.

Lemma 2.29: Lindelöf's Lemma

$A_2 \Rightarrow$ Lindelöf.

Proof:

Suppose that X is A_2. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable basis for the topology on X. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Then,

(i) $\forall x \in X\, \exists n(x) \in \mathbb{N}$ and $\alpha(x) \in A$ such that $x \in B_{n(x)} \subseteq U_{\alpha(x)}$.

(ii) Let $\mathscr{B} := \{B_n \,|\, \exists \alpha \in A \text{ with } B_n \subseteq U_\alpha\}$.

According to (i), $\mathscr{B}$ is a countable open cover of X. $\forall B \in \mathscr{B}$, choose $\alpha(B)$ such that $B \subseteq U_{\alpha(B)}$. Then $\{U_{\alpha(B)}\}_{B\in\mathscr{B}}$ is a countable subcover.

□

Proof of Theorem 2.28:

Assume that X is $T_3\, A_2$. Let $A, B \subseteq X$ be two closed subsets such that $A \cap B = \varnothing$. Since X is regular, $\forall x \in X$ with $x \notin B$, there exist open sets U_x and U'_x such that $x \in U_x$ while $B \subseteq U'_x$ with $U_x \cap U'_x = \varnothing$. Therefore $U_x \subseteq X\backslash U'_x$ hence $\overline{U_x} \subseteq X\backslash U'_x$ since $X\backslash U'_x$ is closed $\Rightarrow \overline{U_x} \cap B = \varnothing$.

Now, let $\{U_\alpha\}_{\alpha\in A} \cup \{X\backslash A\}$ be an open cover of X. Since X is A_2, according to **Lemma 2.29**, X is Lindelöf, thus this cover has a countable subcover, i.e. $\{U_n\}_{n\in\mathbb{N}} \cup \{X\backslash A\}$ is countable. Then $\{U_n\}_{n\in\mathbb{N}}$ is an open cover and $\overline{U_n} \cap B = \varnothing\ \forall n \in \mathbb{N}$. Similarly, there exists an open cover $\{V_n\}_{n\in\mathbb{N}}$ of B with $\overline{V_n} \cap A = \varnothing$.

Note that if W is open in X and C is a closed subset in X. Then $W\backslash C = W \cap (X\backslash C)$ is open. Let $G_1 := U_1\backslash\overline{V_1}$, by the above argument, G_1 is open; with the same fashion, $G_2 := U_2\backslash(\overline{V_1} \cup \overline{V_2})$, so on and so forth, and this fashion terminates at $G_n := U_n\backslash(\bigcup_{n\geq 1} \overline{V_1})$. Similarly, set $H_1 := V_1\backslash\overline{U_1}$, $H_2 := V_1\backslash(\overline{U_1} \cup \overline{U_2})$, so on and so forth, and this fashion again terminates at $H_n := V_n\backslash(\bigcup_{n\geq 1} \overline{U_n})$.

Let now $G := \bigcup_{i=1}^{\infty} G_i$ and $H := \bigcup_{j=1}^{\infty} H_i$ where both of them are open sets. Since $\overline{V_k} \cap A = \varnothing$ and $\bigcup_{n=1}^{\infty} U_n \supseteq A$ then $G \supseteq A$. Similarly, H is an open set and $H \supseteq B$. It left us to argue that $G \cap H = \varnothing$ and then we are done.

[Claim]: $G \cap H = \varnothing$.

Suppose not, then $G \cap H \neq \varnothing \Rightarrow \exists z \in G \cap H \Rightarrow z \in G_n \cap H_m$ for some $n, m \in \mathbb{N}$. We may assume that $n \geq m$. Since

$$H_m := V_m\backslash(\overline{U_1} \cup \cdots \cup \overline{U_m}) \text{ and } G_n := U_n\backslash(\overline{V_1} \cup \cdots \cup \overline{V_n}),$$

$n \geq m$ implies that $G_n \cap H_m = \varnothing$, contradiction.

Therefore, $G \cap H$ is empty and result follows.

□

The Tietze Extension Theorem is a fundamental result in topology that deals with extending continuous functions defined on a closed subset of a topological space to continuous functions defined on the entire space. This theorem has important applications in various areas of mathematics, including analysis, functional analysis, and topology. The theorem is named after the German mathematician Heinrich Tietze, who proved this result in the early 20th century.

Theorem 2.30: Tietze Extension Theorem

Suppose that X is a normal topological space. Let $F \subseteq X$ be a closed subset and let $f : F \to [0,1]$ be a continuous mapping. Then there exists a continuous function $\tilde{f} : X \to [0,1]$, which is an extension of f such that $\tilde{f}|_F = f$.

The Tietze Extension Theorem guarantees that a continuous function defined on a closed subset A of a normal space X can be extended to a continuous function defined on the entire space X.

Remark:

(i) If F is not closed, then this theorem fails to be true.

(ii) One can use this theorem to prove that Moore's plane is not normal.

Example 2.7: Counterexample

Consider the real line $\mathbb{R}$ with the standard topology. Let $A := (0,1)$ and define a function $f : A \to \mathbb{R}$ with $f(x) = \dfrac{1}{x}$.

The function f is continuous on (0,1) and it's a valid candidate for extension to the entire real line. However, if we try to extend f we run into problems near 0, where the value of the function goes to infinity.

If we attempt to extend f to a continuous function on the entire real line, we would need to define $f(0)$ as $\lim_{x \to 0} f(x)$. However, this limit doesn't exist in the real numbers. ||

This is why the closed subset condition in the Tietze Extension Theorem is crucial. The theorem relies on the properties of closed subsets and the normality of the space to ensure that an extension exists and is continuous. Removing the closed subset condition can lead to situations where extensions are not possible or cannot be guaranteed to be continuous.

Now we proceed to the proof of Tietze Extension.

Definition: Uniform Convergence

Let X be a topological space and let (Y, d) be a metric spaec. A sequence of functions $\{f_n : X \to Y\}_{n \in \mathbb{N}}$ is said to be uniformly convergent to $f : X \to Y$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that if $n \geq N$ then $d(f(x), f_n(x)) < \varepsilon \forall x \in X$.

Lemma 2.31:

Suppose that X is a topological space and let (Y, d) be a metric sapce. IF $\{f_n : X \to Y\}$, a sequence of continuous functions, converges uniformly to $f : X \to Y$, then f is also continuous.

Proof:

We need to show that:

$\forall x_0 \in X\ \forall \varepsilon > 0$ there exists an open neighbourhood U of x_0 such that $x \in U \Rightarrow d(f(x), f(x_0)) < \varepsilon$.

Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$,

$d(f_n(x), f(x)) < \dfrac{\varepsilon}{3}\ \forall x \in X$. Since f_N is continuous at x_0, there exists a neighbourhood U of x_0 such that if $x \in U$ then $d(f(x), f_N(x_0)) < \dfrac{\varepsilon}{3}$. Therefore,

$$\forall x \in U, d(f(x_0), f(x)) \le d(f(x_0), f_N(x_0)) + d(f_N(x_0), f(x)) + d(f_N(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \epsilon.$$

□

Proof of **Theorem 2.30**:

Without loss of generality, we may assume that $0 = \inf_{x\in F} f(x)$ and $1 = \sup_{x \in F} f(x)$. Let $A := f^{-1}([0,\frac{1}{3}])$ and $B := f^{-1}([\frac{2}{3},1])$. By this construction, both A and B are nonempty closed subsets in F with $A \cap B = \varnothing \Rightarrow A, B$ is closed in X. According to Urysohn's Lemma, $\exists g_1 : X \to [0,\frac{1}{3}]$ continuous with $g_1|_A \equiv 0$ and $g_1|_B \equiv \frac{1}{3}$. Then $\forall x \in F, f(x) \le \frac{1}{3} \Rightarrow g_1(x) = 0, f(x) \ge \frac{2}{3} \Rightarrow g_1(x) = \frac{1}{3}$. Let $f_1 := f - g_1|_F$, then f_1 is continuous and $0 \le f_1(x) \le \frac{2}{3}$ by construction. Now repeat this fashion by replacing f_1 with f: there exists $g_2 : X \to [0,\frac{1}{3} \times \frac{2}{3}]$ continuous such that $\forall x \in F$, if $f_1(x) \le \frac{1}{3} \cdot \frac{2}{3} \Rightarrow g_2(x) = 0$ and if $f_1(x) \ge \frac{1}{3} \cdot \frac{2}{3} \Rightarrow g_2(x) = \frac{1}{3} \cdot \frac{2}{3}$. Let $f_2 := f_1 - g_2|_F$ with $0 \le f_2(x) \le \frac{1}{3} \cdot \frac{2}{3}$.

Continue this fashion, we finally terminate at a continuous function f_n with $f_n : X \to [0,(\frac{2}{3})^n]$ and there exists $g_{n+1} : X \to [0,\frac{1}{3} \cdot (\frac{2}{3})^n]$ such that $\forall x \in F$, $f_n(x) \le \frac{1}{3} \cdot (\frac{2}{3})^n \Rightarrow g_{n+1}(x) = 0, f_n(x) \ge \frac{2}{3} \cdot (\frac{2}{3})^n \Rightarrow g_{n+1}(x) = \frac{1}{3} \cdot (\frac{2}{3})^n$. Now let $f_{n+1} = f_n - g_{n+1}|_F$ with $f_{n+1} \in [0,\frac{2}{3} \cdot (\frac{2}{3})^{n+1}]$ and $0 \le g_n(x) \le \frac{1}{3} \cdot (\frac{2}{3})^{n-1}$, thus, $g(x) = \sum_{n=1}^{\infty} g_n(x)$ converges uniformly on X therefore it is continuous.

Then, $\forall x \in F, f(x) - g_1(x) = f_1(x), f_1(x) - g_2(x) = f_2(x), \cdots$, $f_{n-1}(x) - g_n(x) = f_n(x)$. If we add all these equalities, we obtain

$$f(x) - \big(g_1(x) + \cdots + g_n(x)\big) = f_n(x),$$

$\forall n \in \mathbb{N}, 0 = \lim_{n\to\infty} f_n(x) = f(x) - \sum_{n=1}^{\infty} g_n(x) = f(x) - g(x)$, thus, $g(x)$ is the desired extension of $f(x)$ and we are done.

□

We now use Tietze's Extension Theorem to show that Moore/Nemyski's plane X is not normal.

The Moore Plane is an example of a topological space that demonstrates the counterintuitive properties that can arise in topology. It is named after the American mathematician Robert Lee Moore. The Moore Plane is a famous example of a topological space that is connected and completely regular (or Tychonoff), yet it is not normal. This highlights the fact that normality is a stronger separation property than mere complete regularity.

The following version of the definition of Moore plane follows from [32], with detailed treatment available in both [33] and [34]:

Definition: Moore Plane

If Γ is the closed upper half-plane $\Gamma := \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\}$, then a topology may be defined on Γ by taking a local basis, namely, $B(p, q)$ as follows:

(i) Elements of the local basis at points (x, y) with $y > 0$ are the open discs in the plane which are small enough to be included by Γ.

(ii) Elements of the local basis with the form $p = (x,0)$ are defined to be the set $\{p\} \cup A$ where A is an open disc in the uppre half-plane which is tangent to x-axis at the point p.

That is to say, the local basis is given by

$$\begin{cases} \left\{U_\varepsilon(p, q) = \{(x, y) \,|\, (x-p)^2 + (y-q)^2 < \varepsilon^2\} \,\middle|\, \varepsilon > 0\right\}, q > 0 \\ \left\{V_\varepsilon(p) := \{(p,0)\} \cup \{(x, y) \,|\, (x-p)^2 + (y-\varepsilon)^2 < \varepsilon^2\} \,\middle|\, \varepsilon > 0\right\} q = 0 \end{cases}.$$

Thus the subspace topology inherited by $\Gamma \backslash \{(x,0) \,|\, x \in \mathbb{R}\}$ is the same as the subspace topology inherited from the standard topology of the Euclidean space.

The Moore Plane is connected, meaning that it cannot be partitioned into two disjoint nonempty open sets. The Moore Plane is completely regular, which means that for any closed set A and a point $x \notin A$ there exists a continuous function $f : \mathbb{R}^2 \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1 \,\forall y \in A$. The most remarkable property of the Moore Plane is that it is not a normal space. This means that there exist disjoint closed sets that cannot be separated by disjoint open sets. In other words, normality fails in the Moore Plane.

[Claim]: The Moore/Nemyski's plane X is not normal.

Define any continuous function $f : L := \{(x, y) \in \mathbb{R}^2 \,|\, y = 0\} \to [0,1]$. We now introduce a notation which is often used in mathematics, especially in functional analysis and PDE. We use $C(L, [0,1])$ to denote the collection of all the continuous functions $f : L \to [0,1]$.

Since L is closed in X if X is normal (In particular, L is a discrete topology). Then any function $f : L \to [0,1]$ extends to a continuous function, namely, $\tilde{f} : X \to [0,1]$. On the other hand, consider the set given by

$$R := \{(x, y) \in X \,|\, x, y \text{ are rational}\}.$$

Then $\overline{R} = X$ and R is countable. If $f, g : X \to [0,1]$ are two continuous functions such that $f\,|_R = g\,|_R \Rightarrow f = g$. Therefore,

$$\left|C(X, [0,1]\right| \leq \left|[0,1]^R\right| \leq \left|[0,1]^{\mathbb{N}}\right|,$$

but this is impossible since this is equivalent to say $\left|[0,1]^{\mathbb{R}}\right| \leq \left|[0,1]^{\mathbb{N}}\right|$.

Remark:

One important fact we used is that every functions from the discrete topologies are automatically continuous. ||

3.1 Metric Space

A metric space is a fundamental concept in mathematics, particularly in the field of analysis and topology. It provides a way to define and understand notions of distance, convergence, continuity, and other important properties within a given set.

We have introduced the notion of metric and the open ball in metric spaces, recall we used $B_r(x) := \{y \in X \mid d(x, y) < r\}$ to denote the open ball; in some literature people use $B(r, x)$ or $B(x, r)$ and sometimes call it r-ball or ball of radius r around x in X.

Definition: Subspace of Metric Space

Let (X, d) be a metric space. A subspace of X is a subset $Y \subseteq X$ with the metric obtained by restricting the one on X to Y.

Example 3.1:

We can view the set of rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the standard Euclidean metric $d(x, y) := |x - y|$. Note the following: the ball of radius 1 in $\mathbb{R}$ around $r \in \mathbb{Q}$ is the interval $(r-1, r+1)$, but the ball of radius 1 in $\mathbb{Q}$ is $(r-1, r+1) \cap \mathbb{Q}$. ||

Definition: Bounded

A subset U of a metric space (X, d) is said to be bounded if there exists a positive $r > 0$ and a point $x \in X$ such that $U \subseteq B_r(x)$.

Boundedness can also apply to functions defined on metric spaces.

Definition: Bounded (functions)

A function $f : (X, d) \to \mathbb{R}$ is said to be bounded if there exists a real number $N \in \mathbb{R}$ such that $|f(x)| \leq M \; \forall x \in X$.

Understanding boundedness is crucial for various reasons: It's a key concept in the study of compactness: A subset of a metric space is compact if and only if it's both closed and bounded. It affects the behavior of sequences and functions: Boundedness can impact the existence of limits, convergence, and continuity.

In many metric spaces, boundedness and compactness are closely related. A subset of $\mathbb{R}$ is compact if and only if it's closed and bounded. This connection highlights the role of boundedness in understanding the compactness of subsets.

Recall the notion of convergent sequences in a metric space, we assumed its uniqueness in the previous discussion, now we prove it is valid.

Proposition 3.1:

In any metric space, limits of convergent sequences are unique.

Proof:

Suppose that (X, d) is a metric space and that the sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converges to x and y. Let $\varepsilon > 0$.

Since $x_n \to x$, there exists an index $N_1 \in \mathbb{N}$ such that $d(x_n, x) < \dfrac{\varepsilon}{2}$ for $n \geq N_1$.

Similarly, if $x_n \to y$ there exists an $N_2 \in \mathbb{N}$ such that $d(x_n, y) < \dfrac{\varepsilon}{2}$ for $n \geq N_2$.
Take $N := \max\{N_1, N_2\}$. Then both inequalities above hold, and so the triangle inequality yields $d(x, y) \leq d(x, x_N) + d(x_N, y) < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$.
Since ε is chosen arbitrarily, let $\varepsilon \downarrow 0$, then $d(x, y) = 0 \Rightarrow x = y$.

□

Recall that we did not build up the same argument for the "nets" in the topological space, this is because: In general, the limit of a convergent net in a topological space is not necessarily unique. Unlike sequences in metric spaces, where limits are unique, nets can have multiple accumulation points in a topological space. This is because nets are more general than sequences and can capture more intricate convergence patterns.

In a non-Hausdorff topological space (where distinct points might not have disjoint open neighborhoods), a net can converge to multiple points. This is because there may be overlapping neighborhoods of different points. Even in Hausdorff spaces, a net can have multiple limit points, meaning that it converges to more than one point.

Remark:

It's important to note that in some cases, topological properties such as Hausdorffness or compactness can ensure unique convergence. For example, in a compact Hausdorff space, the convergence of nets is unique. ||

Proposition 3.2:

A convergent sequence in a metric space is bounded.

Proof:

Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in X such that $x_n \to x$. Since $x_n \to x$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < 1$ for $n \geq N$. Thus the terms in the sequence for $n \geq N$ are contained in $B_1(x)$.
Now we relax the condition on x_n, i.e. we enlarge the radius to include all the elements of the sequence $\{x_n\}_{n\in\mathbb{N}}$. To that end, set $r := 1 + \max\{d(x_1, x), \cdots, d(x_N, x), 1\}$. Take the open ball to be $B_r(x)$ yields the boundedness, result follows.

□

Proposition 3.3:

If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence such that $x_n \to x$ then $\forall y \in Y$, $d(x_n, y) \to d(x, y)$.

Proof:

Let $y \in X$. Since $x_n \to x \Rightarrow d(x_n, x) \to 0$. Let $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$d(x_n, y) \leq d(x_n, x) + d(x, y) < \varepsilon + d(x, y).$$

Similarly we can achieve the lower bound:

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \Rightarrow d(x, y) - d(x_n, x) \leq d(x_n, y).$$

For $n \geq N$ one has $d(x, y) - \varepsilon < d(x_n, y)$. Therefore,

$$d(x, y) - \varepsilon < d(x_n, y) < d(x, y) + \varepsilon \Rightarrow |d(x_n - y) - d(x, y)| < \varepsilon.$$

□

Corollary 3.3.1:

Let $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$.

Therefore it is natural to introduce the notion of boundedness, which we should derive after **Propositio 3.2**.

Definition: Bounded Sequence

A sequence $\{x_n\}$ in (X, d) is said to be bounded if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that $d(x_n, p) \leq B \, \forall n \in \mathbb{N}$.

Remark:

Similarly, we can derive a competitive definition for a subset to be bounded in a metric space: A subset $A \subseteq X$ is said to be bounded if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that $d(x, p) \leq B \, \forall x \in A$. ||

Definition: Cauchy sequence

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, d) is said to be a Cauchy sequence if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, $d(x_n - x_m) \leq \varepsilon$.

Proposition 3.4:

Every convergent sequence is a Cauchy sequence.

Proof:

Let $x_n \to x$ and let $\varepsilon > 0$. Then there exists an N such that $\forall n \geq N$ $d(x_n, x) < \frac{\varepsilon}{2}$. Hence, $\forall n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

A Cauchy complete metric space, often referred to as a complete metric space or just a complete space, is an important concept in the field of analysis and topology. It captures the idea of "completeness" of a metric space, which relates to the convergence of Cauchy sequences.

Definition: Cauchy Complete

A metric space (X, d) is said to be Cauchy complete if every Cauchy sequences are convergent.

Definition: Subsequence

A subsequence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space is a sequence $\{x_{n_k}\}_{n,k\in\mathbb{N}}$ consisting of terms of the sequence $\{x_n\}_{n\in\mathbb{N}}$ with $n_k > n_{k'} \Leftrightarrow k > k'$.

The final condition simply means that the terms in the subsequence are arranged in the same way as in the original sequence.

The limit of a subsequence, if it exists, is unique. This is consistent with the uniqueness of limits for sequences in metric spaces. Moreover, the limit of the subsequence coincides with the limit of the original sequence. But one should always bear in mind that a subsequence can diverge even if the original sequence converges.

Proposition 3.5:

Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in a metric space. Then any subsequence converges to the same limit.

Proof:

Suppose $x_n \to x$ in X and $\{x_{n_k}\}$ is a subsequence. Choose $\varepsilon > 0$. Since $x_n \to x$

there exists $N_1 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon \forall n \geq N_1$. Choose $N \in \mathbb{N}$ large enough so that $n_k \geq N_1$ for $k \geq N$ then $d(x_{n_k}, x) < \varepsilon$ and result follows.

□

Remark:

This proposition also implies that every subsequence of a convergent sequence is convergent. ||

Example 3.2: Singletons are closed

Let (X, d) be a metric sapce and $x \in X$. We now argue that $\{x\}$ is a closed set in X. That is to say, according to the definition of openness, $\forall y \in X \backslash \{x\}$ there is an open ball such that $U_r(y) \not\ni x$. Let $r := \dfrac{d(x,y)}{2}$, if $x \in B_r(y)$ then $d(x, y) < r < d(x, y)$, contradiction, therefore $B_r(y) \subseteq X \backslash \{x\}$. ||

Remark:

Similarly, one can prove that any finite subset of a metric sapce is closed. ||

Proposition 3.6: Convergence Criterion

Let $\{x_n\}$ be a sequence in (X, d). Then $\{x_n\}$ is convergent $\Leftrightarrow \forall \varepsilon > 0$, all but finitely many terms in $\{x_n\}$ are in $(x - \varepsilon, x + \varepsilon)$.

Proof:

"⇒":

Given $x_n \to x$ and $\varepsilon > 0$ $\exists N$ such that $\forall n \geq N$, $|x_n - x| < \varepsilon$. Therefore, $\forall n \geq N.\ x_n \in (x - \varepsilon, x + \varepsilon)$.

"⇐":

For the other direction, fix an arbitrary $\varepsilon > 0$ and consider the open interval $(x - \varepsilon, x + \varepsilon)$. Given that all but finitely many terms in $\{x_n\}$ are in the interval, it follows that $\exists M$ such that $\forall n \geq M$, $x_n \in B_\varepsilon(x)$ therefore x_n is convergent.

□

Theorem 3.7:

Let $\{x_n\}$ be a sequence in a metric space (X, d). Then x_n is convergent to $x \Leftrightarrow$ for every neighbourhood of x, all but finitely many terms in $\{x_n\}$ are not in the neighbourhood of x.

Remark:

It is not hard to derive that every closed set has the property that every convergent sequence converges in the set. ||

Recall the notion of continuous in metric space: a function $f : (X, d_X) \to (Y, d_Y)$ is said to be continuous if $\forall x \in X, \forall \varepsilon < 0 \exists \delta > 0$ such that

$$d_X(x, y) \leq \delta \Rightarrow d_Y(f(x), f(y)) \leq \varepsilon.$$

In metric spaces, continuity is defined using the metric structure to capture the notion of "closeness." Topological spaces generalize the concept of metric spaces by considering open sets instead of distances. In a topological space, a function is considered continuous if it preserves open sets. The relationship between continuity in metric spaces and continuity in topological spaces is as follows:

Every continuous function in a metric space is also continuous in the associated topological space. The topology induced by the metric defines open sets, and a function that preserves distances also preserves open sets.

In general, not every continuous function in a topological space can be equipped with a metric such that it remains continuous. This is because the metric topology is a specific case of the general topological structure and might not capture all possible continuous functions in the topological space.

3.2 Compact Metric Space

Let us first talk about the compactness in the space $\mathbb{R}^n$ since most of the materials are familiar. Recall the notion $C(\mathbb{R})$, which represents the collection of all continuous function over the field $\mathbb{R}$.

Definition: Support

The support of a function $f \in C(\mathbb{R})$ is the set $\overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$.

Note that in general, the support of a function is taken to be $\{x \in \mathbb{R} \mid f(x) \neq 0\}$. Different from the one in $\mathbb{R}$, the support of a function is defined as the closure of the set of points where the function is nonzero to capture both the nonzero points and the points where the function approaches zero in a continuous manner.

Definition: Sequentially Compact

Let (X, d) be a metric space. A set $A \subseteq X$ is said to be sequentially compact if every sequence in A has a convergent subsequence in A.

Definition: Topologically Compact

Let (X, d) be a metric space. A set $A \subseteq X$ is said to be topologically compact if every open cover of A has a finite subcover.

Notaion:

In some literature, the authors may use $A \Subset X$ to denote that A is a compact subset of X.

Example 3.3:

(i) $\mathbb{R}$ is not a compact subset of $\mathbb{R}$.
(ii) $(0,1]$ is not compact or sequentially compact in $\mathbb{R}$.
(iii) $[0,1]$ is compact subset of $\mathbb{R}$.

We now introduce some important result along without proof. Then we will discuss more general situations in metric spaces.

Theorem 3.8:

Compact sets in $\mathbb{R}$ are closed and bounded.

Lemma 3.9:

A compact set in a metric space (X, d) is closed and bounded.

Is the converse also true? The answer is, in general, the converse does not hold. In fact, if this metric space is $\mathbb{R}$, then there is a theorem called Heine-Borel tells us the converse is true. However, in general metric spaces, the Heine-Borel theorem doesn't necessarily hold, so closed bounded sets are not guaranteed to be compact.

Theorem 3.10: Heine-Borel

Let K be a subset of $\mathbb{R}$. Then K is compact $\Leftrightarrow$ K is closed and bounded.

Proof:

We konw that compact implies closed and bounded according to **Theorem 3.8**. We thus need to prove the other direction. Let K be a closed and bounded subset of $\mathbb{R}$. Then, given K is bounded, K is contained in some closed interval, namely $[a, b]$, which we have shown to be compact. Hence $K$is a closed subset of a compact set, therefore compactness of K follows.

□

At this point, one may wonder why we mention the idea of sequential compactness, and how this actually relates to the idea of topological compactness. Firstly, recall the Bolzano-Weierstrass Theorem, which is a powerful result that guarantees the existence of a convergent subsequence for any bounded sequence of real numbers.

Theorem 3.11: Bolzano-Weierstrass

Every bounded sequenece in $\mathbb{R}^n$ has a convergent subsequence.

Lemma 3.12:

Consider $A \subseteq \mathbb{R}^n$ such that A is closed and bounded. Then A is sequentially compact.

Proof:

Let $\{x_n\}$ be a sequence in A. Then $\{x_n\}$ is bouded as A is bounded, and thus by Bolzano-Weierstrass, there exists a convergent subsequence of $\{x_n\}$. Then use the fact that A is closed and every sequence in A has a convergent subsequence, then the result follows.

□

In fact, the converse of **Lemma 3.12** is also true.

Theorem 3.13: Bolzano-Weierstrass

Let K be a subset of $\mathbb{R}$. Then K is sequentially compact $\Leftrightarrow$ K is closed and bounded.

Proof:

We have shown the $\Leftarrow$ direction. Let K be a sequentially compact subset of $\mathbb{R}$. Let $\{x_n\}$ be a sequence in K that converges to x in $\mathbb{R}$. Then every subsequence of $\{x_n\}$ converges to x. Therefore $x \in K$, since x is chosen arbitrarily, K contains all the limit points, hence closed.

Suppose K is not bounded, then there exists a sequence $\{x_n\}$ in K such that $|x_n| \to \infty$ as $n \to \infty$. Then every subsequence of $\{x_n\}$ is unbounded and diverges, thus $\{x_n\}$ has no convergent subsequence, contradiction.

□

Remark:

In fact, one can generalize the result of **Theorem 3.13** into $\mathbb{R}^n$. ||

Corollary 3.13.1:

Given $A \subseteq \mathbb{R}$, A is sequentially compact $\Leftrightarrow$ A is topological compact.

We now know that a set in $\mathbb{R}^n$ is sequentially compact $\Leftrightarrow$ it is topologically compact, this is true by showing

sequentially compact $\Leftrightarrow$ closed and bounded $\Leftrightarrow$ topologically compact,

where the last "$\Leftrightarrow$" uses the Heine-Borel Theorem. However, as the previous remark shows, we do not have Heine-Borel in the general metric spaces. Then the question arises: is sequentially compact equivalent to topologically compact in metric spaces?

The answer is yes. To prove this, recall we proved Lebesgue Number Lemma in **Lemma 1.58**.

The Lebesgue Number Lemma states that for any open cover of a compact metric space, there exists a positive real number (the Lebesgue number) such that every subset of the metric space with diameter less than the Lebesgue number can be completely covered by a single set from the open cover.

Now we introduce the concept called totally bounded, which extends this concept to metric spaces in general, whether they are compact or not. It ensures that the entire space can be covered by small subsets (open balls or open sets) with diameters less than a given positive number.

Definition: Totally bounded

A metric space X is totally bounded if $\forall \varepsilon > 0$ there exist $x_1, x_2, \cdots, x_n \in X$ such that $\{B_\varepsilon(x_i) \mid 1 \leq i \leq n\}$ is an open cover of X.

Properties:

(i) Totally bounded spaces are always bounded (since the diameter of any subset in the cover is limited).

(ii) Totally boundedness is a crucial concept when defining completeness and compactness in metric spaces.

Lemma 3.14:

If a metric space X is sequentially compact then it is totally bounded.

Proof:

Assume that X is sequentially compact and not totally bounded. Then there exists an $\varepsilon > 0$ such that X cannot be covered by a collection of open sets of finitely many ε-balls. Hence $\forall x_1 \in X, x_1 \in X \backslash B_\varepsilon(x_1), x_3 \in (X \backslash B_\varepsilon(x_1)) \backslash B_\varepsilon(x_2)$ and so on. Then $\forall i \neq j, d(x_i, x_j) \geq \varepsilon$ and $\{x_n\}$ has no convergent subsequence as otherwise it would be Cauchy, contradiction.

□

Theorem 3.15:

A metric space X is (topologically) compact ⇔ it is sequentially compact.

Proof:

"⇐":

Let X be sequentially compact and let $\{U_i\}_{i \in I}$ be an open cover of X. By the Lebesgue number lemma(**Lemma 1.58**), there exists an $r > 0$ such that $\forall x \in X, B_r(x) \subseteq U_i$ for some $i \in I$. Now by **Lemma 3.14**, X is totally bounded. Hence there exist $y_1, \cdots, y_n \in X$ such that

$$X \subseteq B_r(y_1) \cup \cdots \cup B_r(y_k).$$

However, $\forall i \in I, B_r(y_i) \subseteq U_{j(i)}$ for some $j(i) \in I$. Thus $\{U_{j(1)}, \cdots, U_{j(n)}\}$ is a finite subcover for X. Since $\{U_i\}$ is chosen arbitrarily, compactness follows.

"⇒":

Assume that there exists a sequence $\{x_n\}$ in X with no convergent subsequence. Notice that no term in the sequence can appear infinitelymany times, as otherwisethere would be a trivial subsequence of $\{x_n\}$. Hence, we may assume, without loss of generality, that $x_i \neq x_j \forall i \neq j$. Since for every n

there exists an $\varepsilon_n > 0$ such that $B_{\varepsilon_n}(x_n)$ contains no other terms in the sequence. If not, there would again be a convergent subsequence, hence for all i, there exists an open ball U_i centered at x_i such that $x_j \notin U_i$ for all $i \neq j$.

Now consider $U_0 := X \backslash \{x_n | n \in \mathbb{N}\}$. U_0 is open since $U_0^c = \{x_n | n \in \mathbb{N}\}$ is closed (it contains all of its limit points). Hence $U_0 \cup \{U_n | n \in \mathbb{N}\}$ is an open cover of X. However, this open cover has no finite subcover as any finite collection of the cover fail to include infinitely many terms from the sequence $\{x_i\}$, contradiction.

□

Recall that for X and Y two metric spaces, a continuous function $f : X \to Y$, then for all open subset $U \subseteq Y, f^{-1}(U)$ is also open in X.

Theorem 3.16:

Let X and Y be metric spaces and $f : X \to Y$ be a continuous map. If $K \subseteq X$ is a compact subset then $f(K) \subseteq Y$ is also compact.

Proof:

Let $\{U_i\}$ be an open cover of $f(K)$. Then define $V_i := \{f^{-1}(U_i)\}$ which is open by the continuity of f. Therefore $\{f^{-1}(U_i)\}$ is an open cover of K. Hence there exists a finite subcover $\{V_{i_1}, \cdots, V_{i_n}\}$ of K as K is compact. Thus $\{U_{i_1}, \cdots, U_{i_n}\} = \{f(V_{i_1}), \cdots, f(v_{i_n})\}$ is a finite subcover.

□

Corollary 3.16.1:

Let X be a metric space and $K \subseteq X$ be a compact subset. Then given a continuous function $f : X \to \mathbb{R}$, f obtains a maximum and minimum finite value on K.

Remark:

Sometimes in particular we want to study bounded continuous functions, and this corollary gives us a nice property: Given a compact metric space X, every continuous function f is bounded. ||

Now we state another useful result to end this subsection.

Theorem 3.17:

Given a metric space (X, d), the followings are equivalent:

(i) X is compact.
(ii) X is sequentially compact.
(iii) X is Cauchy complete and totally bounded.
(iv) Every collection of closed subsets of X with F.I.P. has a non-empty intersection.

Proof: One may consult [42].

3.3 Complete Metric Spaces

Recall that in the last subsection introduced the notion of Cauchy complete metric spaces. We remarked before that the completeness is a property in metric space rather than a topological one, however, there are still a number of theorems involving complete metric spaces that are topological.

Completeness is a fundamental concept in the theory of metric spaces, serving as a generalization of the idea of convergence. A metric space is complete if every Cauchy sequence in the space converges to a point within the space itself. Completeness captures the idea of "filling in the gaps" in a metric space, ensuring that no points are missing from the space even when considering sequences that come arbitrarily close to each other.

Definition: Complete (of metric spaces)

A metric space (X, d) is said to be complete if every Cauchy sequence in X converges in X.

Any convergence sequence in X is necessarily a Cauchy sequence, of course; completeness requires that the converse hold. Note that a closed subset A of a complete metric space (X, d) is necessarily complete in the restricted metric. For a Cauchy sequence in A is also a Cauchy sequence in X, hence it converges in X. Because A is a closed subset of X, the limit must lie in A. Now we introduce the first completeness criterion:

Lemma 3.18:

A metric space (X, d) is complete if every Cauchy sequence in X has a convergent subsequence.

Proof:

Let $\{x_n\}$ be a Cauchy sequence in (X, d). We show that if $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to a point x, then the sequence $\{x_n\}$ itself converges to x. Given $\varepsilon > 0$, first choose N large enough such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ $\forall n, m \geq N$. Then choose an integer i large enough such that $\forall n_i \geq N\ d(x_{n_i}, x) < \frac{\varepsilon}{2}$. Then we have the desired inequality:

$$d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x) < \varepsilon.$$

□

Remark:

The Euclidean space $\mathbb{R}^k$ is complete in the Euclidean metric d. ||

Now we deal with the product space $\mathbb{R}^\omega$. Before that, we need a lemma about sequences in a product space.

Lemma 3.19:

Let X be a product space $X := \prod X_\alpha$ and let $\{x_n\}$ be a squence of points of X. Then $x_n \to x \Leftrightarrow \pi_\alpha(x_n) \to \pi_\alpha(x)\ \forall \alpha$.

Proof:

"⇐":

Since the projection mapping $\pi_\alpha : X \to X_\alpha$ is continuous, it preserves the convergent sequences.

"⇒":

Suppose that $\pi_\alpha(x_n) \to \pi_\alpha(x)\ \forall \alpha$. Let $U := \prod U_\alpha$ be a basis element for X that contains x. For each α for which U_α does not equal to the entire space X_α, choose N_α so that $\pi_\alpha(x_n) \in U_\alpha\ \forall n \geq N_\alpha$. Let N be the largest of the numbers

N_α, then $\forall n \geq N$, one has $x_n \in U$, result follows.

□

Theorem 3.20:

There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is complete.

Proof:

Let $\bar{d}(a,b) := \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. Let D be the metric on $\mathbb{R}^\omega$ defined by

$$D(x,y) := \sup\{\bar{d}(x_i, y_i)/i\}.$$

Then D induces the product topology on $\mathbb{R}^\omega$; we verify that $\mathbb{R}^\omega$ is complete under D.

[Claim]: $\mathbb{R}^\omega$ is complete under D.

Let $\{x_n\}$ be a Cauchy sequence in $(\mathbb{R}^\omega, D)$, since

$$\bar{d}(\pi_i(x), \pi_i(y)) \leq iD(x,y).$$

We see that for fixed i the sequence $\pi_i(x_n)$ is a Cauchy sequence in $\mathbb{R}$ hence the convergence is for certain, namely, say a_i. Then the sequence x_n converges to the point $a = (a_1, a_2, \cdots)$ of $\mathbb{R}^\omega$.

□

Remark: the space $\mathbb{R}^\omega$

For those who are not familiar with the choice of notion "$\mathbb{R}^\omega$" and the notion "$\mathbb{R}^n$", we now give the clarification:

$\mathbb{R}^\omega$ represents the infinite Cartesian product of real number spaces. Each element of $\mathbb{R}^\omega$ is an infinite sequence of real numbers. The topology on $\mathbb{R}^\omega$ is typically given by the product topology, where open sets are generated by cylinders (sets of sequences that agree with a given finite sequence at the first n terms). Moreover, $\mathbb{R}^\omega$ is a very large space with specific properties. It is not locally compact, not separable, and not metrizable under its product topology.||

Although both the product spaces $\mathbb{R}^n$ and $\mathbb{R}^\omega$ have metrics relative to which they are complete, one cannot hope to prove the same result for the product space $\mathbb{R}^J$ in general, since $\mathbb{R}^J$ is not even metrizable if J is uncountable. There is, however, another topology on the set $\mathbb{R}^J$, the one given by the "uniform metric". Relative to this metric, $\mathbb{R}^J$ is complete. ($\mathbb{R}^J : \{f : J \to \mathbb{R} \mid f \text{ continuous}\}$.)

The uniform metric, also known as the supremum metric or the L^∞ metric, is a way of measuring the distance between functions in a function space. It is commonly used in the context of spaces of bounded functions. The uniform metric defines convergence and distance based on the supremum (least upper bound) of the pointwise differences between functions.

Definition: Uniform Metric

Given a set X and a space of functions F defined on X, the uniform metric d_∞ defined on F is given by: For two functions $f, g \in F$,

$$d_\infty(f,g) := \sup_{x \in X} |f(x) - g(x)|.$$

In other words, the distance between two functions is the supremum (maximum) of the absolute differences between their values at each point in the domain.

We now talk more on the uniform metric as well as the so-called "uniform topology". The following discussion is a work of [44]:

The uniform topology on X^J is the metric topology induced by the uniform metric. The uniform topology is finer than the pointwise topology but coarser than the compact-open topology. Moreover, the uniform topology provides a framework for discussing concepts related to uniform convergence and uniform continuity of functions. Now we introduce the first result on this topic, that the product topology on X^J is weaker than the uniform topology on X^J.

Theorem 3.21:

If J is a set and (X, d) is a metric space, then the uniform topology on X^J is finer than the product topology on X^J.

Proof:

If $x \in X^J$, let $U := \prod_{j \in J} U_j$ be a basic open set in the product topology with $x \in U$. Thus, there is a finite subset $J_0 \subseteq J$ such that if $j \in J \backslash J_0$ then $U_j = X$. If $j \in J_0$, then since U_j is an open subset of (X, d) with the metric topology and $x_j \in U_j$, there is some $0 < \varepsilon_j < 1$ such that $B^d_{\varepsilon_j}(x_j) \subseteq U_j$. Let $\varepsilon := \min_{j \in J_0} \varepsilon_j$. If $d_\infty(x, y) < \varepsilon$ then $d(x_j, y_j) < \varepsilon \forall j \in J$ and hence $d(x_j, y_j) < \varepsilon_j \forall j \in J_0$, which implies that $y_j \in U_j$. Therefore, if $y \in B^{d_\infty}_\varepsilon(x)$ then $y \in U$, i.e. $B^{d_\infty}_\varepsilon(x) \subseteq U$ $\forall j \in J_0$. If $j \in J \backslash J_0$ then $U_j = X$ and $y_j \in U_j$ thereafter. Therefore, if $y \in B^{d_\infty}_\varepsilon(x)$ then $y \in U$, i.e. $B^{d_\infty}_\varepsilon(x) \subseteq U$. It follows that the uniform topology on X^J is finer than the product topology on X^J.

□

The following theorem shows that if we take the product of a complete metric space with itself, then the uniform metric on this product space is complete.

Theorem 3.22:

If J is a set and (X, d) is a complete metric space, then X^J with the uniform metric d_∞ is a complete metric space.

Proof:

It is straightforward to check that (X, d) being a complete metric space implies that $(X, \bar{d})$ is a complete metric space (recall $\bar{d}(a, b) := \min\{|a - b|, 1\}$). Let f_n be a Cauchy sequence in (X^J, d_∞): if $\varepsilon > 0$ then there is some N such that $\forall n, m \geq N$ one has $d_\infty(f_n, f_m) < \varepsilon$. Thus, if $\varepsilon > 0$, there is some N such that $\forall n, m \geq N$ and $j \in J$ then $\bar{d}(f_n(j), f_m(j)) \leq d_\infty(f_n, f_m) < \varepsilon$. Thus if $j \in J$ then $f_n(j)$ is a Cauchy sequence in $(X, \bar{d})$, which therefore converges to some $f(j) \in X$, and thus $f \in X^J$. If $n, m \geq N$ and $j \in J$, then

$$\begin{aligned}\bar{d}(f_n(j), f_m(j)) &\leq \bar{d}(f_n(j), f_m(j)) + \bar{d}(f_m(j), f(j)) \\ &\leq d_\infty(f_n, f_m) + \bar{d}(f_m(j), f(j)) \\ &< \varepsilon + \bar{d}(f_m(j), f_n(j)).\end{aligned}$$

As the LHS does not depend on m and $\bar{d}(f_m(j), f(j)) \to 0$, one gets that if

$n \geq N$ and $j \in J$ then $\bar{d}(f_n(j), f(h)) \leq \varepsilon$. Therefore, if $n \geq N$ then $d_\infty(f_n, f) \leq \varepsilon$. This means that f_n converges to f in the uniform metric, showing that (X^J, d_∞) is a complete metric space as we desire.

□

Now let us generalize somewhat, and consider the set Y^X where X is a topological space rather than merely a set. Of course, this has no effect on what has gone so far; the topology of X is irrelevant when considering the set of all functions $f : X \to Y$. But suppose that we consider the subset $C(X, Y)$ of Y^X consisting of all continuous functions $f : X \to Y$. It turns out that if Y is complete, this subset is also complete in the uniform metric. The same holds for the set $B(X, Y)$ of all bounded functions $f : X \to Y$. (A function f is said to be bounded if its image $f(X)$ is a bounded subset of the metric space (Y, d)).

Theorem 3.23:

Let X be a topological space and let (Y, d) be a metric space. The set $C(X, Y)$ of continuous functions is closed in Y^X under the uniform metric. So is the set $B(X, Y)$ of bounded functions. Therefore, if Y is complete, these spaces are complete in the uniform metric.

Proof: Consult [1], **Theorem 43.6**.

Now we arrive at the most important result of this subsection. We will offer a result declaring the existence of an isometric between a metric space and a complete metric space.

Theorem 3.24:

Let (X, d) be a metric space. There is an isometric imbedding of X into a complete metric space.

Proof: Consult [1], **Theorem 43.7**.

We introduced the notion "isometric" in the above theorem, which is a type of mapping between metric spaces that preserves the distances between points. Intuitively, an isometry is a function that doesn't distort the geometric shape of the space it operates on. Isometries are used to study the preservation of geometric properties under certain transformations.

Definition: Isometry

Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is said to be an isometry if $d_X(x, y) = d_Y(f(x), f(y)) \,\forall x, y \in X$.

Isometries are distance-preserving, meaning they preserve the metric structure of the spaces they operate on. Moreover, Isometries are injective (one-to-one) since distinct points in X must map to distinct points in to preserve distances. However, isometries need not be surjective (onto).

Remark:

There are two types of isometry:

(i) Isometric Embedding

An isometry $f : X \to Y$ that is also a surjective map is called an isometric embedding. It essentially preserves the entire geometric structure of X within Y.

(ii) Isometric Isomorphism

If an isometry $f : X \to Y$ is both injective and surjective, it is called an isometric isomorphism. Isometric isomorphisms establish a bijective correspondence between the two spaces while preserving distances. ||

Hence we can give an important remark corresponding to **Theorem 3.24**.

Remark: of **Theorem 3.24**.

The isometry between a metric space and its completion is unique up to isomorphism. This property is known as the universal property of the completion of a metric space. ||

Definition: Completion

Let (X, d) be a metric space. If $h : X \to Y$ is an isometric imbedding of X into a complete metric space Y, then the subspace $\overline{h(X)}$ of Y is a complete metric space. It is called the completion of X.

We now give two important results to conclude this subsection. In fact, in the following theorem, one may use the notion of $\overline{M}$ as an alternative defintion for the completion.

Theorem 3.25:

Every metric space X with fixed metric has a unique metric space $\overline{M}$ (with respect to the metric) such that:

(i) $M \subseteq \overline{M}$.

(ii) The metric on $\overline{M}$ restricts to the metric on M.

(iii) $\overline{M}$ is Cauchy complete, and the closure of M is $\overline{M}$.

The proof of this theorem relies on the following lemma:

Lemma 3.26:

For any metric space M, the space of all bounded continuous functions denoted as $C_\infty(M) := \{f : M \to \mathbb{R} \,|\, f \text{ is continuous and } \sup_{m \in M} |f(m)| < \infty\}$ is a metric space with the uniform metric d_∞. In fact, it is Cauchy complete.

Proof:

We start by mapping M to a subset of $C_\infty(M)$. Choose a point $m' \in M$ and consider the map

$$M \ni m \mapsto g_m(p) = d(p, m) - d(p, m').$$

Notice that this map is bijective and

$$d(m_1, m_2) = \sup_{p \in M} |g_{m_1}(p) - g + m_2(p)| = d_\infty(g_{m_1}, g_{m_2}).$$

This means that the map is isometric and what this allows us to do is view M as a subset of$C_\infty(M)$.

Now we consider the closure of M'. Recall that a closed subset of a complete metric space is complete itself, so the closure of M', denoted by $\overline{M'}$, is Cauchy complete. By the isometry this implies that there exists the completion of the metric space M.

□

Comment:

Completing a metric space is a fundamental concept in the study of metric spaces. The completion of a metric space involves constructing a larger metric space that includes the original space as a dense subspace while ensuring that all Cauchy sequences in the original space converge within the completion. This process is used to address the issue of incomplete metric spaces and to create a space where all Cauchy sequences have limits.

Completion Process:

Given an incomplete metric space (X, d_X), the completion of X to $\overline{X}$ is:

Dense Embedding:

There exists an isometric embedding $i : X \to \overline{X}$ such that $i(X)$ is dense in $\overline{X}$.

Cauchy Sequence Convergence:

For any Cauchy sequence $\{x_n\}$, the sequence converges in $\overline{X}$ to a limit that also belongs to $\overline{X}$.

To the author's own perspective, using $\overline{X}$ to represent the completion of the original metric space X could cause confusion. That is, is it true that the completion of the metric space is equivalent to taking the closure of the original space? The answer is NO, the completion of a metric space is not equivalent to taking the closure of the original space. The completion process involves more than just taking the closure.

While both concepts involve considering limits, they address different issues: The completion of a metric space is about ensuring that all Cauchy sequences have limits by introducing new elements to the space. Taking the closure of a set is about considering the limit points of the set itself, ensuring that no limit points are "missing" from the set.

In summary, the completion of a metric space and taking the closure of a set are distinct concepts that serve different purposes. The completion process is about creating a larger space to accommodate all Cauchy sequence limits, while taking the closure focuses on the limit points of a given set within the same space.

3.4 Metric Topology

It might be ambiguous for beginners to think about the exact differences between metric spaces and topological spaces at the first sight. We have introduced one important difference between these two concepts: that is, with convergence being part of the topological properties, the completeness, however, is not a topological property, but a metric space property.

The construction of topology, as we see in the first chapter, depends on the construction of open sets, and the open sets cannot be identified without the notion of metric. These two concepts have been entwined so far, in fact, the topological spaces are "bigger" than the metric spaces since metric spaces are special types of topological spaces.

In this subsection, we first offer the formal definition of the metric topology, which should be very familiar to you already, then we proceed to the discussion of metrizability, which, again, we have proved in **Urysohn's Metrization Lemma**, the next goal of this subsection is the discussion in boundedness, and we argue that the standard bounded metric induces the same topology as the original metric does. Then we shall enclose this section by discussion about some results derived from the metrizability.

Definition: Metric Topology

Let (X, d_X) be a metric space, then the collection T_X of open balls $B_\varepsilon(x) := \{y \in X \mid d_X(x, y) < \varepsilon\}$ with the radius $\varepsilon > 0$ is a basis for a topology on X called the metric topology induced by the metric d_X.

Every metric spaces automatically induce a topology, however, as we noted in the first chapter, there are topologies coming from no metric. Loosely speaking, the topological spaces are "bigger" than the metric spaces, but this does not necessarily mean that the topological spaces are more complicated than metric spaces. They are derived and studied due to different purposes:

Topological Spaces:

Topological spaces are more general than metric spaces. They define open sets and neighborhoods without relying on a specific notion of distance (as in metrics). This generality allows them to capture a broader range of mathematical objects and phenomena. Moreover, topological spaces provide an abstract framework for studying continuity, convergence, and connectedness. Furthermore, in topological spaces, the topology can be very complicated, leading to exotic topological properties like non-metrizability, non-Hausdorff spaces, and non-separability. This complexity can make the study of certain topological spaces challenging.

Metric Spaces:

Metric spaces are more specific than topological spaces because they rely on a metric, which defines a notion of distance between points. This added structure makes them suitable for studying concepts related to distance, convergence, and continuity. Moreover, metric spaces have a uniform structure due to the metric, which can simplify proofs and calculations. For example, in a metric space, one can use concepts like open balls to analyze neighborhoods and limits.

Remark:

Metric topology provides a specific and well-defined connection between the concepts of distance, convergence, continuity, and open sets in the context of topological spaces. The metric structure gives rise to a particular topology, which is often referred to as the "metric topology." ||

As we see, there are some topologies equipped with no metric structures. Therefore, the motivation behind studying metrizability in topology is rooted in the desire to understand the relationship between topological spaces and metric spaces, and to determine which topological spaces can be endowed with a metric structure. We have proved the **Urysohn's Metrization Theorem**, now we give a formal definition of being metrizable:

Definition: Metrizable

If X is a topological spcae, then X is said to be metrizable if there exists a metric d_X on the set X that induces the topology on X. In particular, a metric space is a metrizable space X together with a specific metric d that gives the topology of X.

We have introduced the convergence, which is a metric space property, now we shall introduce another one, which might be surprising, that the boundedness of a set is not a topological property, but a metric space property, for it depends on the particular metric d_X that is used for X.

Definition: Bounded

Let (X, d_X) be a metric space with metric d_X. A subset A of X is said to be bounded if there is some number M such that

$$d(x_1, x_2) \leq M \ \forall x_1, x_2 \in X.$$

Theorem 3.27: Standard Bounded Metric

Let (X, d_X) be a metric space. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation given by

$$\bar{d}(x, y) := \min\{d(x, y), 1\}.$$

Then $\bar{d}$ is a metric that induces the same topology as d. Then $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof:

Checking the first two conditions for a metric is trivial. Let us check the triangle inequality:

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z).$$

If now either $d(x, y) \geq 1$ or $d(y, z) \geq 1$, then the RHS is at least 1, then it holds since the LHS is at most 1. It remains to consider the case in which $d(x, y) < 1$ and $d(y, z) < 1$.

In this case, we have

$$d(x, z) \leq d(x, y) + d(y, z) \leq \bar{d}(x, y) + \bar{d}(y, z).$$

Now we note that in any metric space, the collection of ε-balls with $\varepsilon < 1$ forms a basis for the metric topology, for every basis element containing x contains such an ε-ball centered at x. It follows that d and $\bar{d}$ induce the same topology on X, since the collections of ε-balls under these two metrics coincide.

□

Now we offer a criterion in telling what metric topology is finer than the other:

Lemma 3.28:

Let d and d' be two metrics defined on the set X; let T and T' be the topologies they induce, respectively. Then T' is finer than $T \Leftrightarrow \forall x \in X \ \forall \varepsilon > 0 \, \exists \delta > 0$ such that

$$B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon).$$

Proof:

"⇒":

Suppose T is finer than T'. Given the basis element $B_d(x, \varepsilon)$ for T, there is a basis element B' for the topology T' such that $x \in B' \subseteq B_d(x, \varepsilon)$. Within B' one can find a ball $B_{d'}(x, \delta)$ centered at x.

"⇐":

Conversely, suppose the RHS holds. Given a basis element B of T containing x, we can find within B a ball $B_d(x, \varepsilon)$ centered at x. By the given condition, there is a δ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$, result follows.

□

Subspaces of metric spaces behave the same way one would expect; if A is a subspace of the topological space X and d_X is a metric for X, then the restriction of d to $A \times A$ is a metric for the topology of A.

About order topologies there is nothing to say; some are metrizable ($\mathbb{Z}^+$ and $\mathbb{R}$) and others are not.

The Hausdorff axiom is satisfied by every metric topology. If $x \neq y$ are distinct points of the metric space (X, d_X), let $\varepsilon := \frac{1}{2}d(x, y)$, then the triangle inequality implies that $B_{d_X}(x, \varepsilon)$ and $B_{d_X}(y, \varepsilon)$ are disjoint.

We now state and prove some results by assuming a topological space to be also metrizable.

Lemma 3.29: The Sequence Lemma

Let X be a topological space; let $A \subseteq X$ be a subset. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.

Proof:

Suppose that such a sequence is $x_n \to x$ where $x_n \in A$. Then every neighbourhood U_x contains a point of A, so $x \in \overline{A}$. Conversely, suppose that X is metrizable and $x \in \overline{A}$. Let d be a metric for the topology of X. For each positive integer n, take the neighbourhood $B_{d_X}(x, \frac{1}{n})$ of radius $\frac{1}{n}$ of x, and choose x_n to be a point of its intersection with A. We assert that the sequence x_n converges to x. Any open set U containing x contains an ε-ball $B_{d_X}(x, \varepsilon)$; if we choose N so that $\frac{1}{N} < \varepsilon$ then U contains $x_i \forall i \geq N$.

□

Theorem 3.30:

Let $f : X \to Y$ be a function. If f is continuous then every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Proof:

Assume that f is continuous. Given $x_n \to x$, we wish to show that $f(x_n) \to f(x)$. Let V be a neighbourhood of $f(x)$. Then $f^{-1}(V)$ is a neighbourhood of x, and so there is an N such that $x_n \in f^{-1}(V) \forall n \geq N$. Then $f(x_n) \in V \ \forall n \geq N$.

Conversely, assume that the convergent sequence condition is satisfied. Let A be a subset of X. We wish to show that $f(\overline{A}) = \overline{f(A)}$.

[Claim]: $f(\overline{A}) \subseteq \overline{f(A)}$.

If $x \in \overline{A}$, then there is a sequence x_n of points of A converging to x by **Lemma 3.29**. By assumption, the sequence $f(x_n) \to f(x)$. Since $f(x_n) \in f(A)$, by **Lemma 3.29** again, $f(x) \in \overline{f(A)}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$ as we desired.

□

3.5 Compactification

Recall that a topological space is said to be locally compact if every point has a compact neighbourhood.

Lemma 3.31:

Let X be a locally compact Hausdorff space (L.C.H.) and $x \in X$ is a point. Then for all neighbourhood U of x, there is a compact neighbourhood N of x with $N \subseteq U$.

Proof:

Since X is locally compact, there exists a compact neighbourhood C of x, thus there exists an open neighbourhood V of x with $V \subseteq C$.
Let $W = V \cap U$, since X is Hausdorff and C is compact, C is closed. Therefore $W := V \cap U \subseteq C \Rightarrow \overline{W} \subseteq C \Rightarrow \overline{W}$ is compact. Moreover, since X is compact and Hausdorff, it is automatically regular, thus $\overline{W}$ is also regular.
Regular spaces have neighbourhood bases of closed sets. There exists a compact neighbourhood N of x in $\overline{W}$ which is closed in $\overline{W}$, with

$$x \in N \subseteq W \subseteq \overline{W},$$

since $\overline{W}$ is compact, then N is also compact.
[Claim]: N is a compact neighbourhood of x in X.

Since N is a neighbourhood of x in $\overline{W}$, there exists an open subset $T \subseteq \overline{W}$ such that $x \in T \subseteq U$. Since T is open in $\overline{W}$, there exsists an open subset $O \subseteq X$ such that $T = \overline{W} \cap O$ (subspace topology). $x \in N \subseteq W$, $x \in W \cap O \subseteq \overline{W} \cap O = T \subseteq N$, since $W \cap O$ is open in X and $x \in W \cap O$, then N is a compact neighbourhood of x in X.

□

Corollary 3.31.1:

Let X be a locally compact Hausdorff (L.C.H.) space. Then $\forall x \in X$ and for all neighbourhood U of x, there exists an open neighbourhood V of x with $\overline{V} \subseteq U$ and $\overline{V}$ is compact.

Theorem 3.32:

A locally compact Hausdorff (L.C.H.) space is completely regular.

Corollary 3.32.1:

An A_2 L.C.H. space X is normal and metrizable.

Proof:

According to **Theorem 3.32**, X is completely regular, hence regular. Thus X is regular and second countable by assumption, hence X is normal. Then by Urysohn's Metrization Theorem, X is metrizable.

□

Compact spaces have nice properties and when we are given a non-compact space, it is naturally to ask can we make this space compact so that the tools we apply to the compact spaces fail to be false. This leads to the following definitinon called compactification.

Definition: Compactification

A compactification of a space X is an embedding $f : X \to Y$ so that

(i) Y is compact.
(ii) $f(X)$ is dense in Y, i.e. $\overline{f(X)} = Y$.

Definition: One-Point Compactification

A compactification $f : X \to Y$ is said to be a one-point compactification if $Y \backslash f(X)$ is a single point.

Recall the definition of embedding:

Definition: Embedding

A continuous map $f : X \to Y$ is called an embedding if $f : X \to f(X)$ is a homeomorphism where $f(X)$ is given a subspace topology.

We now prove two lemmas which in turn prove that the definition of the compactification is well-defined.

Lemma 3.33: "⇒"

Let $f : X \to Y$ be an embedding where Y is a compact Hausdorff space and $Y \backslash f(X)$ is a single point. Then X is L.C.H.

Proof:

Since $f : X \to Y$ is a homeomorphism, we may, without loss of generality, assume that $X \subseteq Y$. Since Y is Hausdorff then X is also Hausdorff (subspace of Hausdorff is Hausdorff). Let $\infty := Y \backslash f(X) = Y \backslash X$. Then $\forall x \in X$, there exists a compact neighbourhood of x in X. Since $x \in X$ and $\infty \notin X$, $x \neq \infty$. Since Y is Hausdorff, there exist open neighbourhoods U of x and V of ∞ such that $U \cap V = \emptyset$. Moreover, since V is open and $Y \backslash V$ is closed and $U \subseteq Y \backslash V$, therefore, $\overline{U} \subseteq Y \backslash V \subseteq Y \backslash \{\infty\} =: X$. Furthermore, since Y is compact and $\overline{U} \subseteq Y$ is a closed subset thus $\overline{U}$ is compact. Taking $N = \overline{U}$ yields a compact neighbourhood of x in X and result follows.

□

Lemma 3.34: "⇐"

Let X be a L.C.H. space, there exists a compact Hausdorff space X^+ and an embedding $f : X \to X^+$ such that $X^+ \backslash f(X)$ is a single point.

If X is compact, then $X^+ \cong X \cup \{\infty\}$. If X is not compact, then $\overline{f(X)} = X^+$ and $f : X \to Y$ is a compactification.

Proof:

Let us use the same notation as we did in the proof of **Lemma 3.33**, denote the single point $Y \backslash X =: \infty$. Let $X^+ = X \cup \{\infty\}$ and define

$$T := \{U \subseteq X \,|\, U \text{ open}\} \cup \{(X \backslash C) \cup \{\infty\} \,|\, C \subseteq X \text{ compact}\}.$$

To make our lives easier, let us denote Type I set to be $\{U \subseteq X \,|\, U \text{ open}\}$ and use Type II set to denote $\{(X \backslash C) \cup \{\infty\} \,|\, C \subseteq X \text{ compact}\}$.

[Claim]: T is a topology.

(i) $\emptyset \subseteq X$ is open $\Rightarrow \emptyset \in T$. Since $\emptyset \subseteq X$ is compact $\Rightarrow$ $(X \backslash \emptyset) \cup \{\infty\} = X \cup \{\infty\} \in T$.

(ii) If $U, V \in T$, $U, V \subseteq$ Type I set then $U \cap V \subseteq X$ is an open subset, so it is in T.
If $U \subseteq X$ is open and $V := (X \backslash C) \cup \{\infty\}$, then $U \cap V = U \cap (X \backslash C)$ is open in X hence in T.
If $U_1 := (X \backslash C_1) \cup \{\infty\}$, $U_2 := (X \backslash C_2) \cup \{\infty\}$, then $U_1 \cap U_2 = (X \backslash (C_1 \cup C_2)) \cup \{\infty\}$ is compact hence $U_1 \cap U_2$ is

compact and it follows that $U_1 \cap U_2 \subseteq$ Type II set and so is in T.

(iii) Suppose that $\{U_\alpha\}_{\alpha\in A}$ is a collection of open subsets of X and $\{C_\beta\}_{\beta\in B}$ is a collection of compact subsets of X. Then one has

$$\Big(\bigcup_{\alpha\in A} U_\alpha\Big)\bigcup_{\beta\in B}(X\backslash C_\beta)\cup\{\infty\}$$
$$=\bigcup_{\alpha\in A}\cup\Big(X\backslash\bigcap_{\beta\in B} C_\beta\Big)\cup\{\infty\}$$
$$=\Big(X\backslash\Big(\bigcap_{\beta\in B} C_\beta\backslash\bigcup_{\alpha\in A} U_\alpha\Big)\Big)\cup\{\infty\}$$
$$=\Big(\bigcap_{\beta\in B} C_\beta\cap\Big(X\backslash\bigcup_{\alpha\in A} U_\alpha\Big)\Big)\text{ is compact.}$$

Therefore $\Big(X\backslash\Big(\bigcap_{\beta\in B} C_\beta\backslash\bigcup_{\alpha\in A} U_\alpha\Big)\Big)\cup\{\infty\}$ is Type II so in T. The other two situations hold analogously.

Also, $X \subseteq X\cup\{\infty\}$ is an embedding.

[Claim]: X^+ is Hausdorff.

Suppose that $x, y \in X^+$ such that $x \neq y$. We want to separate them by open sets. Since X is Hausdorff, we can choose $y = \infty$ while $x \in X$. Since X is L.C.H. there exists an open neighbourhood V of x such that $\overline{V}$ is compact in X. Therefore, $(X\backslash\overline{V})\cup\{\infty\}$ is an neighbourhood of ∞ and $V\cap\big((X\backslash\overline{V})\cup\{\infty\}\big)=\varnothing$.

[Claim]: X^+ is compact.

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X^+ then $\exists\beta\in A$ such that $\infty\in U_\beta$. Therefore, $C := X^+\backslash U_\beta \subseteq X$ is compact and then $\exists n$ such that for $\alpha_1,\cdots,\alpha_n\in A$ one has $C\subseteq U_{\alpha_1}\cup\cdots\cup U_{\alpha_n}$ hence $X^+ = U_{\alpha_1}\cup\cdots\cup U_{\alpha_n}\cup U_\beta$.

Note that if X is compact, then $X\backslash X\cup\{\infty\}=\{\infty\}\in T$ is open, X is also open and they disjoint. If X is not compact, then for all compact subsets $C\subseteq X$ one has $X\backslash C\neq\varnothing \Rightarrow X\cap\big((X\backslash C)\cup\{\infty\}\big)=X\backslash C\neq\varnothing\Rightarrow\infty\in\overline{X}$.

□

In fact, one can prove that the one point compactification of a non-compact L.C.H space is unique up to a unique homeomorphism.

Lemma 3.35: Uniqueness

If $f : X \to Z$ and $g : X \to Y$ are two one-point compactification, then there exists a unique homeomorphism $\varphi : Z \to Y$ such that $f = \varphi^{-1}\circ g$.

Proof:

Let z_0 be the unique point of $Z\backslash f(X)$ and y_0 be the unique point of $T\backslash g(X)$. Since f and g are bijective then the only way to define such a φ is by the following formula:

$$\varphi(z) := \begin{cases} y_0, \text{ if } z = z_0 \\ g(f^{-1}(z)), \text{ if } z \neq z_0, \text{ i.e. } z \in f(X) \end{cases}.$$

[Claim]: φ is a homeomorphism.

It suffices to show that for any open subsets $U \subseteq Z$, $\varphi(U)$ is open in Y. If $U \subseteq f(X)$, $f^{-1}(U)$ is open, $g(f^{-1}(U))$ is open in Y since f^{-1} is continuous and $g(f^{-1}(U))$ is open in $g(X)$ hence in Y.

If $U \nsubseteq f(X)$ then $z_0 \in U \Rightarrow Z\backslash U \subseteq f(X)$, since $Z\backslash U$ is closed in Z and Z is compact then $Z\backslash U$ is also compact in Z and $f(X)$, since f^{-1} and g are homeomorphism, $g(f^{-1}(Z\backslash U))$ is closed in Y. But $(g \circ f^{-1})(Z\backslash U) = \varphi(Z\backslash U) = Y\backslash\varphi(U)$, φ is a bijection $\Rightarrow \varphi(U)$ is open in Y. Therefore φ is open and by symmetry so is φ^{-1}.

□

Definition: Proper Map

A continuous map $f : X \to Y$ between two topological spaces is said to be proper if for all compact subset $C \subseteq Y, f^{-1}(C)$ is also compact.

A proper map between topological spaces is a concept that captures a notion of “boundedness” or "compactness" of preimages under the map. Proper maps are particularly useful in topology and differential geometry.

Theorem 3.36: Criterion for Proper

A continuous map $f : X \to Y$ between two L.C.H. spaces extends to a continuous map $f^+ : X^+ \to Y^+ \Leftrightarrow f$ is proper.

Proof:

We may assume that $Y \subseteq Y^+$ and $X \subseteq X^+$ where $Y^+ := Y \cup \{\infty_Y\}$ and $X^+ := X \cup \{\infty_X\}$.

“$\Rightarrow$”:

Suppose that $f^+ : X^+ \to Y^+$ is continuous and $C \subseteq Y$ is compact. Then $Y^+\backslash C$ is an open neighbourhood of $\infty_Y \Rightarrow C$ is closed in $Y^+ \Rightarrow (f^+)^{-1}(C)$ is closed in $X^+ \Rightarrow (f^+)^{-1}(C) \cap \{\infty_X\} = \varnothing \Rightarrow (f^+)^{-1}(C) = f^{-1}(C)$. Since $f^{-1}(C)$ is closed in X^+ and X^+ is compact, it follows that $f^{-1}(C)$ is compact in X^+ and hence in X.

“$\Leftarrow$”:

Suppose that $f : X \to Y$ is proper.

[Claim]: $\forall U \subseteq Y^+$ open, $(f^+)^{-1}(U)$ is open in X^+.

Case I:

$U \subseteq Y^+\backslash\{\infty_Y\} = Y \Rightarrow (f^+)^{-1}(U) = f^{-1}(U)$ which is open in X since f is continuous. But X is open in $X^+ \Rightarrow (f^+)^{-1}(U)$ is open in X^+.

Case II:

$\{\infty_Y\} \subseteq U \Rightarrow Y^+\backslash U$ is a compact subset of $Y \Rightarrow f^{-1}(Y^+\backslash U)$ is compact since f is proper, where $f^{-1}(Y^+\backslash U) = (f^+)^{-1}(Y^+\backslash U)$, it follows that $(f^+)^{-1}(U) = X^+\backslash(f^+)^{-1}(Y^+\backslash U)$ is open in X^+.

□

Corollary 3.36.1:

A proper map between two L.C.H. spaces is closed.

The proof is left as an exercise. Just to aviod ambiguity, the openness or closedness we used above for a function f means what types of sets it preserves. For example, if f preserves open sets then we say f is open, vice versa.

3.6 Metrizability and σ-Local Finiteness

In the previous subsections we introduced the concept of metrizability, which is done by the Urysohn's Metrization Theorem, stating that a T_1 A_2 completely regular (in fact, the conditions may vary from case to case, for example, in [58], the Urysohn's Metrization Theorem admits A_2 T_3 being sufficient, in our case however, the regularity is not weakened but strengthened, this is because the T_3 condition is weakened to T_1) topological space is metrizable. In that theorem, while the regularity being a neces-sary condition, however, the countable basis (A_2) is somewhat expendable. This subs-ection and the upcoming one focus on (i) weaken the A_2 condition and arrive at the same metrizability and (ii) extend this terminology into a bigger class, i.e. introduce the locally metrizability then discuss its properties as well as some important results. The following literature comes from [1], [2], [9], [39], and [44]. A more detailed treatment could be viewed via [57] and [58].

We now introduce the first notion: Local finiteness, which focuses on the behavior of open sets near individual points. It does not impose global constraints on the entire space but instead looks at local neighborhoods of each point.

Definition: Locally Finite

A cover $\{U_\alpha\}_{\alpha \in A}$ of a space X is said to be locally finite if $\forall x \in X$, there exists a neighbourhood N of x such that $N \cap U_\alpha = \varnothing$ for all but finitely many $\alpha \in A$.

As we shall see later: Local finiteness is a more general property than paracompact-ness. While every paracompact space is locally finite, there exist locally finite spaces that are not paracompact. Paracompactness imposes additional conditions related to open covers and their refinements. Moreover, local finiteness is related to but distinct from local compactness. Locally compact spaces have compact neighborhoods around each point, whereas locally finite spaces only require finite intersections of open sets near each point.

Definition: Partition of Unity

Let $\{U_1, \cdots, U_n\}$ be a finite indexed open covering of the space X. An indexed family of continuous functions

$$f_i : X \to [0,1] \text{ for } i = 1,\cdots,n,$$

is said to be a partition of unity by $\{U_i\}$ if

(i) $\operatorname{supp} f_i \subseteq U_i \ \forall i = 1,\cdots,n$.

(ii) $\sum_{i=1}^{n} f_i(x) = 1 \ \forall x \in X$.

Definition: Refinement

A cover $\{U_\alpha\}_{\alpha \in A}$ is a refinement of a cover $\{V_\beta\}_{\beta \in B}$ if $\forall \alpha \in A$ there exists a $\beta := \beta(\alpha) \in B$ such that $U_\alpha \subseteq V_{\beta(\alpha)}$.

Paracompactness is intimately connected to the existence of partitions of unity, which are functions that assign non-negative values to open sets in a cover and sum up to 1 on the entire space.

Definition: Paracompact

A topological space X is said to be paracompact if X is Hausdorff and any open cover of x has a locally finite open refinement.

Yes, paracompactness is indeed a topological property. In the context of general topology, a property is considered "topological" if it is preserved under homeomorphisms, which are continuous bijections with continuous inverses between topological spaces. Paracompactness is one such property.

Now we prove a useful result of the local finiteness.

Lemma 3.37:

Let $\mathscr{A}$ be a locally finite collection of subsets of X. Then

(a) Any subcollection of $\mathscr{A}$ is locally finite.

(b) The collection $\mathscr{B} := \{\overline{A}\}_{A\in\mathscr{A}}$ of the closures of the elements of $\mathscr{A}$ is locally finite.

(c) $\overline{\bigcup_{A\in\mathscr{A}} A} = \bigcup_{A\in\mathscr{A}} \overline{A}$.

Proof:

Statement (a) is trivial. To prove (b), note that any open set U that intersects the set $\overline{A}$ necessarily intersects A. Therefore, if U is a neighbourhood of x that intersects only finitely many elements A of $\mathscr{A}$, then U can intersect at most the same number of sets of the collection $\mathscr{B}$. (It might intersect fewer sets of $\mathscr{B}$, since $\overline{A_1}$ and $\overline{A_2}$ can be equal even though A_1 and A_2 are not).

To prove (c), let Y denote the union of the elements of $\mathscr{A}$: $\bigcup_{A\in\mathscr{A}} A =: Y$. In general, $\bigcup_{A\in\mathscr{A}} \overline{A} \subseteq \overline{Y}$; we now proceed to the other direction, under the assumption of local finiteness. Let $x \in \overline{Y}$; let U be a neighbourhood of x that intersects only finitely many elements of $\mathscr{A}$, say $A_1, \cdots, A_k$. We assert that x belongs to one of the sets $\overline{A_1}, \cdots, \overline{A_k}$, and hence belongs to the union. For otherwise, the set $U \backslash (\bigcup_{i=1}^{k} \overline{A_i})$ would be a neighbourhood of x that intersects no element of $\mathscr{A}$ and hence does not intersect Y, contradiction to $x \in \overline{Y}$.

□

If our goal is to use locally finite sets to help describe a given topology, obtain a property that is weaker than second countable but stronger than first countable, and prove to metrizability of a topological space, then we would likely want locally finite collections of open sets that describe a basis; that is, we would like a locally finite basis of a topological space. However, given a metrizable topological space, it is unlikely that we will be able to find a locally finite basis since, as Example 6.4.3 shows, the requisite of having arbitrary small neighbourhoods around each point is an immediate obstacle to having a locally finite basis. However, as we can consider balls of a fixed radius at a given time and as we only need to consider rational radii, the following is not out of reach.

Definition: σ-locally finite

A collection $\mathscr{A}$ of subsets of X is said to be σ-locally finite (countably locally finite) if $\mathscr{A}$ can be written as the countable union of collections $\mathscr{A}_n$, each of which is locally finite.

As an example of obtaining a σ-finite refinement, we demonstrate the following lemma. Note this is the best analogue of 'every open cover of a compact topological space has a finite subcover' that we can possibly obtain for a metrizable topological space. Therefore, as compactness is such a nice property, we are perhaps on the right track to study metrizable topological spaces.

Lemma 3.38:

Let (X, T_X) be a metrizable space and let $\mathscr{A}$ be an open cover of (X, T_X). Then there exists an open refinement $\mathscr{A}'$ of $\mathscr{A}$ that is σ-locally finite and covers (X, T_X).

Proof: Consult [58] **Lemma 6.4.9**.

Using **Lemma 3.38**, we can actually prove that metrizable spaces have nice bases thereby showing that having a σ-locally finite basis is a requirement of being metrizable.

Corollary 3.38.1:

Every metrizable topological space has a σ-locally finite basis.

Proof:

Let (X, T_X) be a metrizable topological space and let d be a metric that induces T_X. For every $n \in \mathbb{N}$, let $\mathscr{A}_n := \left\{ B_d(x, \frac{1}{n}) \middle| x \in X \right\}$. Since $\mathscr{A}_n$ is clearly an open cover of (X, T_X). **Lemma 3.38** implies that there exists an open refinement $\mathscr{B}_n$ of $\mathscr{A}_n$ that is σ-locally finite and covers (X, T_X). Since $\mathscr{B}_n$ is a refinement of $\mathscr{A}_n$, notice that if $B \in \mathscr{B}_n$ then $B \subseteq B_d(x, \frac{1}{n})$ for some $x \in X$ and thus $\text{diam}(B) \leq \frac{2}{n}$. Let now $\mathscr{B} := \bigcup_{n=1}^{\infty} B_n$.

[Claim]: $\mathscr{B}$ is a σ-locally finite basis of (X, T_X).

To see this, note $\mathscr{B}$ is clearly σ-locally finite being the countable union of σ-locally finite subset of (X, T_X). To see that $\mathscr{B}$ is a basis for (X, T_X), let $x \in X$ and $\varepsilon > 0$ be chosen arbitrarily. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since $\mathscr{B}_n$ covers (X, T_X), there exists $B \in \mathscr{B}_n \subseteq \mathscr{B}$ such that $x \in B$. Therefore, since $\text{diam}(B) \leq \frac{2}{n}$, it must be the case that

$x \in B \subseteq B_d[x, \dfrac{2}{n}] \subseteq B_d(x, \varepsilon)$. Therefore, since $x \in X$ and $\varepsilon > 0$ were chosen arbitrarily and since d induces T_X, it follows that $\mathscr{B}$ is σ-locally finite basis of (X, T_X) as we desire.

□

As **Corollary 3.38.1** shows that every metrizable topological space must have a σ -locally finite basis, it is natural to ask whether we can obtain a converese. Of course we must add the condition that the topological space under investigation is normal as every metrizable space is normal. However, as verifying a topological space is normal is often difficult, we desire to replace the assumption of being normal with being regular.

Now we arrive at the second metrization theorem, also the main goal of this subsection. Recall in **Urysohn's Metr-ization Therorem** we require the space to be A_2 T_1 . The **Nagata-Smirnov Metrization Theorem s**tates that every regular topological space with a σ-locally finite basis is metrizable. To proceed, we begin by developing additional properties of regular topological spaces with σ-locally finite basis.

Lemma 3.39:

Let (X, T_X) be a regular topological space with a σ-locally finite basis. If $V \in T_X$, then there exists $\{U_n\}_{n=1}^{\infty} \subseteq T_X$ such that

$$V = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \overline{U_n}.$$

Proof:

By assumption there exists a basis $\mathscr{B}$ of (X, T_X) such that $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$ where each $\mathscr{B}_n$ is a locally finite subset of (X, T_X). For each $n \in \mathbb{N}$, define

$$\mathscr{A}_n := \{B \in \mathscr{B}_n \,|\, \overline{B} \subseteq V\}.$$

Since clearly $\mathscr{A}_n \subseteq \mathscr{B}_n$, $\mathscr{A}_n$ is a locally finite subset of (X, T_X) $\forall n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$, let $U_n := \bigcup_{B \in \mathscr{A}_n} B$. Cleraly $U_n \in T_X$ $\forall n \in \mathbb{N}$. Furthermore, we see that, according to **Lemma 3.37**, that

$$U_n \subseteq \overline{U_n} = \bigcup_{B \in \mathscr{A}_n} \overline{B} \subseteq V \;\forall n \in \mathbb{N}.$$

Therefore,

$$\bigcup_{n=1}^{\infty} U_n \subseteq \bigcup_{n=1}^{\infty} \overline{U_n} \subseteq V.$$

To see the inverse inclusion, let $x \in V$ be chosen arbitrarily. Since (X, T_X) is regular and $\mathscr{B}$ is its basis, there then exsits a $B \in \mathscr{B}$ such that

$$x \in B \subseteq \overline{B} \subseteq V.$$

Hence $B \in \mathscr{A}_n$ for some $n \in \mathbb{N}$, so $x \in U_n$ for some $n \in \mathbb{N}$, and thus $x \in \bigcup_{n=1}^{\infty} U_n$. Therefore, as $x \in V$ chosen arbitrarily, the inverse inclusion holds.

□

Lemma 3.40: T_3+σ-locally finite $\Rightarrow T_4$

Let (X, T_X) be a regular topological space with a σ-locally finite basis. Then (X, T_X) is normal.

Proof:

Let A and B be two arbitrary closed subsets of (X, T_X) such that $A \cap B = \emptyset$. Since $X \backslash B$ and $X \backslash A$ are opens sets in (X, T_X), **Lemma 3.39** tells us that there exists $\{U_n\}_{n=1}^{\infty}, \{V_n\}_{n=1}^{\infty} \subseteq T_X$ such that

$$X \backslash B = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \overline{U_n} \text{ and } X \backslash A = \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} \overline{V_n}.$$

By taking complements, we see that $\{U_n\}_{n=1}^{\infty}$ is an open cover of A such that $B \cap \overline{U_n} = \emptyset \forall n \in \mathbb{N}$ and $\{V_n\}_{n=1}^{\infty}$ is an open cover of B such that $A \cap \overline{V_n} = \emptyset \forall n \in \mathbb{N}$. Now we have open covers for disjoint closed sets, our next goal is to let them be disjoint.

[Claim]: $U_n \cap V_n = \emptyset \forall n$.

For every $n \in \mathbb{N}$, let

$$U_n' = U_n \backslash \big(\bigcup_{k=1}^{n} \overline{V_k}\big) \text{ and } V_n' = V_n \backslash \big(\bigcup_{k=2}^{n} \overline{U_k}\big).$$

Clearly $\{\overline{U_n}\}_{n\geq 1}$ and $\{\overline{V_n}\}_{n\geq 1}$ are collections of closed subsets of (X, T_X) so that $\{\bigcup_{k=1}^{n} \overline{U_k}\}_{n\geq 1}$ and $\{\bigcup_{k=1}^{n} \overline{V_k}\}_{n\geq 1}$ are closed subsets of (X, T_X).

Therefore, since $\{U_n\}_{n\geq 1}$ and $\{V_n\}_{n\geq 1}$ are collections of open subsets of (X, T_X) and since $D \backslash E = D \cap (X \backslash E)$ for all $D, E \subseteq X$, we see that $\{U_n'\}_{n\geq 1}$ and $\{V_n'\}_{n\geq 1}$ are collections of open subsets of (X, T_X). Let now

$$U = \bigcup_{n=1}^{\infty} U_n' \text{ and } V = \bigcup_{n=1}^{\infty} V_n',$$

which are open subsets of (X, T_X). Notice that since $\overline{V_n} \cap A = \emptyset$ $\forall n \in \mathbb{N}$ that $U_n' \cap A = U_n \cap A$. It follows that $A \subseteq \bigcup_{n=1}^{\infty} U_n$ hence $A \subseteq U$. Furthermore, similar arguments allow $B \subseteq V$. Suppose the contrary that $U \cap V \neq \emptyset$ so that there exists an $x \in U \cap V$. By definition of U and V, there must exist $n, m \in \mathbb{N}$ such that $x \in U_n'$ and $x \in V_m'$, if $n \geq m$, then $x \in V_m' \Rightarrow x \in V_m$ and $x \in U_n'$ implies that $x \in U_n \backslash \big(\bigcup_{k=1}^{n} \overline{V_k}\big) \subseteq U_n \backslash V_m$, contradition. The other side will also lead to a contradiction, therefore $U \cap V = \emptyset$ as we desired.

□

It is well-known that the second countability axiom is sufficient for a regular topological space to be metrizable, but it is not necessary. That is, $T_3 + A_2 \Rightarrow$ Metrizable but Metrizable $\not\Rightarrow T_3 + A_2$. For example, any discrete space X is metrizable, but if X consists of uncountably many points it does not have a countable basis. It is natural to ask if there exists a necessary and sufficient condition for a topological space to be metrizable.

Eiichi Nagata, a Japanese mathematician, made significant contributions to metrization theory. In 1947, he proved a metrization theorem that extended Urysohn's theorem and addressed when a topological space is metrizable in terms of a base for the topology. This result is known as Nagata's Metrization Theorem. In 1953, Anatoly Smirnov, a Russian mathematician, improved and extended Nagata's theorem, providing a more general and widely applicable characterization of metrizable spaces. This result became known as the Nagata-Smirnov Metrization Theorem.

Before we prove this theorem, we first introduce a notion called the G_δ sets.

Definition: G_δ Set

A subset A of a space X is said to be a G_δ set in X if it equals to the intersection of a countable collection of open subsets of X.

Example 3.4: G_δ Sets

In a metric space X, each closed set is a G_δ set. Given $A \subseteq X$, let $U_\varepsilon(A)$ denote the ε-neighbourhood of A. If A is closed, one can check that $A = \bigcap_{n \in \mathbb{Z}^+} U_{\frac{1}{n}}(A)$ is the desired G_δ set. ||

Lemma 3.41:

Let (X, T_X) be a regular topological space with a σ-locally finite basis. Then every closed subset of (X, T_X) is a G_δ subset of (X, T_X).

Proof:

Let F be an arbitrary closed subset of (X, T_X). Since $X \backslash F$ is open, **Lemma 3.39** tells us that there exists $\{V_n\}_{n=1}^{i} nfty \subseteq T_X$ such that

$$X \backslash F = \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} \overline{V_n}.$$

For each $n \in \mathbb{N}$, let $U_n := X \backslash \overline{V_n} \in T$. We claim that $F = \bigcap_{n=1}^{\infty} U_n$ thereby showing that F is a G_δ set. Indeed this follows directly from the above set equality due to **De Morgan's Laws**. Therefore, as F was arbitrary, every closed subset of (X, T_X) is G_δ.

□

Lemma 3.42:

Let (X, T_X) be a normal topological space and let A be a closed G_δ subset of (X, T_X). There exists an $f \in C(X, [0,1])$ such that $f(a) = 0 \ \forall a \in A$ and $f(x) > 0 \ \forall a \in X \backslash A$.

Proof: Consult [58], **Lemma 6.5.4**.

Now we have enough tools to prove the main result of this subsection.

Theorem 3.43: Nagata-Smirnov Metrization Theorem

A topological space (X, T_X) is metrizable $\Leftrightarrow$ it is regular and has a σ-locally finite base.

There exist several equivalent formulations of this theorem in the literature but are rather complicated. We do not present a proof here, the readers could consult [1] **Theorem 40.3**, [58] **Theorem 6.5.5**, while [62] **Theorem 3.2.1** provides a proof with the notion "perfectly normal", in [59], Athanasios Andrikopoulos offers a new proof of the Nagata-Smirnov Metrization Theorem based n Rudin's Proof of Stone's result on paracompactness (see [65]). One may also consult the original work of Nagata [64], and an overview on Nagata's contribution to theory of generalized metric spaces [60] is also considered helpful.

It is a fact that Urysohn's Metrization is "more popular" than Nagata-Smirnov Metrization Theorem and this is due to some historical reasons. We recommend [63] for readers who are interested in.

We state a very helpful interpretation of the metrizability we have explored so far, perhaps we can make it an equivalent (in fact, loosely-defined) definition for a topological space being metrizable.

Definition: Metrizable

A metrizable space is a topological space that is homeomorphic to a metric space.

Hence the exploration of the metrizability turns out to be, in some cases, finding the homeomorphisms between topological spaces and specific metric spaces. This is not an easier task than the methods we have introduced so far, but it offers a great insight for other possible approaches.

3.7 Metrizability and Paracompactness

Of course the **Nagata-Smirnov Metrization Theorem** has one limitation in that one needs to verify that a topological space has a σ-locally finite basis, which is often not an easy task. As the idea of a σ-locally finite basis was motivated by trying to weaken second countability via an idea similar to compactness, in this subsection we will introduce a generalization of compactness called paracompactness. It turns out that paracompactness is particularly useful for applications in topology and differential geometry. However, our only goal will be to relate paracompactness to the existence of the σ-locally finite bases.

On the other hand, the **Nagata-Smirnov Metrization Theorem** gives one set of necessary and sufficient conditions for metrizability of a space. The theorem we are going to prove, called Smirnov Metrization Theorem, offers another such set of conditions. It is a corollary of the **Nagata-Smirnov Metrization Theorem** and was first proved by Smirnov. For detailed description and treatment one may consult [61], [62], [66], and [67].

This subsection, the same as the previous one, is divided into two parts. In the first part we introduce the paracompactness and its properties as well as some results on it; in the second part, we shall state and prove the **Smirnov Metrization Theorem**.

Definition: Paracompact

A topological space X is said to be paracompact if X is Hausdorff and any open cover of x has a locally finite open refinement.

Recall that paracompactness is indeed a topological property and it is related to the refinement we introduced in the previous subsection. Same as its motivation, compactness is a stronger property than paracompactness. Every compact space is paracompact, but not every paracompact space is compact. In other words, paracompact spaces exhibit some of the desirable properties of compactness without being necessarily compact. Moreover, every metrizable space is paracompact. This means that in metric spaces (spaces that can be equipped with a metric), paracompactness is a general property that holds. However, paracompactness extends beyond metrizable spaces to a broader class of topological spaces.

Theorem 3.44: T_2+Paracompactness $\Rightarrow T_4$

Every paracompact Hausdorff space is normal.

Proof:

The proof is somewhat similar to the proof that a compact Hausdorff space is normal.

Step I: First one proves regularity.

Let a be a point of X and let B be a closed set of X disjoint from a. The Hausdorff condition enables us to choose, for each $b \in B$, an open set U_b about b whose closure is disjoint from a. Cover X by the open sets U_b, along with the open set $X \setminus B$; take a locally finite open refinement $\mathscr{C}$ that covers X. Form the subcollection $\mathscr{D}$ of $\mathscr{C}$ consisting of every element of $\mathscr{C}$ that intersects B. Then $\mathscr{D}$ covers B. Furthermore, if $D \in \mathscr{D}$ then $\overline{D}$ is disjoint from a. For D intersects B, so it lies in some set U_b, whose closure is disjoint from a. Let

$$V := \bigcup_{D \in \mathscr{D}} D.$$

Then V is an open set in X containing B. Because $\mathscr{D}$ is locally finite, it follows

$$\overline{V} = \bigcup_{D \in \mathscr{D}} \overline{D}.$$

Therefore $\overline{V}$ is disjoint from a. Thus regularity follows.

Step II: Derive Normality.

To prove the normality, one merely repeats the same argument, replacing a by the closed set A throughout and replacing the Hausdorff condition by regularity.

□

Theorem 3.45:

Every closed subspace of a paracompact space is paracompact.

Proof:

Let Y be a closed subspace of the paracompact space X; let $\mathscr{A}$ be a covering of Y by sets open in Y. For each $A \in \mathscr{A}$, choose an open set A' of X such that $A' \cap Y = A$. Cover X by the open sets A', along with the open set $X \setminus Y$. Let $\mathscr{B}$ be a locally finite open refinement of this covering that covers X. Then the collection $\mathscr{C} := \{B \cap Y \mid B \in \mathscr{B}\}$ is the desired locally finite open refinement.

□

Let us now turn to the motivation of using the idea of paracompactness to simplifying the task of finding σ-locally finite bases. In particular, our goal in a regular topological space is to relate paracompactness and the existence of a σ-locally finite bases.

Lemma 3.46:

Let X be a regular topological space. Then the following conditions on X are equivalent: Every open covering of X has a refinement that is:

(i) An open covering of X and σ-locally finite.

(ii) A covering of X and locally finite.

(iii) A closed covering of X and locally finite.

(iv) An open covering of X and locally finite.

Proof: Consult [1] **Lemma 41.3** or [58] **Lemma 6.6.6**.

Theorem 3.47:

Every Metrizable space is paracompact.

Proof:

Let X be a metrizable space. We already know from **Lemma 3.38** that given an open covering $\mathscr{A}$ of X, it has an open refinement that covers X and is σ-locally finite. The preceding lemma then implies that $\mathscr{A}$ has an open refinement that covers X and is locally finite.

□

Theorem 3.48:

Every regular Lindelöf space is paracompact.

Proof:

Let X be a regular Lindelöf space. Given an open covering $\mathscr{A}$ of X, it has a countable subcollection that covers X, this subcollection is automatically σ-locally finite. Then applying the preceding lemma shows that $\mathscr{A}$ has an open refinement that covers X and is locally finite.

□

However, the product of two paracompact spaces need not be paracompact. A famous conterexample could be viewed in[68].

We now introduce another terminology and then introduce the **Shrinking Lemma**.

Definition: σ-compact

A space X is said to be σ-compact if X is a union of countably many compact sets.

Just like the σ-local finiteness, the notion σ-compactness is also a combination between the countability and a familiar term compactness. The σ-compact spaces often have nice topological properties. For example, they are Lindelöf spaces (every open cover has a countable subcover) and paracompact spaces (every open cover has a locally finite open refinement). Moreover, just like the paracompactness, a compact space is automatically paracompact, and, yes, σ-compact. Now we build up bridges between σ-compactness and paracompactness.

Proposition 3.49:

A locally compact σ-compact topological space is paracompact.

Proof: Consult [69].

Remark:

Moreover, we can prove that a closed subset of a paracompact space is compact. ||

Now we prove a useful lemma: The Shrinking Lemma, known as the shrinking criterion, is an important result in topology, particularly in the context of proving paracompactness. It's a tool used to show that given a certain collection of open sets, you can find a "shrunken" subcollection that retains certain properties.

Lemma 3.50: Shrinking Lemma

Suppose that X is paracompact and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover. Then there exists a locally finite open cover $\{V_\alpha\}_{\alpha\in A}$ with $\overline{V_\alpha} \subseteq U_\alpha$ $\forall\alpha$. ($V_\alpha = \varnothing$ for some α is allowed).

Proof:

Since paracompact spaces are regular. Then $\forall\alpha, \forall x \in U_\alpha$, there exists an open neighbourhood O of x such that $\overline{O} \subseteq U_\alpha$. We get a collection of open cover:

$$\mathcal{O} := \{O \subseteq X \text{ open} \mid \overline{O} \subseteq U_\alpha \text{ for some } \alpha\}.$$

Since X is paracompact, $\mathcal{O}$ has a locally finite open refinement, for each $W \in \mathcal{O}$, we can choose an α such that $\overline{W} \subseteq U_\alpha$, i.e. choose a function

$$f : W \to A,$$

so that $\overline{W} \subseteq U_{f(W)}$. $\forall\alpha \in A$, we define $V_\alpha := \bigcup_{f(W)=\alpha} W$.

Since $\{W \mid f(W) = \alpha\} = \varnothing \Rightarrow V_\alpha = \varnothing$. Since W is locally finite, it follows that

$$\overline{V_\alpha} = \overline{\bigcup_{f(W)=\alpha} U_\alpha} = \bigcup_{f(W)=\alpha} \overline{U_\alpha}.$$

Since $\overline{W} \subseteq U_\alpha$ $\forall W$ with $f(W) = \alpha$. One has $\overline{V_\alpha} = \overline{U_\alpha}$.

[Claim]: $\{V_\alpha\}_{\alpha\in A}$ is locally finite.

Choose a point $x \in X$. Since W is locally finite, there exists a neighbourhood N of x such that $N \cap W = \emptyset$ for all but finitely many $w \in W$. Let $A' := \{f(W) \,|\, N \cap W \neq \emptyset\}$, then it is finite. Since $V_\alpha \cap W \neq \emptyset$ only for $\alpha \in A'$, result follows.

□

Now we introduce the last metrization theorem in this chapter. Moreover, the connection between the **Shrinking Lemma** and the **Smirnov Metrization Theorem** lies in their roles in proving metrizability:

In practice, when proving that a space is metrizable, you may start by showing that it is paracompact (often using tools like the **Shrinking Lemma**) and Hausdorff. Once these conditions are met, the Smirnov Metrization Theorem guarantees the existence of a metric that makes the space metrizable.

The Nagata-Smirnov metrization theorem gives one set of necessary and sufficient conditions for metrizability of a space. The Smirnov's Metrization Theorem, on the other hand, is a corollary of the Nagata-Smirnov Theorem and was first proved by Smirnov. Before that, we shall introduce a notion called local metrizability, which is a topological property of a space that means that every point in the space has a neighbourhood that is homeomorphic to a metric space.

Definition: Locally Metrizable

A space X is locally metrizable if every point x of X has a neighbourhood U that is metrizable in the subspace topology.

Theorem 3.51: Smirnov Metrization Theorem

A space X is metrizable $\Leftrightarrow$ it is a paracompact Hausdorff space that is locally metrizable.

Proof:

"$\Rightarrow$":

Suppose that X is metrizable. Then X is locally metrizable and Hausdorff; it is also paracompact, by **Theorem 3.47**.

"$\Leftarrow$":

Conversely, suppose that X is a paracompact Hausdorff space that is locally metrizable. We shall show that X has a basis that is σ-locally finite. Since X is regular, it will then follow from the **Nagata-Smirnov Theorem** that X is metrizable.

Cover X by open sets that are metrizable; then choose a locally finite open refinement $\mathscr{C}$ of this covering that covers X. Each element C of $\mathscr{C}$ is metrizable; let the function $d_C : C \times C \to \mathbb{R}$ be a metric that gives the

topology of C. Given $x \in C$, let $B_C(x, \varepsilon)$ denote the set of all points y of C such that $d_C(x, y) < \varepsilon$. Being open in C, the set $B_C(x, \varepsilon)$ is also open in X. Given $m \in \mathbb{Z}^+$, let $\mathscr{A}_m$ be the covering of X by all these open balls of radius $1/m$; that is, let $\mathscr{A}_m := \{B_C(x, 1/m) \,|\, x \in C \text{ and } C \in \mathscr{C}\}$.
Let $\mathscr{D}_m$ be a locally finite open refinement of $\mathscr{A}_m$ that covers X (Here we use paracompactness.) Let $\mathscr{D}$ be the union of the collections $\mathscr{D}_m$. Then $\mathscr{D}$ is σ -locally finite.
[Claim]: $\mathscr{D}$ is a basis for X. Then our theorem follows.

Let x be a point of X and let U be a neighbourhood of x. We seek to find an element D of $\mathscr{D}$ such that $x \in D \subseteq U$. Now x belongs to only finitely many elements of $\mathscr{C}$, say to $C_1, \cdots, C_k$. Then $U \cap C_i$ is a neighbourhood of x in the set C_i, so there is an $\varepsilon_i > 0$ such that

$$B_{C_i}(x, \varepsilon) \subseteq U \cap C_i.$$

Choose m so that $2/m < \min\{\varepsilon_1, \cdots, \varepsilon_k\}$. Because the collection $\mathscr{D}_m$ covers X, there must be an element D of $\mathscr{D}_m$ containing x. Because $\mathscr{D}_m$ refines $\mathscr{A}_m$, there must be an element $B_C(y, 1/m)$ of $\mathscr{A}_m$, for some $C \in \mathscr{C}$ and some $y \in C$, that contains D. Because

$$x \in D \subseteq B_C(y, 1/m),$$

the point x belongs to C, so that C must be one of the sets $C_1, \cdots, C_k$. Say $C := C_i$. Since $B_C(y, 1/m)$ has diameter at most $2/m < \varepsilon_i$, it follows that

$$x \in D \subseteq B_{C_i}(y, 1/m) \subseteq B_{C_i}(x, \varepsilon_i) \subseteq U.$$

□

Smirnov Metrization Theorem is a fundamental result in topology that establishes a connection between metrizability and topological properties, providing a criterion for when a topological space can be equipped with a metric structure.

4.1 Manifold Introduction

It may arise the consideration that since the Euclidean spaces possess so many good properties and behaviours, in practice, can we always treat our spaces as Euclidean? The question is no since not every spaces have such a behaved structure. Then a natural question is that how do we use the properties in Euclidean when we are not working on it? This motivates the invention of manifolds. In fact, the motivation for defining manifolds lies in the need to create a mathematical framework that can capture and describe the geometry and topology of spaces in a flexible and versatile way. Manifolds provide a bridge between local geometry, which can be understood using calculus and linear algebra, and global topology, which focuses on the broader properties of spaces. The following introduction of the manifolds is from [37], we only include the necessary parts in it in order to derive the embedding theorems; for those who are interested in manifolds, we recommend [35], [36], [37], and [38], where in the last one more concrete results and examples are offered.

Our desired construction should contain these features:

(i) Capture the idea that in the vicinity of any point on it, the space behaves like a Euclidean space.
(ii) Provide a framework for studying smooth and continuous transformations between spaces.
(iii) Allow us to classify and distinguish spaces based on their topological properties.

To this end, we adapt the following definition of an n-dimensional manifold met all the expectations:

Definition: n-dimensional Topological Manifold

An n-dimensional topological manifold is defined to be a $T_2\,A_2$ topological space, for which every point has an open neighbourhood homeomorphic to an open set in $\mathbb{R}^n$.

Loosely speaking, manifolds are the mathematical objects that are used to model the abstract shapes of "phisical spaces". A d-dimensional manifold is a topological space that locally looks like $\mathbb{R}^d$ as the definition. For example, the surface of the Earth looks locally flat, like a piece of the plane, but globally its topology is that of a sphere. The universe is modeled by a 3-dimensional manifold because locally it looks like a piece of $\mathbb{R}^3$, but its global topology might be more complicated. Space-time is a 4-dimensional manifold. The space of possible positions of a ball rolling on a plane is a 5-dimensional manifold.

The last condition in the n-dimensional topological manifold means that a topological manifold looks locally like $\mathbb{R}^n$, as we desired. Therefore the following terminology is used.

Definition: n-dimensional Chart

Let M be a topological space. An n-dimensional chart (U, φ) on M consists of a homeomorphism $\varphi : U \to \tilde{U}$ from an open set $U \subseteq M$ to an open set $\tilde{U} \subseteq \mathbb{R}^n$.
The **components** of φ are denoted by

$$\varphi := (x_\varphi^1, \cdots, x_\varphi^m), \text{ where } x_\varphi^i : U \to \mathbb{R}, \text{ for } 1 \le i \le m,$$

and are called the **local coordinates** on M corresponding to the chart (U, φ).
The inverse map $\varphi^{-1} : \tilde{U} \to U$ is called a **local parameterization** of M.

Definition: Topological Atlas

Let M be a topological space. An m-dimensional topological atlas on M (or C^0 -atlas on M) consists of a collection of m-dimensional charts
$\mathscr{A} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ covering M, i.e. $M = \bigcup_{\alpha\in I} U_\alpha$.

With this terminology, note that a Hausdorff, second countable topological space is an n-dimensional topological manifold if and only if it admits an n-dimensional topological atlas.

When we define a new space, it is important to know the isomorphisms on it. In topological spaces, we define such an isomorphism as homeomorphism; in manifolds, we define such an isomorphism as "diffeomorphism".

Definition: Diffeomorphism

A map $f : U \to V$ between open sets $U, V \subseteq \mathbb{R}^n$ is called a homeomorphism if f is bijective and both f and f^{-1} are smooth.

We used the term "smooth", in order to make this lecture note as self-contained as it can be, we now offer a brief review on the smoothness.

The smoothness of a function is a fundamental concept in calculus and analysis that describes how nicely a function behaves with respect to its derivatives. Smoothness is often associated with the concept of continuity, but it goes further by considering higher-order derivatives.

Definition: Smooth Function

A function is considered smooth if it has derivatives of all orders (i.e., it is infinitely differentiable).

Remark:

(i) Smooth functions have derivatives at every point and exhibit no abrupt changes in their behavior.

(ii) The smoothness of a function is often indicated by the notation C^∞, which represents the class of smooth functions.

(iii) Examples of smooth functions include polynomial functions, trigonometric functions, and exponential functions. ||

This definition is important since it can be related to not only the concept of analytic functions, but also to the concept of curvature: Analytic functions are a subset of smooth functions that can be expressed as a convergent power series (An analytic function is smooth and can be approximated with high precision using Taylor series expansions.) In the context of curves, the smoothness of a curve is related to the curvature. A curve is smoother if its curvature varies more slowly along its length.

Let us now go back to the discussion of diffeomorphism. Note that (as in the case of homeomophisms) the condition that f^{-1} be smooth is not automatically satisfied. The standard example of a smooth bijection which is not a diffeomorphism is given by:

$$f : \mathbb{R} \to \mathbb{R}, f(t) := t^3.$$

In a local chart a topological manifold is described as an open piece of $\mathbb{R}^n$. To develop analysis on manifolds, one needs to introduce derivatives and integration of functions, notions which, locally in charts, should coincide with those from multivariable calculus. However, a function that is differentiable in one chart may fail to be differentiable in the other chart. To circumvent this, on a smooth manifold one only works with mutually compatible charts.

Definition: Compatible Charts

Let M be a topological space. Let (U, φ) and (V, ψ) be two n-dimensional charts on M. The map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is called the **change of coordinates map** or the **transition map** between the two charts. The two charts are said to be **compatible** if the transition map is a diffeomorphism.

Definition: Smooth Atlas (C^∞-atlas)

An n-dimensional topological atlas $\mathscr{A}$ on an n-dimensional topological manifold M is said to be an n-dimensional smooth atlas (C^∞-atlas) if every two charts in $\mathscr{A}$ are compatible.

Example 4.1: C^∞-Atlases

On $\mathbb{R}^n$ there is a smooth atlas with only one chart $\mathscr{A} := \{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$.
Another smooth atlas is the collection of all diffeomorphisms between open subsets of $\mathbb{R}^n$:

$$\mathscr{B} := \left\{(U, \varphi) : U, V \subseteq \mathbb{R}^n \text{ are open and } \varphi : U \to V \text{ diffeomorphism}\right\}.$$

In fact, $\mathscr{B}$ consists of all charts compatible with $(\mathbb{R}^n, Id_{\mathbb{R}^n})$. ||

We now introduce a relation on atlases and we will prove that such a relation is in fact an equivalence relation.

Definition: Equivalence Relation

Let M be an n-dimensional topological manifold. Consider the following relation on n-dimensional C^∞-atlases on M :

$$\mathscr{A}_1 \sim \mathscr{A}_2 \Leftrightarrow \mathscr{A}_1 \cup \mathscr{A}_2 \text{ is a } C^\infty\text{-atlas.}$$

Proposition 4.1:

The relation $\sim$ is an equivalence relation on the set of n-dimensional C^∞ -atlases of M.

Proof:

Reflexivity and symmetry of the relation $\sim$ are obvious. We will check that transitivity also holds. Consider three n-dimensional C^∞-atlases on M: $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ such that $\mathscr{A}_1 \sim \mathscr{A}_2$ and $\mathscr{A}_2 \sim \mathscr{A}_3$. We need to show that each pair of the charts $(U_1, \varphi_1) \in \mathscr{A}_1$ and $(U_3, \varphi_3) \in \mathscr{A}_3$ are compatible, i.e. that the map

$$\varphi_1 \circ \varphi_3 : \varphi_3(U_1 \cap U_3) \to \varphi_1(U_1 \cap U_3) \tag{4.1}$$

is a homeomorphism. This map is clearly a bijective, with inverse being:

$$\varphi_3 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_3) \to \varphi_3(U_1 \cap U_3). \tag{4.2}$$

So, it suffices to show that:

[Claim]: (4.1) and (4.2) are smooth.

Let $p \in U_1 \cap U_3$. Since $\mathscr{A}_2$ is an atlas, there exists a chart $(U_2, \varphi_2) \in \mathscr{A}_2$ such that $p \in U_2$. Since $\mathscr{A}_1 \sim \mathscr{A}_2$ and $\mathscr{A}_2 \sim \mathscr{A}_3$ it follows that the following maps are diffeomorphisms:

$$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2) \tag{4.3}$$

and

$$\varphi_2 \circ \varphi_3^{-1} : \varphi_3(U_2 \cap U_3) \to \varphi_2(U_2 \cap U_3). \tag{4.4}$$

In particular, the restriction of their composition is a diffeomorphism:

$$\varphi_1 \circ \varphi_3^{-1} : \varphi_3(U_1 \cap U_2 \cap U_3) \to \varphi_1(U_1 \cap U_2 \cap U_3).$$

Hence, (4.1) and (4.2) are smooth when restricted to the open neighbourhoods $\varphi_3(U_1 \cap U_2 \cap U_3)$ and $\varphi_1(U_1 \cap U_2 \cap U_3)$, respectively, of $\varphi_3(p)$ and $\varphi_1(p)$.

Since p is chosen arbitrarily, result follows.

□

In the context of topological manifolds, a maximal atlas is a concept related to the structure and smoothness of the manifold. A maximal atlas on a topological manifold M is an atlas that is maximal in the sense that it cannot be extended by adding more compatible charts. In other words, it's the largest collection of charts that can be defined on M such that they cover M and are compatible with each other.

Definition: Maximal Atlas

An n-dimensional C^∞-atlas $\mathscr{A}$ on the topological manifold M is said to be maximal if $\mathscr{A} \sim \mathscr{B} \Rightarrow \mathscr{B} \subseteq \mathscr{A}$ for any n-dimensional C^∞-atlas $\mathscr{B}$ on M.

Proposition 4.2:

Any equivalence class of smooth atlases has a unique maximal representative. The maximal C^∞-atlas $\mathscr{A}_{\max}$ equivalent to the C^∞-atlas $\mathscr{A}$ is given by

$$\mathscr{A}_{\max} := \big\{(U,\varphi) : A \cup \{(U,\varphi)\} \text{ is a } C^\infty \text{ atlas}\big\}.$$

Proof: Consult [37].

Definition: Differentiable Structure (Smooth Structure)

An n-dimensional differentiable structure (or smooth structure) on a topological manifold M is an equivalence class of n-dimensional C^∞-atlas on M. Equivalently, by **Proposition 4.2**, a differentiable structure is the same as a maximal atalas on M.

A smooth manifold is defined by its maximal smooth atlas, which is a maximal collection of charts (diffeomorphisms) that make the manifold a smooth space. Every smooth manifold has a maximal smooth atlas, and this atlas uniquely defines the smooth structure of the manifold.

Definition: Smooth Manifold

An n-dimensional smooth manifold is a topological manifold endowed with an n-dimensional smooth structure.

Remark:

The axiom of being second countable insures that manifolds are not "too big". Let us mention that there are examples of Hausdorff topological spaces, endowed with a smooth atlas, but which are not second countable. An easy example is the disjoint union of an uncountable collection of manifolds of the same dimension; e.g. an uncountable collection of points with the discrete topology is not second countable, and has a 0-dimensional atlas. ||

4.2 Embedding Manifolds

Embedding of manifolds is a fundamental concept in differential geometry and topology. It involves the inclusion of one manifold into another in a way that preserves certain topological and geometric properties. The most basic form of embedding is a topological embedding, where the map f is required to be a homeomorphism onto its image $f(M)$. In this case, $f(M)$ is a topological manifold, and f serves as a topological isomorphism between M and $f(M)$. This is the main discussion we shall hold in this subsection and the main resources could be found in [1] and [46]. In differential geometry, we often work with smooth manifolds and smooth embeddings. We shall not talk about this concept and we recommend [35], [37], [38], and [45].

Recall that a 1-manifold is often called a curve, and a 2-manifold is called a surface. We shall prove that if X is a compact manifold then X can be embedded into a finite-dimensional Euclidean space. The theorem holds without the assumption of compactness, but the proof is a good deal harder.

First we introduce some termnilogies. Recall that the support of $f : X \to \mathbb{R}$ is defined to be the closure of the set $f^{-1}(\mathbb{R}\backslash\{0\})$. Thus if x lies outside the support of f, there is some neighbourhood of x on which f vanishes.

Definition: Partition of Unity

Let $\{U_1, \cdots, U_n\}$ be a finite indexed open covering of the space X. An indexed family of continuous functions

$$f_i : X \to [0,1] \text{ for } i = 1, \cdots, n,$$

is said to be a partition of unity by $\{U_i\}$ if

(i) $\operatorname{supp} f_i \subseteq U_i \ \forall i = 1, \cdots, n$.

(ii) $\displaystyle\sum_{i=1}^{n} f_i(x) = 1 \ \forall x \in X$.

Remark: Possible Connection to Probability Theory

While there can be conceptual connections between partitions of unity and probability, they are distinct mathematical concepts with their own formalisms and applications. Partitions of unity are primarily used in topology and geometry, while probability theory deals with uncertainty and randomness. The connection between the two arises in specific applications where smooth and continuous functions play a role in probabilistic modeling or density estimation. ||

Theorem 4.3: Existence of Finite Partitions of Unity

Let $\{U_1, \cdots, U_n\}$ be a finite open covering of the normal space X. Then there exists a partition of unity dominated by $\{U_i\}$.

Proof: Consult [1] **Theorem 36.1**.

We have now equipped with all the backgrounds of proving the main theorem of this chapter:

Theorem 4.4: Embedding

If X is a compact n-manifold then X can be imbedded in $\mathbb{R}^N$ for some positive integer N.

Proof:

Cover X by finitely many open sets $\{U_1, \cdots, U_n\}$, each of which may be embedded in $\mathbb{R}^n$. Choose embeddings $g_i : U_i \to \mathbb{R}^n \ \forall i$. Being compact and Hausdorff, X is normal. Let $f_1, \cdots, f_n$ be a partition of unity dominated by $\{U_i\}$; let $A_i := \operatorname{supp} f_i$. For each $i = 1, \cdots, n$, define a function $h_i : X \to \mathbb{R}^n$ by the rule

$$h_i(x) := \begin{cases} f_i(x) \cdot g_i(x), & \text{for } x \in U_i \\ 0 := (0, \cdots, 0), & \text{for } x \in X \backslash A_i \end{cases}.$$

The function h_i is well-defined since the two functions of h_i agree on the intersection of their domains, and h_i is continuous because its restrictions to the open sets U_i and $X \backslash A_i$ are continuous. Now define

$$F : X \to (\mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}),$$

where there are $2n$ times Cartesian product. By the rule, one has

$$F(x) = (f_1(x), \cdots, f_n(x), h_1(x), \cdots, h_n(x)).$$

Clearly F is continuous.

[Claim]: F is an embedding

It suffices to show that F is injective since X is compact. Suppose that $F(x) = F(y)$. Then $f_i(x) = f_i(y)$ and $h_i(x) = h_i(y) \ \forall i$. Suppose $f_i(x) > 0$

for some $i, f_i(y) > 0$ also, so that $x, y \in U_i$. Then

$$f_i(x) \cdot g_i(x) = h_i(x) = h_i(y) = f_i(y) \cdot g_i(y).$$

Since $f_i(x) = f_i(y) > 0$, it follows that $g_i(x) = g_i(y)$. But $g_i : U_i \to \mathbb{R}^n$ is injective, so that $x = y$ as we desire.

□

Let us now turn this into a definition:

Definition: Embedded Manifold

A subset $M \subseteq \mathbb{R}^n$ is called an n-diensional embedded manifold in $\mathbb{R}^n$ if around every point in M there exists an open set $U \subseteq \mathbb{R}^n$ and there exists a homeomorphism $f : U \to V$ where $V \subseteq \mathbb{R}^n$ is open, such that

$$f(M \cap U) = (\mathbb{R}^n \times \{0\}) \cap V.$$

We will call a diffeomorphism (U, f) as above a chart adopted to M.

We enclose this subsection via the introduction of Whitney's Embedding Theorem. Whitney's Embedding Theorem, also known as the Whitney Embedding Theorem, is a fundamental result in differential topology. It was proved by the American mathematician Hassler Whitney in 1936. This theorem is a significant milestone in the study of smooth manifolds and provides insights into their embedding in higher-dimensional Euclidean spaces. The motivation behind Whitney's Embedding Theorem and the study of embedding smooth manifolds into higher-dimensional spaces lies in the desire to understand the structure and properties of smooth manifolds.

Theorem 4.5: Whitney's Embedding Theorem

Any n-dimensional smooth manifold for $n \geq 1$ is diffeomorphic to an embedded manifold in $\mathbb{R}^{2m}$, which is a closed subset.

Proof: Consult [47].

5.1 Group Theory

In this subsection, we will offer fundamental concepts in group theory, which, for most of the readers, could be seen as a review session, therefore skipping this subsection will does no harm to further exploration.

We follow the general introduction of the group theory. Our description of the general abstract algebra follows from [56] and [73], for comprehensive treatment on group theory one may consult [71], [72], and [75].

Definition: Group

A group is a set G, together with a binary operation, namely $*$, such that $\forall a, b, c \in G$, one has

(i) $a * b \in G$. (Closure)

(ii) $a * (b * c) = (a * b) * c$. (Associativity)

(iii) $\exists i \in G$ such that $a * i = a = i * a$. (Existence of Identity)

(iv) $\exists a^{-1} \in G$ s.t. $a * a^{-1} = i = a^{-1} * a$. (Existence of Inverse)

Although group structure seems easily to establish, however, its structure is far from simple. An impotant reason is that we do not assume $\forall a, b \in G$, $a * b = b * a$. This is an algebraic property and any group $(G, *)$ with this additional condition will be called an Abelian group:

Definition: Abelian Group

A group $(G, *)$ is said to be Abelian if $\forall a, b \in G, a * b = b * a$.

One of the original motivations for studying Abelian group, according to [74], was number theory, in particular the study of the Ideal Class Group of a number ring and the group of units $(\mathbb{Z}/n\mathbb{Z})^*$ notably by C. F. Gauss. More generally, abelian groups arise naturally in terms of Cohomology Theories, which serve to distinguish (geometric) objects by algebraic invariants.

Now we introduce a way in constructing a new group with existing ones, this is called the direct product of groups. Distinct from the Cartesian product, this operation preserves the group structure. It provides a way to combine groups in a systematic and structured manner.

Definition: Direct Product

Let $(G, *)$ and $(H, \circ)$ be two groups. On the Cartesian product given by $G \times H$ we deine an operation $\diamond$ via

$$(g_1, h_1) \diamond (g_2, h_2) = (g_1 * g_2, h_1 \circ h_2) \ \forall g_1, g_2 \in G \text{ and } \forall h_1, h_2 \in H.$$

Under this operation, we call $G \times H$ the direct product of G and H.

The Cartesian product is a set-theoretical concept used to create a set containing all possible ordered tuples formed by taking one element from each of the component sets. It's a way to describe the combinations of elements from different sets. The direct product, on the other hand, is a concept used in group theory to create a new group from existing groups. It involves defining a binary operation on ordered tuples of group elements to form a new group.

In summary, both Cartesian product and direct product involve combining elements from multiple sets or groups, but they serve different purposes and have distinct mathematical structures. The Cartesian product focuses on forming sets of ordered tuples, while the direct product is used to create new groups with specific algebraic properties.

In the above context, we know another structural property called algebraic property, since we introduced the topological properties before, now we talk about their difference:

Topological Properties:

These properties are concerned with the geometric and spatial aspects of mathematical spaces, particularly topological spaces. They describe how points are related to each other in terms of proximity, continuity, and connectedness. Examples of topological properties include openness, compactness, continuity, and connectedness.

Algebraic Properties:

These properties are concerned with the algebraic structure of mathematical objects, such as groups, rings, fields, and vector spaces. Algebraic properties describe how elements in these structures interact under algebraic operations like addition, multiplication, and inverses. Examples of algebraic properties include associativity, commutativity, the existence of inverses, and distributivity.

In summary, topological properties and algebraic properties represent different aspects of mathematical structures. Topological properties are concerned with spatial relationships and continuity, while algebraic properties focus on algebraic operations and their behavior within specific algebraic structures. Both types of properties are essential for understanding different mathematical contexts and structures.

We admitted that the direct product between two groups is also a group, let us now prove this is well-defined.

Theorem 5.1:

The direct product of two groups is also a group.

Proof:

Let us adopt the same notation as in the definition. First we must check that

Step I: Direct product is closed.

Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$, then

$$(g_1, h_1) \diamond (g_2, h_2) = (g_1 * g_2, h_1 \circ h_2) \in G \times H$$

since $g_1 * g_2 \in G$ and $h_1 \circ h_2 \in H$, closure follows.

Step II: Associativity.

Let $g_1, g_2, g_3 \in G$ and $h_1, h_2, h_3 \in H$, then

$$\begin{aligned}\big((g_1, h_1) \diamond (g_2, h_2)\big) \diamond (g_3, h_3) &= (g_1 * g_2, h_1 \circ h_2) \diamond (g_3, h_3)\\ &= \big((g_1 * g_2) * g_3, (h_1 \circ h_2) \circ h_3\big)\\ &= \big(g_1 * (g_2 * g_3), h_1 \circ (h_2 \circ h_3)\big)\\ &= (g_1, h_1) \diamond \big((g_2, h_2) \diamond (g_3, h_3)\big).\end{aligned}$$

Step III: Existence of Identity

Since $(G, *)$ and $(H, \circ)$ are groups. We may denote i_G and i_H to be their identity elements, respectively. Then $\forall g \in G$ and $\forall h \in H$, one has

$$(g, h) \diamond (i_G, i_H) = (g * i_G, h \circ i_H) = (g, h).$$

The other side holds analogously.

Step IV: Existence of Inverse.

$(g, h) \diamond (g^{-1}, h^{-1}) = (g * g^{-1}, h \circ h^{-1}) = (i_G, i_H)$, similarly, we also have $(g^{-1}, h^{-1}) \diamond (g, h) = (i_G, i_H)$, existence of inverse follows thereafter.

□

In fact, **Theorem 5.1** could be generalized into "The direct product of countable groups is still a group". Moreover, this statement is still valid when one replaces the countable condition by arbitrary. Therefore, the direct product of arbitrarily many groups, countable or not, finite or not, is still a group. We may say that, as we explained before, the group structure is preserved under the operation of taking direct product.

We now state some theorems without proof. These theorems contribute to the fundamental properties of a group. Readers who are not familiar with them could consult [56].

Theorem 5.2: Uniqueness of Identity and Inverse

Let $(G, \circ)$ be a group. Then the identity i_G of G is unique, as well as the inverse a^{-1} for arbitrary element $a \in G$.

Theorem 5.3: Invariant under Permutations

Let $(G, \circ)$ be a group and let $a_1, \cdots, a_n \in G$ be its elements. Then regardless of how the product $a_1 a_2 \cdots a_n$ is bracket, the result equals

$$(\cdots(((a_1 a_2) a_3) a_4) \cdots a_{n-1}) a_n. \tag{5.1}$$

Theorem 5.4: Inverse Operation

Let $(G, \circ)$ be a group and let $a, b \in G$ be its elements. Then:

(i) $(a^{-1})^{-1} = a$.

(ii) $(ab)^{-1} = b^{-1}a^{-1}$.

Theorem 5.5: Cancellation Law

Let $(G, \circ)$ be a group and let $a, b, c \in G$ be its elements. If either $ab = ac$ or $ba = ca$, then $b = c$.

Like the treatment of the size of a set in set theory, we are also interested in the size of a given group. This motivates us to define the size of a group, namely, the order of a group. In practice, most groups share the same order are isomorphic with each other.

Definition: Order

If $(G, \circ)$ is a group. Then the **order** of G, denoted by $|G|$, is the number of elements in the set G. We say that G is a **finite group** if its order is finite; otherwise, we say G is an infinite group.

In fact, just like the relationship between the structure of a topology on a given set A and the set-theoretical structure of itself. We also concern the relationships between the structure of a group on A with the corresponding binary operation $\circ$ and structure purely on A. Loosely speaking, a simple set A has no structure, it is barely a collection of elements, endowed with either topological structure or algebraic structure, we are able to study the behaviour of its elements via the given operations. However, since these structures are endowed to the set A, it is natural to ask if the underlying structure (if any) provides insights for additional possible structure like topological structures. This is a rather philosophical study in mathematics, one need to consider under what conditions the underlying system should possess so that the endowed structures could lead to no contradictions, e.g. the Russell's Paradoxes, where the statement that there is no such a set contanining all the sets. Sometimes we call it descriptive set theory, where the study of the axiomatic systems matters (e.g. Axiom of Choice). We shall go through this special topic in the sixth chapter.

In order to derive one of our main topics in this section, we shall offer some important results on the order of a given set, still, without proof. Readers could consult [56] for detailed descriptions.

Theorem 5.6: Order Operation

Let $(G, \circ)$ be a group and let $a \in G$ be its element. Pick $n, m \in \mathbb{Z}$. Then

(i) $a^m a^n = a^{m+n}$.

(ii) $(a^m)^n = a^{mn}$, and, in particular,

(iii) $a^{-n} = (a^{-1})^n$.

Definition: Cyclic Group

Recall that in a given group $(G, \circ)$, the power of $a \in G$ is defined to be $a^n := a \cdot \cdots \cdot a$ where there are n many a's being multiplied. In particular, note that $a^0 = i_G$. A group G is said to be **cyclic** if there exists an element $a \in G$ such that $\forall b \in G$, $\exists n \in \mathbb{Z}$ such that $a^n = b$. In particular, we say that G is **generated** by a, and denote $G := \langle a \rangle$.

A very important reulst on cyclic group is that every cyclic groups is abelian, the proof is rather trivial.

Theorem 5.7:

Every cyclic groups is Abelian.

In the previous chapter, we studied the concept of subspace topology, which is the restriction of the original space but preserves all the topological structures. In the algebraic aspects, we wish the same behaviour to hold.

Definition: Subgroup

Let $(G, \circ)$ be a group. Then a subset $H \subseteq G$ is called a subgroup of $(G, \circ)$ if H is a group under the same operation, namely $\circ$. In this case, we use $H \leq G$ to denote that H is a subgroup of G. Similarly as we did in the elementary set theory, if $H < G$ we say that H is a **proper subgroup** of G.

Remark:

Every group is a subgroup of itself, and $\{i\}$ is a subgroup of every group. ||

The isomorphisms between topological spaces that preserve the topological structures is called homeomorphism, in algebraic aspect, we have the another such a morphism, called simply the isomorphism, which preserve the algebraic structures.

Definition: Homomorphism (Group Homomorphism)

Let $(G, \circ)$ and $(H, *)$ be two groups. Then a group homomorphism (or, simply, a homomorphism) from G to H is a function $\alpha : G \to H$ such that

$$\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2) \ \forall g_1, g_2 \in G.$$

In particular, if $G = H$ and $\alpha = Id_G$ then α is the identity homomorphism.

Definition: Isomorphism

A homomorphism $\alpha : G \to H$ which is bijective is called an isomorphism.
Two groups are said to be isomorphic if there exists an isomorphism between them.

We make two remarks here. First recall a question we asked when proposed the homemorphism, when do we say that two topological spaces are the same? The answer is that if there exists a homeomorphism between them, this is the same thing we concern in group isomorphisms. Secondly do bear in mind that all homemorphisms are continuous bijection, but a continuous bijection may not be a homeomorphism; but in group isomorphisms, according to the definition, the inverse, however, holds. Moreover, one may notice that when we define a new mathematical object, we always consider build up a morphism to preserve the underlying structures, but such a morphism may behave differently.

Definition: Endomorphism

A homomorphism from a group to itself, e.g. $\alpha : G \to G$, is called an endomorphism.

Definition: Automorphism

An endomorphism which is also an isomorphism is called an automorphism.

In topology, we did not assign special names for the mapping inside the same space. But in group theory, we do concern these mappings. This is because the different mathematical objects we are dealing with. In summary, group theory and topology have different objectives and deal with different mathematical structures. Group theory focuses on algebraic structures and internal mappings (automorphisms) that preserve the algebraic properties of groups. Topology focuses on the topological properties of spaces and continuous mappings between different spaces. The concept of homeomorphisms in topology serves a similar role to automorphisms in group theory but within the context of topological spaces.

Proposition 5.8: Homomorphisms are closed under composition

Let $(G, \circ)$, $(H, *)$, and $(M, \diamond)$ be three groups. Let $\alpha : G \to H$ and $\beta : H \to M$ be two homomorphisms. Then the composition $\beta\alpha : G \to M$ is also a homomorphism.

Proof:

Let $x, y \in G$. Then $\beta\alpha(x \circ y) = \beta(\alpha(x) * \alpha(y)) = \beta\alpha(x) \diamond \beta\alpha(y)$.

□

The kernel of a group homomorphism is a subgroup that captures the elements mapping to the identity element in the target group. It plays a fundamental role in group theory, helping to detect homomorphisms, understand injectivity, and relate the domain and codomain groups through important theorems like the First Isomorphism Theorem. It is a valuable tool for studying the structure and relationships between groups.

Definition: Kernel

If $\alpha : G \to H$ is a homomorphism. Then the kernel of α is defined to be the set

$$\ker(\alpha) := \{g \in G \,|\, \alpha(g) = i_H\}.$$

Theorem 5.9:

Let $\alpha : G \to H$ be a homomorphism and let $g \in G$ be an element. Then

(i) $\alpha(i_G) = i_H$,

(ii) $\alpha(g^n) = (\alpha(g))^n$.

Proof: Consult [56] **Theorem 4.10**.

Definition: Center

If $(G, \circ)$ is a group. Then the center of G, denoted by $Z(G)$, is defined to be the set of elements in G that commute with everything in G. That is to say,

$$Z(G) := \{z \in G \,|\, az = za \,\forall a \in G\}.$$

Theorem 5.10:

If G is a group, then $Z(G)$ is a subgroup of G.

Proof:

Certainly $i_G a = a = a i_G \ \forall a \in G$ hence $i_G \in Z(G)$, existence of identity follows. If $y, z \in Z(G)$ and $a \in G \ \Rightarrow yza = yaz = ayz$; thus $yz \in Z(G)$, closure holds. Furthermore, if $z \in Z(G)$ and $a \in G \ \Rightarrow a^{-1}z = za^{-1}$. Inverting both sides yields $z^{-1}a = az^{-1} \ \Rightarrow z^{-1} \in Z(G)$ hence the existence of the inverse is valid, result follows.

□

Theorem 5.11: Subgroup Criterion

Let G be a group and let $H \subseteq G$ be a subset. Then H is a subgroup of $G \Leftrightarrow$

(i) $i \in H$;

(ii) $ab^{-1} \in H \ \forall a, b \in H$.

Proof: Consult [56] **Theorem 3.13**.

Theorem 5.12: Alternative Subgroup Criterion

Let G be a group and let H be a finite subset of G. Then $H \leq G \Leftrightarrow$

(i) $i \in H$;

(ii) $ab \in H \; \forall a, b \in H$.

Proof: Consult [56] **Theorem 3.14**.

We shall introduce cosets, normal subgroup, and quotient group to enclose this subsection.

Definition: Congruent Modulo

Let G be a group and let H be a subgroup. If $a, b \in G$, then we say a is congruent to b modulo H, and we denote it as $a \equiv b(\mod H)$ if $a^{-1}b \in H$ (or, in the case of an additive group $-a + b \in H$).

Lemma 5.13: Equivalence Relation

Let G be a group and H be a subgroup. Then the congruence modulo H is an equivalence relation on G.

Proof:

To show the equivalence is to show the following:

Step I: Reflexivity

If $a \in G$, then $a^{-1}a = i \in H$, and therefore $a \equiv a(\mod H)$.

Step II: Symmetry

If $a, b \in G$ and $a \equiv b(\mod H)$, then $a^{-1}b \in H$ and therefore $(a^{-1}b)^{-1} = b^{-1}a$ lies in H as well. But this means that $b \equiv a(\mod H)$.

Step III: Transitivity

Suppose that $a, b, c \in G$, where $a \equiv b(\mod H)$ and $b \equiv c(\mod H)$. Then $a^{-1}b, b^{-1}c \in H$. But in this case, H contains their product, $a^{-1}bb^{-1}c = a^{-1}c$. Thus, it follows that $a \equiv c(\mod H)$.

□

Lemma 5.14: Equivalence Class

Let G be a group and let H be a subgroup. If $a \in G$, then its equivalence class with respect to congruence modulo H is the set $\{ah | h \in H\}$.

Proof:

If $a \equiv b(\mod H)$, then $a^{-1}b \in H$, so $a^{-1}b = h$, for some $h \in H$. Thus, $b = ah$, which is in our set. Conversely, if $b = ah$, for some $h \in H$, then $a^{-1}b = h \in H$, and therefore $a \equiv b(\mod H)$.

□

Definition: Left Coset

Let G be a group, $H \leq G$ and $a \in G$. Then the left coset of a with respect to H is the set $\{ah | h \in H\}$, which is denoted aH. (If the group operation is addition, then we will write $a + H := \{a + h | h \in H\}$.)

In summary, cosets are an important concept in group theory, providing a way to partition a group into sets that share certain properties. They are a foundational concept for understanding group structure, Lagrange's theorem, and the formation of factor groups, which are critical tools in group theory and its applications.

Remark:

Cosets provide a natural way to partition a group into distinct sets that share certain properties. This partitioning helps organize the elements of the group based on their relationships with a given subgroup.

Theorem 5.15:

Let G be a group and H be a subgroup. Then the left cosets of H in G partition G. In particular,

(i) Each $a \in G$ is in exactly one left coset, namely aH;

(ii) If $a, b \in G$, then either $aH = bH$ or $aH \cap bH = \emptyset$.

Two points should be kept in mind here. First, left cosets are not subgroups! Remember, the left cosets partition G, and therefore the identity can only be in one of them, namely, $iH = H$. The rest cannot possibly be subgroups. Second, as we have already seen, when we write aH, the element a is not unique. Indeed, since the left cosets are equivalence classes, we have $aH = bH$ if and only if $a^{-1}b \in H$.

We can now prove our first big result on finite groups, due to Joseph-Louis Lagrange.

Theorem 5.16: Lagrange's Theorem

Let G be a finite group and H a subgroup. Then $|H|$ divides $|G|$.

Proof:

We have already seen that G is partitioned into left cosets; in particular, $|G|$ is the sum of the sizes of these cosets. But for any $a \in G$, $aH = \{ah \mid h \in H\}$. Now, if $ah_1 = ah_2$, with $h_1, h_2 \in H$, then by the cancellation law, $h_1 = h_2$. Therefore, aH consists of precisely $|H|$ distinct elements. It now follows that the order of G is $|H|$ multiplied by the number of left cosets. In particular, $|H|$ divides $|G|$.

□

Definition: Index

Let G be a group and $H \leq G$. Then the index of H in G, denoted by $[G : H]$, is the number of left cosets of H in G.

Corollary 5.16.1:

If G is a finite group and H is a subgroup, then $[G : H] = |G|/|H|$.

Definition: Right Coset

Let G be a group and $H \leq G$. Then for any $a \in G$, the right coset of a with respect to H is $Ha = \{ha \mid h \in H\}$. (If G is an additive group, then we write $H + a = \{h + a \mid h \in H\}$.)

If G is abelian, then there is no distinction between left and right cosets. In nonabelian groups, right cosets also partition G, but possibly in a different way.

Let H be a subgroup of G. We would like to form a group whose elements are the left cosets aH. Unfortunately, not just any subgroup will suffice; we need an extra condition. This is where normal subgroups come in. Recall that if $H \leq G$, then the left cosets of H do not necessarily coincide with the right cosets. We need to consider subgroups for which they do coincide.

Definition: Normal Subgroup

Let G be a group and N a subgroup. We say that N is a normal subgroup of G if $aN = Na \ \forall a \in G$.

Remember, when we say that $aN = Na$, we do not necessarily mean that $an = na$ $\forall n \in N$. Indeed, we could have $an = n_1 a$ for some different $n_1 \in N$.

Theorem 5.17:

If H is a subgroup of G and $a \in G$. Then $a^{-1}Ha$ is a subgroup of G. Furthermore, $|a^{-1}Ha| = |H|$.

Proof:

We have $i \in H$ and therefore $i = a^{-1}ia \in a^{-1}Ha$. If $a^{-1}h_1a, a^{-1}h_2a \in a^{-1}Ha$, then $(a^{-1}h_1a)(a^{-1}h_2a) = a^{-1}h_1(aa^{-1})h_2a = a^{-1}h_1ih_2a = a^{-1}h_1h_2a \in a^{-1}Ha$, since $h_1h_2 \in H$. Finally, if $a^{-1}ha \in a^{-1}Ha$, then $(a^{-1}ha)^{-1} = a^{-1}h^{-1}a \in a^{-1}Ha$, since $h^{-1} \in H$. Thus, $a^{-1}Ha$ is a subgroup of G. Also, given the definition of $a^{-1}Ha$, it is clear that we can only get one element for each element of H. But if $a^{-1}h_1a = a^{-1}h_2a$, then by cancellation, $h_1 = h_2$. Thus, it follows that $|a^{-1}Ha| = |H|$.

□

Let us now introduce our last terminology in this subsection. The quotient groups, which are a powerful tool in group theory that allow us to simplify the study of group structure by partitioning a group into cosets of a normal subgroup and defining a group operation on these cosets. They have wide-ranging applications in understanding group properties, solving equations in groups, and classifying group actions, making them a central concept in abstract algebra.

Definition: Quotient Group

Let G be a group and N be a normal subgroup. Then the quotient group G/N is the set of all left cosets aN, with $a \in G$, under the operation $(aN)(bN) = abN$.

The fact that the quotient group is indeed a group needs to be proved:

Theorem 5.18:

If G is any group and N is a normal subgroup, then G/N is a group of order $[G : N]$.

Proof: Consult [56] **Theorem 4.6**.

Theorem 5.19:

Let G be a group and N be a normal subgroup. Then the subgroups of G/N are precisely of the form H/N, where H is a subgroup of G containing N. Furthermore, H/N is normal in $G/N \Leftrightarrow H$ is normal in G.

Here is one more rather neat fact about quotient groups:

Theorem 5.20:

Let G be any group. If $G/Z(G)$ is cyclic, then G is abelian.

Proof:

Let $Z = Z(G)$ and suppose that $G/Z = \langle aZ \rangle$. Take any $b, c \in G$. Then $bZ = a^M Z$ for some integer m and $cZ = a^n Z$, for some integer n. Thus, $b = a^m y$ and $c = a^n z$ for some $y, z \in Z$. But noting that powers of a commute with each other and elements of Z commute with everything, we have $bc = a^m y a^n z = a^n z a^m y = cb$. Thus, G is abelian.

□

5.2 Topological Group

In this subsection we offer a method of marrying algebraic and topological structures. The materials are mainly from [72], [76], [77], [78], and [79]. Before we start, let us first introduce some group notations:

A group G is written multiplicatively if the binary operation of the group is written $G^2 \to G$; $(x, y) \mapsto xy$ and called multiplication; the unique inverse is written x^{-1} and the map $G \to G$; $x \mapsto x^{-1}$ is called inversion; and the identity is written 1_G. Given $S, T \subset G$, we write

$$S^{-1} := \{s^{-1} \,|\, s \in S\} \text{ and } ST := \{st \,|\, s \in S, t \in T\}.$$

For $n \in \mathbb{N}_0$ we define S^n inductively by

$$S^0 := \{1_G\} \text{ and } S^{n+1} := S^n S; \text{ and } S^{-n} := (S^{-1})^n.$$

It will also be convenient to write $xS := \{x\}S$ and $Sx := S\{x\}$ for $x \in G$, which aligns the usual notation for left and right cosets when S is a subgroup. This notation has effect that in general $SS^{-1} \neq S^0$ and $S^2 \neq \{s^2 \,|\, s \in S\}$.

We first introduce the semitopological group, then the topologized group, and lastly the topological group. It follows that semitopological group is finer than topologized group and the topologized group is finer than topological group.

Definition: Semitopological Group

We call a triple $(G, T_G, \circ)$ a semitopological group when (G, T_G) is a topological space and $(G, \circ)$ is a group, and the group operation $\circ : G \times G \to G$ that maps (x, y) to $x \circ y$ is continuous in each variable separately. When there is no ambiguity as to what the operation and topology are, we will simply use G to denote a semitopological group.

Note that the function $\circ : G \times G \to G$ is continuous in the variable x when the function $g_{y_0} : G \to G$ defined by $x \to x \circ y_0$ is continuous for all y_0 in G. Similarly, $\circ$ is continuous in y when the function $g_{x_0} : G \to G$ defined by $y \to x_0 \circ y$ is continuous for all x_0 in G.

For G to be a topological group we require G to satisfy all of the conditions for a semitopological group as well as two more requirements. The group operation needs to be continuous in both variables together and the inverse mapping given by $x \to x^{-1}$ needs to be continuous.

Remark:

Any group with the discrete topology is both a topological group and a semitopological group. ||

Theorem 5.21:

A locally compact Hausdorff semitopological group with a group operation that is continuous in both variables together is a topological group.

Proof: Consult [79], **Theorem 3.14**.

Connection: Semitopological Group and Topological Group

A semitopological group is a group equipped with a topology such that the group's multiplication operation is continuous with respect to that topology. In other words, it's a topological space and a group in which the group's binary operation is a continuous map.

A topological group, on the other hand, is a group equipped with a topology such that both the group's multiplication operation and the inverse operation are continuous maps. In other words, it's a topological space and a group in which both multiplication and inversion are continuous.

Similarities:

Both semitopological groups and topological groups combine the algebraic structure of a group (with operations like multiplication and inversion) with the topological structure of a topological space. Moreover, in both cases, the group's multiplication operation is required to be continuous with respect to the given topology. This ensures that group elements can be combined in a topologically coherent way.

Difference:

The primary difference lies in the treatment of the inversion operation. In a semitopological group, only the group's multiplication operation is required to be continuous, while the inversion operation may or may not be continuous. In contrast, in a topological group, both the multiplication and inversion operations must be continuous. Also, a topological group imposes stronger topological conditions on its topology than a semitopological group. Specifically, a topological group's topology must be a Hausdorff (T_2) topology, which ensures the separation of points, while a semitopological group does not require this level of separation.

In summary, both semitopological groups and topological groups combine group theory with topology, but they differ in the continuity requirements placed on the group operations. Semitopological groups require only the multiplication operation to be continuous, while topological groups require both multiplication and inversion to be continuous. Topological groups impose a stronger topological structure by requiring a Hausdorff topology, while semitopological groups allow for more flexibility in the choice of topology.

Definition: Topologized Group

A group G that is also a topological space is called a topologized group.

Without any additional assumptions these are no more than their constituent parts: a group and a topological space. When the group inversion $G \to G$ and the group operation $G^2 \to G$ are both continuous, where G^2 has the product topology, we say that G is a topological group.

Connection: Topologized Group and Semitopological Group

A topologized group is a group equipped with a topology in a way that the group operations (multiplication and inversion) are continuous with respect to this topology.

Both topologized groups and semitopological groups combine the algebraic structure of a group (with operations like multiplication and inversion) with the topological structure of a topological space. Moreover, in both cases, the group multiplication is required to be continuous with respect to the given topology. This ensures that group elements can be combined in a topologically coherent way.

Differences:

The main difference is in the treatment of the inversion operation. In a topologized group, both the multiplication and inversion operations are required to be continuous.

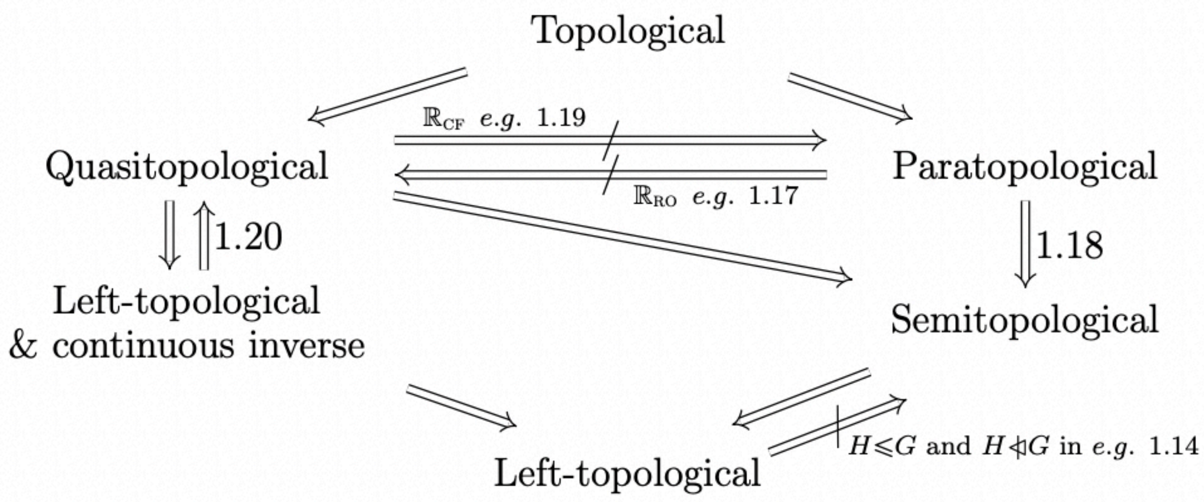

Figure 1: Relationships between types of topologized groups

In a semitopological group, only the multiplication operation needs to be continuous, and there's no continuity requirement for inversion. Also, semitopological groups are a more relaxed concept than topologized groups. While a semitopological group imposes a certain degree of topological structure on the group, it doesn't require as much topological regularity as a topologized group. Topologized groups are a stricter subclass of semitopological groups.

In summary, both topologized groups and semitopological groups blend group theory and topology, but they differ in the strength of their continuity requirements. Topologized groups impose continuity conditions on both multiplication and inversion, while semitopological groups require continuity only for multiplication, allowing more flexibility in the topology of the group.

Comment:

In general, semitopological group is finer than topologized group and the topologized group is finer than topological group.

A semitopological group is a more relaxed concept than a topologized group. In a semitopological group, continuity is required only for the group multiplication, while the inversion operation is not necessarily continuous. This provides more flexibility in the choice of topology.

A topologized group imposes a stricter topological structure than a semitopological group. In a topologized group, both the group multiplication and inversion must be continuous with respect to the chosen topology. This ensures a stronger level of topological regularity.

A topological group is a special case of a topologized group where the topology is required to be Hausdorff (T_2) and both the group multiplication and inversion are continuous. Thus, a topological group imposes the strongest topological conditions among these concepts.

So, in terms of the strength of topological conditions imposed, the hierarchy generally goes from semitopological group (weakest requirements) to topologized group (intermediate requirements) to topological group (strongest requirements). This hierarchy reflects the level of topological structure imposed on the underlying group.

In fact, there are other approaches other than semitopological group or topologized group.

One may consult [79] for a detailed description on these subjects as well as the interactions among them. We now focus on the description of the topological group. As we mentioned before, this is the marriage between topological properties and the algebraic properties. Now let us state the formal definition of a topological group.

Definition: Topological Group

> We say that $(G, \times, T_G)$ is a topological group if $(G, \times)$ is a group and (G, T_G) is a topological space such that, writing $M(x, y) := x \times y$ and $Jx := x^{-1}$ the multiplication map $M : G^2 \to G$ and the inversion map $J : G \to G$ are continuous.

Definition: Isomorphism

> If $(G, \times_G, T_G)$ and $(H, \times_H, T_H)$ are topological groups we say that $\theta : G \to H$ is an isomorphism if it is a group isomorphism and a topological homeomorphism.

We pause a little bit to consider a question: In defining a topological group, which property we focus more? The topological property or the algebraic topology? In fact, both the topological properties and the algebraic properties are equally important and are given significant attention. The goal is to combine the algebraic structure of a group with the topological structure of a topological space in a way that respects both structures.

Algebraic Property:

> The algebraic property of a group is fundamental and central in defining a topological group. The underlying set must form a group, meaning it must satisfy the group axioms (closure, associativity, identity element, and inverses). This algebraic structure is non-negotiable and serves as the foundation of the topological group.

Topological Property:

> Equally important is the topological property, which is the choice of topology on the group that respects the group's algebraic operations. The topology must be compatible with the group structure in a way that ensures the group operations (multiplication and inversion) are continuous functions. This compatibility between the group and the topology is crucial to defining a topological group.

Hausdorff Property (part of topological property):

> In many cases, a Hausdorff (T_2) topology is preferred for a topological group. The Hausdorff property ensures that points can be separated, which is desirable for a topological space. It helps in dealing with limits, continuity, and

convergence in a more convenient manner.

Balance:

The challenge in defining a topological group lies in finding the right balance between the algebraic structure and the topological structure. The topology should be chosen so that it respects the algebraic properties, yet it should be flexible enough to allow for continuous group operations.

In summary, the definition of a topological group places equal importance on both the algebraic and topological properties. The focus is on finding a topology that harmonizes with the group's algebraic structure, ensuring that group operations are continuous while maintaining the group axioms. The choice of topology, its compatibility with group operations, and whether it is a Hausdorff topology are all considered carefully to achieve this balance.

Lemma 5.22:

Let $(G, \times, T_G)$ be a topological group. Then

(i) $xU = \{xu \mid u \in U\}$ is open $\Leftrightarrow$ U is open.

(ii) V is a neighbourhood of $x \Leftrightarrow x^{-1}V$ is a neighbourhood of e.

Here and elsewhere we will use e to denote the unit of a multiplicative group and 0 to denote the unit of an additive one.

Theorem 5.23: Topological Group Criterion

Suppose that $(G, \times)$ is a group and (G, T_G) is a topological space. Then $(G, \times, T_G)$ is a topological group $\Leftrightarrow$, writing $\mathcal{N}_a$ for the set of open neighbourhoods of a, we have

(i) Let $a \in G$. Then $N \in \mathcal{N}_e \Leftrightarrow aN \in \mathcal{N}_a$.

(ii) If $N \in \mathcal{N}_e$ then there exists an $M \in \mathcal{N}_e$ with $M^2 \subseteq N$.

(iii) If $N \in \mathcal{N}_e$ then there exists an $M \in \mathcal{N}_e$ with $M \subseteq N^{-1}$.

(iv) If $N \in \mathcal{N}_e$ and $a \in G$ then there exists an $M \in \mathcal{N}_e$ with $M \subseteq aNa^{-1}$.

From time to time it is useful to have neighbourhood bases with further properties:

Lemma 5.24:

If $(G, \times, T_G)$ is a topological group we can find a neighbourhood basis $\mathcal{N}_e$ for e consisting of open sets N with $N^{-1} = N$.

It is easy to define topological subgroups and quotient groups along the lines given in the next lemma:

Lemma 5.25:

If $(G, \times, T_G)$ is a topological group and H is a subgroup of G, if H is equipped with the standard subspace topology then it is a topological group. Moreover, if H is a normal subgroup of G then G/H equipped with the standard quotient space topology (formally, the finest topology on G/H which makes the map $G \to G/H$ given by $x \to xH$ continuous) is a topological group.

However, it is important to realise that, without further conditions quotient topological groups may not behave well.

Lemma 5.26:

Let $(G, \times, T_G)$ be a topological group and H a subgroup of G. Then

(i) The (topological) closure $\overline{H}$ of H is a subgroup.

(ii) If H is normal, so is $\overline{H}$.

(iii) If H contains an open set then $\overline{H}$ is open.
(iv) If H is open then $\overline{H}$ is closed.
(v) If H is closed and of finite index in G then $\overline{H}$ is open.

Therefore, we have the implications to the topological properties of the given topological group, namely, the subspace topology criterion, the normal criterion, the oepnnes and the closedness criterion. We now claim two results without proof.

Lemma 5.27:

If $(G, \times, T_G)$ is a topological group then $I := \overline{\{i_G\}}$ is a closed normal subgroup.

Lemma 5.28:

Let $(G, \times, T_G)$ be a topological group. Then the followings are equivalent:

(i) $\{i_G\}$ is closed.
(ii) G is T_2.
(iii) G is T_0.

For a detailed and propert treatment on topological groups one may consult [76], [77], [78], and [79].

5.3 Categories and Functors

The main ingredients for this subsection and the upcoming one are from [3], [50], [80], [81], and [82].

Category theory has been around for about half a century now, invented in the 1940's by Eilenberg and MacLane. Eilenberg was an algebraic topologist and MacLane was an algebraist. They realized that they were doing the same calculations in different areas of mathematics, which led them to develop category theory. Category theory is really about building bridges between different areas of mathematics.

Recall that we say a function $f : A \to B$ has its domain in A and we call B the codomain of f. We denote by $\mathrm{dom}(f) := A$ while $\mathrm{cod}(f) := B$. Recall the definition we made in the previous chapters.

Definition: Category

A category $\mathscr{C}$ consists of:

(i) A collection $\mathscr{C}_0$ of objects of $\mathscr{C}$.
(ii) $\forall a, b \in \mathscr{C}_0$, a collection of morphisms between them, namely $\mathscr{C}_1$, the collection of all morphisms, sometimes we denote $\mathrm{Hom}_{\mathscr{C}}(a, b)$ the collection of all morphisms in $\mathscr{C}$ connecting a and b.
(iii) An operation $\circ : (f, g) \to f \circ g$ from pairs of morphisms to objects as long as they are composable. We write $A \xrightarrow{f} B$ or $f : A \to B$ for $f \in \mathscr{C}_1$.

Remark:

These data is subject to two axioms:

(i) $\forall a, b \in \mathscr{C}_0$, $\forall f \in \mathscr{C}_1$ such that $f : a \to b$ or $f : b \to a$, one has the identity morphism i such that $i \circ f = f = f \circ i$.
(ii) The composition "$\circ$" is associative, i.e. $\forall f, g, h \in \mathscr{C}_1$ that are composable, one has $(h \circ g) \circ f = h \circ (f \circ g)$. ||

In the above statement, we used the term "composable", this is defined as:

Definition: Composable

Two morphisms $f, g \in \mathscr{C}_1$ are said to be composable if $\text{dom}(f) = \text{cod}(g)$, or the other way around.

Remark:

If $f, g \in \mathscr{C}_1$ are composable, then they must have a composition $f \circ g$.

Moreover, every object a has an identity morphism i_a. ||

Example 5.1: Categories

(i) $\mathscr{C}$:= Set, where the objects are sets and the morphisms are functions.

(ii) $\mathscr{C}$:= Topology, where the objects are topological spaces and the morphisms are continuous maps.

(iii) $\mathscr{C}$:= LCH, where the objects are locally compact Hausdorff spaces and the morphisms are proper maps. ||

Definition: Small Category

Since we do not require $\mathscr{C}_0$ or $\mathscr{C}_1$ to be sets, we call our cateogry $\mathscr{C}$ a small category when they are sets.

Theorem 5.29: The Duality Principal

If φ is a valid statement about categories, so is the statement φ^{-1} obtained by reversing all the morphisms.

Therefore, the category $\mathscr{C}^{\text{op}}$ obtained reversing all the direction of morphisms in $\mathscr{C}$ is also a category. Furthermore, the duality principal tells us that every categorical concepts, theorems, definitions, and proofs have a dual conterpart obtained by reversing all the morphisms.

Definition: Opposite Category

Let $\mathscr{C}$ be a category, then the opposite category $\mathscr{C}^{\text{op}}$ is defined by setting $(\mathscr{C}^{\text{op}})_0 := \mathscr{C}_0$ and $\forall a, b \in \mathscr{C}$, $\text{Hom}_{\mathscr{C}^{\text{op}}}(b, a) := \text{Hom}_{\mathscr{C}}(a, b)$. That is, $\forall f : a \to b$ in $\mathscr{C}$, one has $f^{\text{op}} : b \to a$ in $\mathscr{C}^{\text{op}}$. Given $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathscr{C}$, we have $c \xrightarrow{g^{\text{op}}} b \xrightarrow{f^{\text{op}}} a$ in $\mathscr{C}^{\text{op}}$. We define $(f^{\text{op}} \circ g^{\text{op}}) := (g \circ f)^{\text{op}}$.

Definition: Equivalence Relation on $\mathscr{C}_1$

In general, an equivalence relation $\sim$ on $\mathscr{C}_1$ is called a congruence if:

(i) $f \sim g \Rightarrow \text{dom}(f) = \text{dom}(g)$ and $\text{cod}(f) = \text{cod}(g)$.

(ii) $f \sim g \Rightarrow fh \sim gh$ and $kf \sim kg$ for all h, k such that the composition is valid.

Remark:

There is a category $\mathscr{C}/\sim$ with the same objects as $\mathscr{C}$ but $\sim$-equivalence classes as morphisms. ||

Definition: Isomorphism

The morphism f from a to b ($a \xrightarrow{f} b$) in a category $\mathscr{C}$ is said to be an isomorphism if there is a $g \in \mathscr{C}_1$ such that $g \circ f = i_a$ and $f \circ g = i_b$.

Example 5.2:

If $\mathscr{C}$:= Set, then an isomorphism $f : a \to b$ is an invertible map.

If $\mathscr{C}$:= Topology, then an isomorphism is a homeomorphism. ||

Definition: Terminal

An object t in a category $\mathscr{C}$ is a terminal if for all object a in $\mathscr{C}$, there exists a unique morphism $f : a \to t$.

Example 5.3: Terminal

(i) Set has terminal objects, any one element category $\{ \cdot \}$ is terminal since for any set X, $\exists! f : X \to \{ \cdot \}$ such that $f(x) = \cdot \; \forall x \in X$.

(ii) Group has terminal objects, in the category Group of groups and homomorphisms, one element groups $\{i\}$ is a terminal. Since for all group G $\exists! f : G \to \{i\}$ such that $f(g) = i \; \forall g \in G$.

Note that not all categories have terminal objects. ||

Proposition 5.30: Uniqueness

Two terminal objects $t_1, t_2 \in \mathscr{C}$ are "uniquely" isomorphic.

Proof:

Since t_1 is a terminal then $\exists! g : t_2 \to t_1$. Similarly $\exists! h : t_1 \to t_2$. Applying definition of terminal again yields a unique morphism in $\mathrm{Hom}_{\mathscr{C}}(t_1, t_2)$ but $i_{t_1} \in \mathrm{Hom}_{\mathscr{C}}(t_1, t_1)$ so this morphism has to be i_{t_1}. Since $g \circ h : t_1 \to t_1$ one has $g \circ h = i_{t_1}$. Similarly, $h \circ g : t_2 \to t_2$ so $h \circ g = i_{t_2}$. Therefore f and g are isomorphic.

□

Definition: Initial

An object i in a category $\mathscr{C}$ is said to be an initial if $\forall a \in \mathscr{C}_0 \exists! i \xrightarrow{f} a$.

In later description, if there are no ambiguities, we shall denote $a \in \mathscr{C}$ to state the fact that an object $a \in \mathscr{C}_0$. Similar to the uniqueness of the terminal objects, initial objects are also "uniquely" isomorphic. This is a direct result of **Theorem 5.29**, in fact, we can even state that an object $a \in \mathscr{C}$ is a terminal ⇔ it is an initial in $\mathscr{C}^{\mathrm{op}}$.

Example 5.4: Initial

(i) In the category Set, the empty set is initial since for all set X there exists a unique function $\varphi : \varnothing \to X$, which is the empty function.

(ii) In the category Group, the one element group $\{i\}$ is initial since for all group G there exists a unique homomorphism $\varphi : \{i\} \to G$ such that $\varphi(i) = i_G$, which is the identity element in G.

(iii) In the category $\mathrm{Vect}_{\mathbb{R}}$, the zero vector space $\{0\}$ is initial since for all vector space V there exists a unique linear map $T : \{0\} \to V$ such that $T(0) = \vec{0}$. ||

When we introduce a new space, it is natural to ask if there is a mapping between the elements inside.

Definition: Morphisms between Categories

A functor F from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a pair of functions, namely, $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and $F_1 : \mathscr{C}_1 \to \mathscr{D}_1$ such that the followings hold:

(i) So that $\forall a, b \in \mathscr{C}$, $\forall a \xrightarrow{f} b \in \mathrm{Hom}_{\mathscr{C}}(a, b)$,

$$F_1(f) \in \mathrm{Hom}_{\mathscr{D}}(F_0(a), F_0(b)).$$

(ii) $\forall a \in \mathscr{C}, F_1(i_a) = i_{F_0(a)}$.

(iii) F_1 preserves compositions, i.e. $\forall a \xrightarrow{f} b \xrightarrow{g} c, F_1(g \circ f) = F_1(g) \circ F_1(f)$.

Example 5.5: Forgetful Functors

We have the forgetful/underlying set functor U : Group → Set which forgets the group structure: which forgets the group structure for a group G, $U(G)$ is the set of elements of G. Given a homomorphism between the groups:

$$\varphi : G \to H, \text{ such that } U(\varphi) : U(G) \to U(H).$$

The $U(\varphi)$ is the corresponding function. ||

6.1 Paradoxes and Axioms

In 1873, the German mathematician Georg Cantor discovered that the set of algebraic reals is countable. A few weeks later he was able to demonstrate that the set of all real numbers is uncountable. A new mathematical discipline was born: set theory. In the course of the next two decades, Cantor developed the fundamental concepts of this new discipline; the concepts of equipotent sets, order-isomorphic structures, cardinals and ordinals are all due to him.

Generally speaking, set theory is the study of collections of objects. This view was expressed by Cantor in his famous definition of a set:

"By a 'set' we mean any collection *M* into a whole of definite, distinct objects *m* (which are called the 'elements' of *M*) of our perception or of our thought."

If an element m belongs to a set M, we write $m \in M$. It is also quite common to say in this case that m is a member of M, and to refer to $\in$ as the membership relation. As we shall see, all relevant facts about sets can be expressed in terms of the membership relation.

The career of set theory has been impressive. In the first half of the 20th century, the new fields of set-theoretic topology, theory of real functions, and functional analysis evolved. Each of these disciplines is strongly rooted in set theory, albeit not exclusively. Even more importantly, set theory can be regarded as the foundation of all mathematics. It is possible to interpret the other branches of mathematics as the study of sets. This seems at first glance to be an implausible claim. For most people the real number $\sqrt{2}$ is just a single object, perhaps a point on the real line, but certainly not a collection of other objects.

To solve this ambiguity, we shall present compelling evidence for the possibility of founding all of mathematics on set theory. But why bother? Many mathematicians, especially those of the more applied persuation, never use even such basic set-theoretic tools as arithmetic of infinite ordinals in their research. Would it be more reasonable to study set theory just as a separate discipline, rather than trying to fit all other disciplines into a set-theoretic straightjacket? Or, if the topologiests really cannot live without the straightjacket, shouldn't at least the applied areas be spared?

This suggestion misses the point on two counts. First, it is one thing to claim that set theory could in principle serve as a foundation for all of mathematics, including, say, differentail equations; and it is quite another thing to seriously propose that the Navier-Stokes equations should be expressed in the language of set theory. Second, there is a definite advantage to having a single framework for the separate subdisciplines of mathematics. Different branches of mathematics build on each other. Analytic functions are heavily used in number theory, for example. If each branch of mathematics had its own separate foundations, each use of a theorem from another subfield might raise foundational issues. The existence of a common, albeit at times clumsy, framework makes such mathematical cross-breeding entirely unproblematic. It is exactly the existence of the established common framework that allows most practioners to just do mathematics and to leave all foundational issues to the specialists: set theorists, logicians, and philosophers.

The suggestion "to spare at least the applied areas from the straightjacket of set theory" may sound funny, but it expresses a belief that is held in earnest by many mathematicians and science administrators: that one can draw a clear dividing line between applied and pure mathematics. Even those who do not share this view tend to think that there are clear-cut instances of belonging to the realm of either the pure or the applied. Any yet, the distance between the most applied and the most abstract may be surprisingly short. Consider probability theory. This is as applied a field as any. To a mathematician, it is just the study of probability measures. Theses are functions defined on certain σ-fields of sets, for instance on the Lebesgue measurable subsets of the unit interval. Once this framework for doing probability theory has been established, it is very natural to ask whether there exists a probability function defined on all subsets of the unit interval so that, as in the case of the familiar Lebesgue measure, each individual point has probability zero. This is one of the deepest and most perplexing problems in set theory. Not only is it unsolved; there are indications that it may even be unsolvable in a very strong sense.

Let us consider a hypothetical unsolved problem. Since this will be our substitute for a real problem, let us call it the **Virtual Problem**. Let us assume that the problem is to prove or refute the **Virtual Conjecture**. How would you like this:

Theorem 6.1: Virtual Conjecture

The Virtual Problem is unsolvable.

Well, if you have been working hard on the **Virtual Problem** without many luck, **Theorem 6.1** may be a consolation prize. But could one possibly prove a theorem like **Theorem 6.1**? Yes and no. Intuitively speaking, if all of mathematics can be formulized in set theory, then also all modes of mathematical reasoning can be formalized. Thus, "the collected reasonings of all mathematicians of all times" become a mathematical object, and can be studied like any other mathematical object. It may be possible to prove that neither a proof nor a refutation of the **Virtual Conjecture** is among "the collected reasonings of all mathematicians of all times." Does this constitute a proof of **Theorem 6.1**? Almost. The assumption that all mathematics can be formalized in set theory is an act of belief that does not lend itself to mathematical scrutiny. While we can be reasonably sure that set theory encompasses essentially all correct mathematical arguments that have been used by mathematicians up to this point in history[3], there is always the somewhat remote possibility that eventually somebody will discover an immediately recognizable mathematical truth that transcends set theory. Thus, if it can be established that neither a proof nor a refutation of the **Virtual Conjecture** is among "the collected reasonings of all mathematicians of all times," something like the follwoing theorem will have been proved:

Theorem 6.2:

> The Virtual Conjecture is unsolvable, unless currently used foundations of mathematics are changed.

Over the last three decades, set theorists have proved hundreds of theorems like **Theorem 6.2**. Such theorems are called independence results. The **Virtual Conjecture** does not have to be a strictly set-theoretical statement. It may be a problem in topology, algebra, functional analysis, or measure theory.

On the other hand, the foundations of mathematics have remained remarkably stable. Most mathematicians accept the axiomatic version ZFC[4] of set theory as a reasonably good foundation of mathematics and see little reason to exchange it for something else[5]. Thus, evidence is accumulating that many problems in set theory and related fields may be unsolvable in an absolute sense. However, if they are, we can never be entirely sure of this.

Set theory not only serves mathematics by providing a foundation and allowing one to delineate the limits of the knowable. It also is good mathematics. Set-theoretic theorems and techniques can be used in many other branches of mathematics much in the same way as linear algebra is used in differential equations. This is true not only for the concepts and methods known already to Cantor, but also for more recent results like Zorn's Lemma, the Erdös-Rado Theorem, or the Pressing Down Lemma.

The history of set theory has not always been a smooth ride. In fact, the start was rather bumpy. While some mathematicians embraced set theory eagerly, others were openly hostile. For example, David Hilbert said in 1925 that no one shall be able to drive us from the paradise that Cantor created for us. Henri Poincar*é* said in 1908 that

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[3] Of course, this becomes immediately a false statement if the deliberately vague term "set theory" is replaced by a formal incarnation of it, like ZFC. In this case, the "immediately recognizable mathematical truth that transcends ZFC" could be the assertion that ZFC is consistent. However, if the **Virtual Conjecture** is something like the **Continuum Hypothesis**, then the intuitive picture drawn here will do for a reasonably accurate first approximation of the notion of an independence result.

[4] The letters stand for Zermelo and Fraenkel, who developed the system, and for one of the axioms, called the Axiom of Choice.

[5] Not all mathematicians share this view. Mathematics can be developed in other frameworks. Some of these are brands of set theory similar to the version ZFC discussed in this text; others are entirely different approaches. In this book, we concentrate almost exclusively on a presentation of ZFC. (The only exceptions are occasional discussions of set theory without the Axiom of Choice.) We are far from claiming superiority of ZFC over alternative foundations of mathematics. For whatever reason, it won the competition. It does a desent job; so let us stick to it. It should be pointed out through that, to the best of our knowledge, none of the competitors of ZFC resolves the question of truth or falsity of any statement whose independence of ZFC has been established by the method of forcing.

future generations will look at set theory as a sickness, from which mathematicians will have recovered.

The power of the set concept lies in the possibility of treating collections of infinitely many objects m as a single entity M. Many contemporaries of Cantor felt uneasy about this approach. The question as to whether infinity actually exists, or is just an abstraction, a remote possibility that can be considered and approximated, but never attained, is as old as philosophy itself. For the Greek philosopher Plato, infinity was as real[6] as any finite object. His disciple Aristotle took the opposite stand: Infinity exists only as a potential that is never actually attained. The chasm between the Platonist and the Aristotelian approaches has permeated philosophical thought ever since. In essence, Cantor's treatment of infinity followed Plato, whereas his opponents espoused Aristotelian thinking.

To the authors's philosophy, it is hard for us to define the supremacy and assign this terminology to either one of them. Platonist or Aristotelian, whatever we mathematicians choose to trust, or, more precisely, whatever axiomatic system we choose to use, will finaly in some days in the future, fails to convey the importance as it did before. That is to say, if we have to give a definition of better axiomatic system, we need to accept that this definition will not last forever. To this end, the author shares Platonist's view of infinity, that the forever will arrive, the infinity could be attain in an abstract way, that is, it is an idea dependent to the time. However, Bertrand Russel discovered that Cantor's definition of a set leads to a contradiction.

Let us say that an object x has property $\mathscr{P}$, if x is a set, but x is not an element of itself, which will be denoted by $x \notin x$. Let us collect all objects x with property $\mathscr{P}$ into a set M. Does M have property $\mathscr{P}$? The question is no and this is the well-known **Russell's Paradox**.

Theorem 6.3: Russell's Paradox:

The assumption that the collection of all sets leads to a contradiction.

Pf:

Suppose that the collection $\mathscr{S}$ of all sets is a set, let

$$A := \{x \mid x \in \mathscr{S}, \text{ such that } x \notin x\},$$

for example, $\varnothing \notin A$. Since $\mathscr{S}$ is a set by assumption, then A is also a set, which means that $A \in \mathscr{S}$, therefore $A \in A$. But by our construction we have $A \notin A$, contradiction.

□

Can one resolve **Russell's Paradox**, or do we have to accept it as a refutation of set theory? Of course it can be resolved; otherwise this book would not exist. Let us forget the terminologies in **Theorem 6.3** for a moment and observe the statement above it. As one may notice, we forgot to check whether M is a set. Property $\mathscr{P}$ has two clauses. If M is a set, then M has property $\mathscr{P} \Leftrightarrow M \notin M$; but if M is not a set, then M does not have property $\mathscr{P}$. In this latter case, the paradox would disappear.

But is M a set? Let us consider Cantor's definition. In the spirit of his theory, collections of definite, distinct objects of our thought are sets. M satisfies the criterion of distinctness, since its members are distinguished from its nonmembers by a certain

[6] In a sense, infinity was even more real for Plato than the finite objects of our perception.

property. Thus, Cantor's definition implies that M is a set, and we get **Russell's Paradox**.

We have seen that Russell's Paradox disappears if M is not a set. We have also seen that Cantor's definition implies that M is a set. Is there perhaps something wrong with Cantor's definition?

To see its flaw, let us reexamine the process by which the set M of our example w-as constructed. M was the collection of certain definite objects of our thought, distinguished by a property $\mathscr{P}$. But how "definite" are these objects? If M might or might not be one of those, isn't M a bit indefinite? So, maybe, we should disqualify M as a possible element of M on the grounds of its indefiniteness?

Nevertheless, admitting the modification leads us to a similar paradox; this time with the added clause of some vaguely understood "definiteness" in the defining property of M.

But perhaps Cantor's definition could be salvaged by giving a precise meaning to the word "definite?" Think of the elements of a set as building blocks, and the formation of a set as assembling these building blocks into a whole. It is reasonable to require that at the moment a given set M is being formed, all its building blocks must have already attained their final shape; in this sense they should be "definite." Let us call this stance the **architect's view of set theory**. It stipulates that although it is possible to contemplate all sets at once, each set has to be formed at some moment in an abstract "time," and at that moment, all its building blocks must already have been available in their final shape[7]. Also, once a set is formed, one should be able to use it as a building block of other sets.

This view solves **Russell's Paradox** in an unexpected way: M is not a set, because it could never have been established! At no moment in set-theoretic time do all the building blocks for the construction of M exist.

How can **architect's view of set theory** be expressed with sufficient mathematical precision? The approach concentrates not on what sets are, but on how sets are being formed. At the beginning of set-theoretic time, the only set that can be formed is the empty set, since no previous building blocks exist. Once this set is formed, it can be used as a building block for further sets. The modern alternative to Cantor's definition is to describe precisely by which operations new sets can be built from existing ones, and then to apply these operations successively to the empty sets.

Can we get all sets in this way? Perhaps not, but we can construct a universe of sets rich enough to encompass all known mathematics. This will do for starters.

The **architect's view of set theory** can be formalized by axioms, similar to the way in which our space intuitions were formalized by Euclid more than two thousand years ago. The axiom system ZFC that will be studied in this book was proposed by E. Zermelo and A. Fraenkel early in 20th century. Once an axiom system has been formulated, one can ask whether a given mathematical statement or its negation follows from the axioms. The answer may be a "yes", a "no", or an independence result.

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[7] Note that this view is a synthesis of Platonists and Aristotelian elements.

One often talks about "naive" versus "axiomatic" set theory. This may suggest a much deeper partition than there actually is.

The relation between Cantor's view of sets and the axiomatic treatment of the theory is similar to that between our space intuitions and Euclidean geometry. The Euclidian axioms allow us to derive mathematical truths about points, lines, and planes deductively. This is important, because the deductive method accounts for the high confidence mathematicians have in the truth of their theorems. However, our "naive" space intuitions are a valuable guide in guessing these theorems and outlines of their proofs. Also, a similarities between geometric objects and features of the real world are grasped by our intuition, not by deductive reasoning. Without naivity, there would be no applications of mathematics.

When we say that somebody practices naive set theory, all we mean is that his/her arguments are based on Cantor's definition of a set. The naive approach is quite often the most enlightening one. If a mathematician's reasonings are also informed by a careful analysis of how sets are being built, we say that he/she practices axiomatic set theory. Frequently, this will just mean adopting the architect's point of view without being concerned about details of the axiomatization[8].

In this book, both modes of set-theoretical thought will be practices. We start out with a naive treatment of some of the basics: relations, functions, equipotency, order types and induction. As we go along, questions will arise that call for a more careful scrutiny. Of course, the choice of topics reflects our own biases and our desire to keep the number of pages finite.

To close this section, we introduce some ideas from *Tractatus Logico*, which explains the pictures perfectly well:

> "The world is all that is the case. The world is the totality of facts, not of thins. The world divides into facts." Page 57.
>
> "The world is determined by the facts, and by their being all the facts. For the totality of facts determines what is the case, and also whatever is not the case. The facts in logical space are the world." Page 58.

Perhaps, we need to accept that the world is dynamic, the absolute stable and absolute movement of an object will fail to be valid. An existence is valid only for a particular period. For example, our ancients used to believe that the earth is plain and the sky is a big circle surronding the earth, before the development of sciences, this seemingly to be an unchangable fact for all, but when the time of its modification arrives, it arrives its infinity. Admit this concept or not, we end the section with the following saying again in *Tractatus Logico*:

> "What we cannot speak about we must pass over in silence." Page 56.

The quote of Hilbert is taken from [83]. The quote of Poincaré is taken from [84]. Ca-ntor's famous definition of a set is the first sentence of the article [85]. Detailed accounts of the history of set theory in general, and of Cantor's work in particular can be found in the following book: [85], [86], and [87].

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[8] The use of the phrase "axiomatic set theory" in such instances may not be entirely appropriate. But it is commonly used, and there is no need to further complicate the picture by naming additional modes of practicing set theory.

English translations of many of the most influential papers on the foundations of mathematics written between 1879 and 1931 are reprinted in the book [88].

The question to what extent Cantor's personal view of sets included elements of what we call "the architect's view" is a fascinating topic for philosophers and historians of science. It you are interested in this issue, we recommend the book [89].

Bertrand Russell found his Paradox in June 1901. He described it in a letter to Frege written on June 16, 1902, and apparently also in an earlier letter to Peano. The paradox was first published in his book [90].

The original version of the paradox was different from ours. We gave here a formulation of **Russell's Paradox** that very naturally leads to its resolution. If you want to imagine the impression **Russell's Paradox** must have made on his contemporaries, please keep in mind that they were lacking the benefit of hindsight which informed our choice or wording.

Most of the materials could be found in [91], [92], and [93], we follow the routine mostly by the advanced one (in saying advanced one, we only mean the one focused to graduate students) [93]. Some materials in [94] will be introduced to the author's belief, ignoring such assertions will not affect the understanding.

6.2 Axiomatic Set Theory

Axiomatic set theory is a branch of mathematical logic in which one deals with fragments of the informal theory of sets by methods of mathematical logic. Usually, to this end, these fragments of set theory are formulated as a formal axiomatic theory. In a more narrow sense, the term "axiomatic set theory" may denote some axiomatic theory aiming at the construction of some fragment of informal ("naive") set theory.

Set theory, which was formulated around 1900, had to deal with several paradoxes from its very beginning. The discovery of the fundamental paradoxes of G. Cantor and B. Russell gave rise to a widespread discussion and brought about a fundamental revision of the foundations of mathematical logic. The axiomatic direction of set theory may be regarded as an instrument for a more thorough study of the resulting situation.

The construction of a formal axiomatic theory of sets begins with an accurate description of the language in which the propositions are formulated. The next step is to express the principles of "naive" set theory in this language, in the form of axioms and axiom schemes. A brief description of the most widespread systems of axiomatic set theory is given below. In this context, an important part is played by the language which contains the following primitive symbols:

- (i) the variables $x, y, z, u, v, x_1, \cdots$ which play the part of common names for the sets in the language;
- (ii) the predicate symbols $\in$ (sign of incidence) and $=$ (sign of equality);
- (iii) the description operator ι, which means "an object such that $\cdots$";
- (iv) the logical connectives and quantifiers: $\Leftrightarrow$ (equivalent), $\Rightarrow$ (implies), $\vee$ (or), $\wedge$ (and), $\neg$ (not), $\forall$ (for all), $\exists$ (there exists);
- (v) the parantheses (and).

The expressions of a language are grouped into terms and formulas. The terms are the names of the sets, while the formulas express propositions. Terms and formulas are generated in accordance with the following rules:

(1) If τ and σ are variables or terms, then $(\tau \in \sigma)$ and $(\tau = \sigma)$ are formulas.

(2) If A and B are formulas and x is a variable, then $(A \Leftrightarrow B)$, $(A \Rightarrow B)$, $(A \vee B)$, $(A \wedge B)$, $\neg A$, $\forall x A$, and $\exists x A$ are formulas and $\iota x A$ is a term; the variable x is a term.

For instance, the formula $\forall x(x \in y \Rightarrow x \in z)$ is tantamount to the statement "y is a subset of z", and can be written as $y \subseteq z$; the term $\iota w \forall y(y \in w \Leftrightarrow y \subseteq z)$ is the name of all subsets z and, expressed in conventional mathematical symbols, this is Pz. Let the symbol $:=$ mean "the left-hand side(LHS) is a notation for the right-hand side (RHS)". Below a number of additional notations for formulas and terms will be presented:

Notations:

(a) The empty set: $\varnothing := \iota x \forall y \neg y \in x$.

(b) The set of all x such that $A(x)$: $\{x | A(x)\} := \iota z \forall x(x \in z \Leftrightarrow A(x))$,
where z does not enter freely in $A(x)$ (i.e., is not a parameter of the formula $A(x)$).

(c) The unordered pair x and y: $\{x, y\} := \{z | z = x \vee z = y\}$.

(d) The single-element set consisting of x: $\{x\} := \{x, x\}$.

(e) The ordered pair x and y: $\langle x, y\rangle := \big\{\{x\}, \{x, y\}\big\}$,
where $\langle \cdot , \cdot \rangle$ denotes the ordered pair, instead of being an inner product signal.

(f) The union of x and y: $x \cup y := \{z | z \in x \vee z \in y\}$.

(g) The intersection of x and y: $x \cap y := \{z | z \in x \wedge z \in y\}$.

(h) The union of all elements of x: $\cup x := \{z | \exists v(z \in v \vee v \in x)\}$.

(i) The Cartesian product of x and y:

$$x \times y := \{z | \exists uv(z = \langle u, v\rangle \wedge u \in x \wedge v \in y\}.$$

These notations are already familiar to us as basic operations between well-defined set theory. Now we introduce the terminologies as for the functions:

(j) w is a function:

$\mathrm{Fnc}(w) := \exists v(w \subseteq v \times v) \wedge \forall u v_1 v_2(\langle u, v_1\rangle \in w \wedge \langle u, v_2\rangle \in w \Rightarrow v_1 = v_2)$

(k) The values of the functionw on the element x:

$w'x := \iota y\langle x, y\rangle \in w$.

It therefore equips us with the ability to represent the standard infinite set z, which is stated as the following:

(l) The standard infinite set z:

$\mathrm{Inf}(z) := \varnothing \in z \wedge \forall u(u \in z \Rightarrow u \cup \{u\} \in z)$. ||

The axiomatic theory A that follows is the most complete representation of the principles of "naive" set theory. The axioms of A are:

A1: Axiom of extensionality: $\forall x(x \in y \Leftrightarrow x \in z) \Rightarrow y = z$,
that is, if the set x and y contain the same elements, then they are equal.

A2: Axiom scheme of comprehension: $\exists y \forall x(x \in y \Rightarrow A)$.
where A is an arbitrary formula not containing y as a parameter. That is, there exists a set y containing only elements x for which A.

This system is self-contradictory. If, in **A2**, the formula $\neg x \in x$ is taken as A, the formula $\forall x(x \in y \Leftrightarrow \neg x \in x)$ readily yields $y \in y \Leftrightarrow \neg y \in y$, which is a contradiction called Russell's paradox as we introduced in **Theorem 0.3**. In order to have a well-defined axiomatic system to avoid such paradox, we intend to introduce the axiomatic systems of set theory may be subdivided into the following four groups:

(a) The construction of axiomatic systems in the first group is intended to restrict the comprehension axioms so as to obtain the most natural means of formalization of conventional mathematical proofs and, at the same time, to aviod the familiar paradoxes. The first axiomatic system of this type was the system Z, due to E. Zermelo (1908). However, this system does not allow a natural formalization of certain branches of mathematics, and the supplementation of Z be a new principle — the axiom of replacement — was proposed by A. Fraenkel in 1922. The resulting system is known as the Zermelo-Fraenkel system and is denoted by ZF.

(b) The second group is constituted by systems of the axioms of which are selected in the context of giving some explanations for paradoxes, for example, as a consequence of non-predicative definitions. The group includes Russell's remified theory of types, the simple theory of T-types, and the theory of types with transfinite indices (see [114]).

(c) The third group is characterized by the use of non-standard means of logical deduction, multi-valued logic, complementary conditions of proofs and infinite derivation laws. Systems in this group have been developed to the least extent.

(d) The fourth group includes modifications of systems belonging to the first three groups and is aimed at attaining certain logical and mathematical objectives. Only the system NBG of Neumann-Gödel-Bernays (1925) and the system NF of W. Quine (1937) will be mentioned here. The construction of the system NBG was motivated by the desire to have a finite number of axioms of set theory, based on the system ZF. The system NF represents an attempt to overcome the stratification of the concepts in the theory of types.

The systems Z, ZF, and NF can be formulated in the language described above. The derivation rules, and also the so-called logical axioms, of these systems are identical, and form an applied predicate calculus of the first order with equality and with a description operator. Here are the axioms of equality and of the description operator:

$$x = x, x = y \Rightarrow (A(x) \Rightarrow A(y)), \tag{2.3}$$

where $A(x)$ is a formula not containing the bound variable y (i.e., it has no constituents of the type $\forall y, \exists y, \iota y$), while $A(y)$ is obtained from the formula $A(x)$ by replacing certain free entries of the variable x with y:

$$\exists! x A(x) \Rightarrow A(\iota x A(x)),$$

where the quantifier $\exists! x$ means that "there exists one and only one x", while the formula $A(\iota x A(x))$ is obtained from the formula $A(x)$ by replacing all free entries of the variable x with the term $\iota x A(x)$. The quantifier $\exists !x$ can be expressed in terms of the quantifiers $\forall$ and $\exists$ and equality.

Now we shall introduce the non-logical axioms of the system Z:

Z1: The axiom of extensionality **A1**.

Z2: The pair axiom: $\exists u \forall z (z \in u \Leftrightarrow z = x \vee z = y)$. (the set x, y exists)

Z3: The union axiom: $\exists y \forall x (x \in y \Leftrightarrow \exists t (t \in z \wedge x \in t))$. (the set z exists)

Z4: The power set axiom: $\exists y \forall x (x \in y \Leftrightarrow x \subseteq z)$. (the set Pz exists)

Z5: The separation axiom scheme: $\exists y \forall x (x \in y \Leftrightarrow x \in z \wedge A(x))$. (there exists a subset z consisting of the elements x in z for which $A x$ is true).

The axioms **Z2**-**Z5** are examples of axioms of comprehension;

Z6: The axiom of infinity: $\exists z \mathrm{Inf}(z)$.

Z7: The axiom of choice:
$\forall z \exists w (\mathrm{Fnc}(w) \wedge \forall x (x \in z \wedge \neg x = \emptyset \Rightarrow w' x \in x))$.
(for any set z there exists a function w which selects, out of each non-empty element x of the set z, a unique element $w' x$).

The above axioms are complemented by the regularity axiom:

Z8: The axiom of regularity:
$\forall x (\neg x = \emptyset \Rightarrow \exists y (y \in x \wedge y \cap x = \emptyset))$,
which is intended to postulate that there are no descending chains $x_2 \in x_1, x_3 \in x_2, x_4 \in x_3, \cdots$. Axiom **Z8** simplifies constructions in Z, and its introduction does not result in contradictions.

The system Z is suitable for developing arithmetic, analysis, functional analysis and for studying cardinal numbers smaller than $\aleph_\omega$. However, if the alephs are defined in the usual manner, it is no longer possible to demonstrate the existence in Z of $\aleph_\omega$ and higher cardinal numbers.

The system ZF is obtained from Z by adding Fraenkel's replacement axiom scheme, which may be given in the form of the comprehension axiom scheme:

ZF9: $\exists y \forall x (x \in y \Leftrightarrow \exists v (v \in z \wedge x = \iota t (A(t, v))))$.
(there exists a set y consisting of x, $x = \iota t A(t, v)$, where v runs through all the elements of a set z). In other words, y is obtained from z if each element v of z is replaced with $\iota t A(t, v)$.

The system ZF is a very strong theory. All ordinary mathematical theorems can be formalized in terms of ZF. Before we proceed to the NBG axiomatic system, let us pause to give a description of ZFC with a description by English, for which the author chooses to use the work of [93].

Definition: Axiom of ZFC

(1) **Axiom of Extensionality**.
If X and Y have the same elements, then $X = Y$.

(2) **Axiom of Pairing**.
For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.

(3) **Axiom Schema of Separation**.

If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X \mid P(u, p)\}$ that contains all those $u \in X$ that have property P.

(4) **Axiom of Union**.
For any X there exists a set $Y = \cup X$, the union of all elements of X.

(5) **Axiom of Power Set**.
For any X there exists a set $Y = P(X)$, the set of all subsets of X.

(6) **Axiom of Infinity**.
There exists an infinite set.

(7) **Axiom of Choice**.
Every family of nonempty sets has a choice function.

(8) **Axiom of Regularity**.
Every nonempty set has an $\in$-minimal element.

(9) **Axiom Schema of Replacement**.
If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) \mid x \in X\}$.

The system NBG is obtained from ZF by adding a new type of variables — the class variables $X, Y, Z, \cdots$ — and a finite number of axioms for forming classes, by means of which it is possible to prove formulas of the type

$$\exists Y\, \forall x (x \in Y \Leftrightarrow A(x)),$$

where $A(x)$ is a formula of NBG which does not contain bound class variables or the symbol ι. Since any formula $A(x)$ can be used to form a class, the infinite number of ZF axioms can be replaced by a finite number of axioms containing a class variable. The axiom of choice has the form

$$\exists X(\mathrm{Fnc}(X) \wedge \forall x(\neg x = \varnothing \Rightarrow X'x \in x))$$

and confirms the existence of a selection function, which is unique for all sets and which constitutes a class.

The system NF has a simpler axiomatic form, that is:

(i) The axiom of extensionality

(ii) The xaioms of comprehension in which a formula A can be stratified, i.e., it is possible to assign to all variables of the formula A superscript indices so as to obtain a formula of the theory of T-types, i.e., in the subformulas of type $x \in y$ the index of x is one lower than the index of y.

The system NF has the following characteristics:

(1) The axiom of choice and the generalized continuum hypothesis are disprovable.

(2) The axiom of infinity is demonstrable.

(3) The extensionality axiom plays a very important role. Thus, if the extensionality axiom is replaced by the slightly weaker axiom: $(\exists u(u \in y) \wedge \forall u(u \in y \Leftrightarrow u \in z)) \Rightarrow y = z$, which permits a large number of empty sets, while the comprehension axioms of NF remain unchanged, a fairly weak theory is obtained: The consistency of the resulting system can be proved even in formal arithmetic.

Results concerning the interrelationships between the systems we have just described are given below:

(a) Any formula of ZF is demonstrable in NBG ⇔ it is demonstrable in ZF.

(b) In ZF it is possible to establish the consistency of Z, completed by any finite number of examples of the axiom scheme of replacement **ZF9**. Thus, ZF is much stronger than Z.

(c) The consistency of T is demonstrable in Z, so that Z is stronger than T.

(d) NF is not weaker than T in the sense that it is possible to develop the entire theory of types in NF.

The axiomatic approach to the theory of sets has made it possible to state a proposition on the unsolvability in principal (in an exact sense) of certain mathematical problems and has made it possible to demonstrate it rigorously. The general procedure for the utilization of the axiomatic method is as follows:

Consider a formal axiomatic system S of the theory of sets (as a rule, this is ZF or one of its modifications) that is sufficiently universal to contain all the conventional proofs of classical mathematics, and for all ordinary mathematical facts to be deducible from it. A given problem A may be written down as a formula in the language S. It follows that problem A cannot be solved (in either way) by tools of the theory S, but since this theory S was assumed to contain all ordinary methods of proof, the result means that A cannot be solved by ordinary methods of construction, i.e., A is "transcendental".

Results which state that a proof cannot be performed in the theory S are usually obtained under the assumption that S, or some natural extension of S, is consistent. This is because on the one hand, the problem can be non-deducible in S only if S is consistent, but such consistency cannot be established by the tools offered by S (cf. Gödel incompleteness theorem), i.e., cannot be derived by ordinary tools. On the other hand, the consistency of S is usually a very likely hypothesis; the very theory S is based on its truth.

Furthermore, the axiomatic approach to the theory of sets made it possible to accurately pose and solve problems connected with effectiveness in the theory of sets, which had been intensively studied during the initial development of the theory by R. Baire, E. Borel, H. Lebesgue, S.N. Bernstein, N.N. Luzin, and W. Sierpiński. It is said that an object in the theory of sets which satisfies a property $\mathfrak{A}$ is effectively defined in the axiomatic theory S if it is possible to construct a formula $A(x)$ of S for which it can be demonstrated in S that it is fulfilled for a unique object, and that this object satisfies property $\mathfrak{A}$. Because of this definition it is possible to show in a rigorous manner that for certain properties $\mathfrak{A}$ in S it is impossible to effectively specify an object which satisfies $\mathfrak{A}$, while the existence of these objects in S can be established. But since the chosen theory S is sufficiently universal, the fact that the existence of certain objects in S is ineffecitve is also a proof of the fact that their existence cannot be effectively established by ordinary mathematical methods.

Finally, the methods of the axiomatic theory of sets make it possible to solve a number of difficult problems in classical branches of mathematics as well: in the theory of cardinal and ordinal numbers, in descriptive set theory and in topology.

Some of the results obtained by the axiomatic theory of sets are given below. Most of the theorems concern the axiomatic set theory of Zermelo-Fraenkel (ZF), which is now the most frequently employed. Let ZF^- be the system ZF without the axiom of choice **Z7** (or simply ZF, as a terminology of ZFC), the results can be readily adapted to the system NBG as well.

(i) It was shown in 1939 by K. Gödel that if ZF^- is consistent, it will remain consistent after the axiom of choice and the continuum hypothesis in ZF. In order to prove this result, Gödel constructed a model of the theory ZF consisting of the so-called Gödel constructive sets (cf. Gödel constructive set, see [95]), this model plays an important role in modern axiomatic set theory.

(ii) The problem as to whether or not the axiom of choice or the continuum hypothesis is deducible in ZF remained open until 1963, when it was shown by P.J. Cohen, using his forcing method, that if ZF^- is consistent, it will remain consistent after the addition of any combination of the axiom of choice, the continuum hypothesis or their negations. Thus, these two problems are independent in ZF.

The principal method used for establishing that a formula A is not deducible in ZF is to construct a model of ZF containing the negation of A. Cohen's forcing method, which was subsequently improved by other workers, strongly extended the possibilities of constructing models of set theory, and now forms the basis of almost all subsequent results concerning non-deducibility. For instance:

(iii) It has been shown that one can add to ZF, without obtainiing (additional) inconsistencies, the hypothesis stating that the cardinality of the set of subsets of a set x may be an almost arbitrary pre-given function of the cardinality of x on regular cardinals (the only substantial restrictions are connected with König's theorem).

(iv) M.Ya. Suslin (1920) formulated the following hypothesis. Any linearly totally ordered set such that any pairwise non-intersecting family of non-empty open intervals in it is at most countable must contain a countable everywhere-dense subset. The non-deducibility of Suslin's hypothesis in ZF was established by Cohen's method.

(v) It was shown that the following postulate: "Any subset of real numbers is Lebesgue measurable" is unsolvable in ZF^- (without the axiom of choice).

(vi) The interrelationships of many important problems of descriptive set theory with ZF was clarified. The first results relating to this problem were demonstrated by P.S. Novikov. The methods of axiomatic set theory made it possible to discover previously unknown connections between the problems of "naive" set theory.

(vii) It was proved that an effectively totally ordered continuum is absent in ZF. Numerous results proved the absence of effectively defined objects in the descriptive theory of sets and in the theory of ordinal numbers.

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