Potential Theory in the Complex Plane

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1. Harmonic Functions
1.1 Harmonic and Holomorphic Functions

Harmonic functions, namely solutions of Laplace's equation, exhibit many properties remoniscent of those of holomorphic functions. In fact, when working in a plane, as well shell, there is a direct connection between the two classes. We shall unshamedly exploit this to accelerate the initial development of harmonic functions, under the assumption that we already know something about holomorphic ones. Later, potential theory will repay its debt to complex analysis in the form of many beautiful applications.

We begin with the formal definition. A function $h \in C^2(U)$ means it has second derivative in U.

Definition: Harmonic Function

Let U be an open subset of $\mathbb{C}$. A function $h : U \to \mathbb{R}$ is said to be harmonic if $h \in C^2(U)$ and $\Delta h = 0$ on U.

Definition: Holomorphic Function

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be holomorphic at the point $z \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, h \in \mathbb{C}$$

exists. It is said to be holomorphic if this holds for every point $z \in \mathbb{C}$.

Remark 1.1: Some Properties of Holomorphic Functions

(i) If f is holomorphic on D, then for some appropriate closed paths γ in D,

$$\int_\gamma f(z)dz = 0. \qquad \text{(Contour Integration)}$$

(ii) If f is holomorphic, then f is infinitely differentiable. (Regularity)

(iii) If f and g are holomorphic functions on D which are equal in an arbitrarily small disc in D then $f = g$ everywhere on D.

(Identity Principle) ◇

The following basic result not only furnishes numerous examples of harmonic functions, but also provide a useful tool in deriving their elementary properties from those of holomorphic functions. We shall use Re f to denote the real-part of f.

Theorem 1.1: Characterization of Harmonicity as Holomorphy

Let D be a domain in $\mathbb{C}$.

(i) If f is holomorphic on D and $h = \text{Re} f$, then h is harmonic on D.

(ii) If h is harmonic on D, and if D is simply connected, then $h = \text{Re} f$ for some f holomorphic on D. Moreover, f is unique up to adding a constant.

Proof:

Step I: Assertion (i)

Let $f := h + ik$, the Cauchy-Riemann equations give that

$$h_x = k_y \text{ and } h_y = -k_x.$$

Therefore, using Cauchy-Riemann equation in the second equality gives

$$\Delta h := h_{xx} + h_{yy} = k_{yx} - k_{xy} = 0.$$

Thus (ii) follows from the definition of harmonicity.

Step II: Uniqueness in assertion (ii)

If $h = \operatorname{Re} f$ for some holomorphic function f, say $f := h + ik$, then

$$f' = h_x + ik_x = h_x - ih_y. \tag{1.1}$$

Thus, if f exists, then f' is completely determined by h, and hence f is unique up to adding a constant.

Step III: Existence in assertion (ii)

Equation (1.1) suggests how we might construct such a function f. Define $g : D \to \mathbb{C}$ by $g := h_x - ih_y$. Then $g \in C^1(D)$ and g satisfies the Cauchy-Riemann equations by assumption of harmonicity in h:

$$h_{xx} = -h_{yy} \text{ and } h_{xy} = h_{yx}.$$

Therefore g is holomorphic on D since $g \in C^1(D)$ and g satisfies the Cauchy-Riemann equation. Fix $z_0 \in D$, and define $f : D \to \mathbb{C}$ by

$$f(z) := h(z_0) + \int_{z_0}^{z} g(w)dw,$$

the integral being taken over any path in D from z_0 to z. As D is simply connected, Cauchy's theorem (see **Remark 1.1** (i)) ensures that the integral is independent of the particular path chosen. Then f is holomorphic on D and

$$f' = g = h_x - ih_y.$$

Denote $\widetilde{h} := \operatorname{Re} f$, we have

$$\widetilde{h_x} - i\widetilde{h_y} = f' = h_x - ih_y,$$

so that

$$(\widetilde{h} - h)_x = 0 \text{ and } (\widetilde{h} - h)_y = 0.$$

It follows that $\widetilde{h} - h$ is constant on D, putting $z = z_0$ shows that the constant is zero, thus $h = \operatorname{Re} f$, as desired.

□

As a consequence, we obtain a useful result about holomorphic logarithms. Recall that the holomorphic function has the maximum modulus principle.

Remark 1.2: Maximum Modulus Principle for Holomorphy

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be holomorphic/analytic. If $|f|$ has a local maximum on U then f is constant on U. ◇

Corollary 1.1.1: Logarithms for Holomorphic Functions

Let f be holomorphic and non-zero on a simply connected domain D in $\mathbb{C}$. Then there exists a holomorphic function g on D such that $f = e^g$.

Proof:

Put $h := \log |f|$ on D. Because h is locally the real part of a holomorphic function, namely a branch of $\log f$, it is harmonic by **Theorem 1.1** (i). Now using **Theorem 1.1** (ii), there exists a holomorphic function g on D such that $h = \operatorname{Re} g$ there, or in other words,

$$|fe^{-g}| = |ff^{-1}| = 1 \text{ on } D$$

By **Remark 1.2**, fe^{-g} is a constant C. Adding a suitable constant to g, we can suppose that $C = 1$ and therefore $f = e^g$.

□

Corollary 1.1.1 (and, by implication, **Theorem 1.1**(ii)) may fail if D is not assumed to be simply connected.

Example 1.1: **Corollary 1.1.1** Fails When D Is NOT Simply Connected

The function $f(z) = z$ is holomorphic and non-zero on the domain $D := \mathbb{C}\backslash\{0\}$; but ther eis no holomorphic function g such that $z = e^{g(z)}$ on D, for such a g would satisfy $g'(z) = \frac{1}{z}$, and this would then imply that

$$0 = \int_{|z|=1} g'(z)dz = \int_{|z|=1} \frac{1}{z}dz = 2\pi i,$$

where the first equality holds by **Remark 1.1** (i), the second equality holds since $g'(z) = 1/z$, and the last equality by simply calculation. This is impossible since $0 \neq 2\pi i$. ⋄

However, since discs are simply connected, every harmonic function is at least locally the real part of some holomorphic function. This leads to the following results.

Corollary 1.1.2: Regularity of Harmonic Functions

If h is a harmonic function on an open subset U of $\mathbb{C}$ then $h \in C^\infty(U)$.

Corollary 1.1.3: Composition for Harmonic Functions via Holomorphy

If $f : U_1 \to U_2$ is a holomorphic map between open subsets U_1 and U_2 of $\mathbb{C}$, and if h is harmonic on U_2, then $h \circ f$ is harmonic on U_1.

To make our notes self-contained, we state some properties of holomorphic functions into the following remark without proof.

Remark 1.3: More Properties of Holomorphic Functions

If f and g are holomorphic on $D \subset \mathbb{C}$ then

(i) $f + g$ is holomorphic on D and $(f + g)' = f' + g'$.

(ii) fg is holomorphic on D and $(fg)' = f'g + fg'$.

(iii) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and $(f/g)' = \dfrac{f'g - fg'}{g^2}$.

(iv) If $f : D \to U$ and $g : U \to \mathbb{C}$ are holomorphic then the chain rule holds

$$(f \circ g)'(z) = g'\big(f(z)\big)f'(z) \ \forall z \in D. \quad ⋄$$

The result in **Corollary 1.1.3** allows us to extend the notion of harmonicity to the Riemann sphere.

Example 1.2: Extending Harmonicity to Riemann Sphere

Given a function h defined on an open neighbourhood U of ∞, we say h is harmonic on U if $h \circ \varphi^{-1}$ is harmonic on $\varphi(U)$, where φ is a conformal mapping of U onto an open subset of $\mathbb{C}$. It does not matter which map φ is chosen: if φ_1 and φ_2 are two such choices, then

$$(h \circ \varphi_1^{-1}) = (h \circ \varphi_2^{-1}) \circ f,$$

where $f = \varphi_2 \circ \varphi_1^{-1}$, so by **Corollary 1.1.3**, $h \circ \varphi_1^{-1}$ is harmonic on $\varphi_1(U)$ if and only if $h \circ \varphi_1^{-1}$ is harmonic on $\varphi_2(U)$. ⋄

Another simple consequence of **Theorem 1.1** will be of great importance later.

Theorem 1.2: Mean-Value Property of Harmonic Functions

Let h be a harmonic function on an open neighbourhood of the disc $\overline{\Delta}(w, \rho)$. Then

$$h(w) = \frac{1}{2\pi}\int_0^{2\pi} h(w + \rho e^{i\theta})d\theta.$$

Proof:

Choose $\rho' > \rho$ so that h is a harmonic function on $\Delta(w, \rho')$. By **Theorem 1.1** (ii), there exists f holomorphic on $\Delta(w, \rho')$ such that $h = \mathrm{Re}\, f$ there. Now using Cauchy's integral formula in the first equality and change to radial coordinate in the second, one gets

$$f(w) = \frac{1}{2\pi i}\int_{|\xi - w| = \rho} \frac{f(\xi)}{\xi - w} d\xi = \frac{1}{2\pi}\int_0^{2\pi} f(w + \rho e^{i\theta})d\theta.$$

Result follows upon taking real part of both sides.

□

This section ends with two further ways in which harmonic functions behave like holomorphic ones, an identity principle and a maximum principle. We shall deduce the harmonic versions of both these results from their holomorphic counterparts.

Theorem 1.3: Identity Principle for Harmonic Functions

Let h and k be harmonic functions on a domain D in $\mathbb{C}$. If $h = k$ on a non-empty open subset U of D. Then $h = k$ throughout D.

Proof:

Without loss of generality, we may suppose that $k = 0$. Set $g := h_x - ih_y$. Then as in the proof of **Theorem 1.1**, g is holomorphic on D, and also $g = 0$ on U since $h = 0$ there. By **Remark 1.1** (iii), it follows that $g = 0$ throughout D, and hence that $h_x = 0$ and $h_y = 0$ on D. Therefore h is constant on D, and since $h = 0$ on U, this constant must be zero. It follows that $h = k = 0$ on D.

□

For holomorphic functions, a stronger form of identity principle holds: namely, if two holomorphic functions agree on a set with a limit point in the domain D, then they agree throughout D. However, this is not the case for harmonic functions.

Example 1.3: Stronger Identity Principle Fails for Harmonic Functions

The functions $h(z) := \mathrm{Re}\, z$ and $k(z) = 0$ are harmonic function on $\mathbb{C}$ and agree on the imaginary axis without being equal on the whole $\mathbb{C}$. ⋄

Theorem 1.4: Maximum Principle for Harmonic Functions

Let h be a harmonic function on a domain D in $\mathbb{C}$.

(i) If h attains a local maximum on D then h is constant.

(ii) If h extends continuously to $\overline{D}$ and $h \le 0$ on ∂D then $h \le 0$ on D.

This is perhaps a proper moment for a reminder about our convention that all closures and boundaries are taken with respect to the extended complex plane $\mathbb{C}^\infty$ rather than $\mathbb{C}$. Indeed, **Theorem 1.4** (ii) would otherwise be false.

Example 1.4: Without Our Convention **Theorem 1.4** (ii) Fails

The harmonic function $h(z) := \mathrm{Re}\, z$ on the domain $D := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ extends continuously to $\overline{D}$ and $h \le 0$ on ∂D but $h > 0$ on D. ⋄

Proof of **Theorem 1.4**:

Step I: Assertion (i)

Suppose h attains a local maximum at $w \in D$. Then for some $r > 0$ we have $h \leq h(w)$ on $\Delta(w, r)$. By **Theorem 1.1** (ii) there exists a holomorphic function f on $\Delta(w, r)$ such that $h = \text{Re} f$ there. Then $|e^f|$ attains a local maximum at w, so e^f must be a constant. Therefore h is constant on $\Delta(w, r)$ and hence on the whole of D by **Theorem 1.3**.

Step II: Assertion (ii)

As $\overline{D}$ is compact, h must attain a maximum at some point $w \in \overline{D}$. If $w \in \partial D$ then $h(w) \leq 0$ by assumption, and so $h \leq 0$ on D. If $w \in D$ then by (i) we just proved, h is constant on D, hence on $\overline{D}$, and so once again $h \leq 0$ on D.

□

1.2 The Dirichlet Problem on the Disc

The Dirichlet problem is to find a harmonic function on a domain with the prescribed boundary values. It is one of the greatest advantages of harmonic functions over holomorphic ones that for "nice" domains, a solution always exists. This is a powerful tool for many applications. We first formulate the problem.

Definition: Dirichlet Problem

Let D be a subdomain of $\mathbb{C}$ and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. The Dirichlet problem is to find a harmonic function h on D such that

$$\lim_{z \to \xi} h(z) = \varphi(\xi) \ \forall \xi \in \partial D.$$

Theorem 1.5: Uniqueness of Solution to Dirichlet Problem

There exists at most one solution to Dirichlet problem.

Proof:

Suppose h_1 and h_2 are two solutions to the Dirichlet problem. Then $h_1 - h_2$ is harmonic on D (use **Remark 1.3** (i) to see this). Moreover $h_1 - h_2 = 0$ on ∂D by the definition of Dirichlet problem and $h_1 - h_2$ extends continuously to $\overline{D}$ by **Theorem 1.4** (ii). Another application of **Theorem 1.4** (ii) to $\pm(h_1 - h_2)$ respectively yields that $h_1 - h_2 = 0$ on D, as desired.

□

The question for the existence of solution to Dirichlet problem will be discussed in the fourth chapter as we need more tools. There is a special case we can solve based on the current knowledge.

Definition: Poisson Kernel

The Poisson kernel $P : \Delta(0,1) \times \partial\Delta(0,1) \to \mathbb{R}$ is defined by

$$P(z, \zeta) := \text{Re}\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{1 - |z|^2}{|\zeta - z|^2},$$

where $|z| < 1$ and $|\zeta| = 1$.

Definition: Poisson Integral

If $\Delta := \Delta(w, \rho)$ and $\varphi : \partial\Delta \to \mathbb{R}$ is a Lebesgue integrable function. Then the Poisson integral $P_\Delta \varphi : \Delta \to \mathbb{R}$ is defined by

$$P_\Delta \varphi(z) := \frac{1}{2\pi} \int_0^{2\pi} P\Big(\frac{z - w}{\rho}, e^{i\theta}\Big) \varphi(w + \rho e^{i\theta}) d\theta$$

for $z \in \Delta$. More precisely, if $r < \rho$ and $0 \leq t < 2\pi$ then

$$P_\Delta \varphi(w + re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2}\varphi(w + \rho e^{i\theta})d\theta.$$

The following result is fundamental.

Theorem 1.6: Properties of Poisson Integral

(i) $P_\Delta \varphi$ is harmonic on Δ.

(ii) If φ is continuous at $\zeta_0 \in \partial\Delta$ then $\lim_{z\to\zeta_0} P_\Delta\varphi(z) = \varphi(\zeta_0)$.

(iii) In particular, if φ is continuous on the whole of $\partial\Delta$ then $h := P_\Delta\varphi$ solves the Dirichlet problem on Δ.

Proof:

Using an affine transformation if necessary, without loss of generality, we may assume that $w = 0$ and $\rho = 1$, hence $\Delta = \Delta(0,1)$.

Step I: Assertion (i)

By the definition of Poisson kernel, one has, for $z \in \Delta$,

$$P_\Delta\varphi(z) = \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\varphi(e^{i\theta})d\theta\right),$$

so that $P_\Delta\varphi$ is the real part of a holomorphic function and thus by **Theorem 1.1** (i), $P_\Delta\varphi$ is harmonic on Δ.

Step II: Assertion (ii)

To prove the second assertion, we need a lemma.

Lemma 1.7: Properties of Poisson Kernel

The Poisson kernel P satisfies

(i) $P(z,\zeta) > 0$ for $|z| < 1$ and $|\zeta| = 1$. (Non-Negative)

(ii) $\frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})d\theta = 1$ for $|z| < 1$. (Normalization)

(iii) $\sup_{|\zeta-\zeta_0|\geq\delta} P(z,\zeta) \to 0$ as $z \to \zeta_0$, where $|\zeta_0| = 1$ and $\delta > 0$.

Proof:

The first assertion follows immediately from the definition of Poisson kernel.

Step I: Assertion (ii)

Expressing the given integral as a contour integral and using Cauchy's formula in the first and the second equality respctively,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})d\theta &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}\right) \\ &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=1}\left(\frac{2}{\zeta - z} - \frac{1}{\zeta}\right)d\zeta\right). \\ &= \mathrm{Re}\,(2 - 1) = 1\end{aligned}$$

Step II: Assertion (iii)

If $|z - \zeta_0| < \delta$ then

$$\sup_{\zeta-\zeta_0|\geq\delta} P(z,\delta) \leq \frac{1-|z|^2}{(\delta-|\zeta_0-z|)^2},$$
sending $z \to \zeta_0$ yields the last assertion.

□

Proof of **Theorem 1.6**: Continued

Once again, we may assume that $\Delta = \Delta(0,1)$. Then using **Lemma 1.7** (ii) and (i) in the first and the second equality respectively gives

$$\left|P_\Delta\varphi(z)-\varphi(z_0)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\big(\varphi(e^{i\theta})-\varphi(\zeta_0)\big)d\theta\right|$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\left|\varphi(e^{i\theta}-\varphi(\zeta_0)\right|d\theta.$$

Let $\varepsilon > 0$. If φ is continuous at $\zeta_0 \in \partial\Delta$, then there exists a $\delta > 0$ such that
$$|\zeta-\zeta_0| < \delta \Rightarrow \left|\varphi(\delta)-\varphi(\zeta_0)\right| < \varepsilon$$
by the continuity assumption. Hence, using **Lemma 1.7** (i) and (ii) again one obtains
$$\frac{1}{2\pi}\int_{|e^{i\theta}-\zeta_0|<\delta} P(z,e^{i\theta})\left|\varphi(e^{i\theta})-\varphi(\zeta_0)\right|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\varepsilon d\theta = \varepsilon.$$
Moreover, according to **Lemma 1.7** (iii), there exists $\delta' > 0$ such that
$$|z-\zeta_0| < \delta' \Rightarrow \sup_{|\zeta-\zeta_0|\geq\delta} P(z,\zeta) < \varepsilon.$$
Hence if $|z-\zeta_0| < \delta'$ then
$$\frac{1}{2\pi}\int_{|e^{i\theta}-\zeta_0|\geq\delta} P(z,e^{i\theta})\left|\varphi(e^{i\theta})-\varphi(\zeta_0)\right|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \varepsilon\left|\varphi(e^{i\theta})-\varphi(\zeta_0)\right|d\theta$$
$$\leq \varepsilon\left(\frac{1}{2\pi}\int_0^{2\pi}|\varphi(e^{i\theta})|d\theta + |\varphi(\zeta_0)|\right)$$
where we used Minkowski's inequality in the second inequality. Combining these facts, we deduce that if $|z-\zeta_0| < \delta'$ then
$$\left|P_\Delta\varphi(z)-\varphi(\zeta_0)\right| \leq \varepsilon\left(1+\frac{1}{2\pi}\int_0^{2\pi}|\varphi(e^{i\theta})|d\theta + |\varphi(\zeta_0)|\right).$$
Finally, since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ yields (ii) and (iii).

□

As an immediate consequence of this result, we obtain an analogue of the Cauchy integral formula for harmonic functions.

Corollary 1.6.1: Poisson Integral Formula for Harmonic Functions

If h is harmonic on an open neighbourhood of the disc $\overline{\Delta}(w,\rho)$ then for $r < \rho$ and $0 \leq t < \pi$,
$$h(w+re^{it}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-r^2}{\rho^2-2\rho r\cos(\theta-t)+r^2}h(w+\rho e^{i\theta})d\theta.$$

Proof:

Consider the Dirichlet problem on $\Delta := \Delta(w, \rho)$ with $\varphi := h|_{\partial\Delta}$. According to **Theorem 1.6** (iii), h and $P_\Delta h$ are both solutions and by **Theorem 1.5** the solution is unique. Thus $h = P_\Delta h$ on Δ.

□

Note that this result is a generalization of the mean-value property **Theorem 1.2**, for which is the case when $r = 0$. It allows us to recapture the values of h everywhere on Δ from the knowledge of h on ∂D. Exercise 4 gives an analogous formula for f on Δ, where f is the essentially unique holomorphic function such that $h = \operatorname{Re} f$.

The mean value property **Theorem 1.2** actually characterizes harmonic functions. This is proved in the next theorem, which also illustrates well the value of being able to solve the Dirichlet problem.

Theorem 1.8: Mean-Value Property Characterizes Harmonic Functions

Let $h : U \to \mathbb{R}$ be a continuous function on an open subset $U \subset \mathbb{C}$, and suppose that it possesses the local mean-value property, that is, given $w \in U$, there exists $\rho > 0$ such that for $0 \leq r < \rho$,

$$h(w) = \frac{1}{2\pi}\int_0^{2\pi} h(w + re^{it})dt.$$

Then h is harmonic on D.

Proof:

It suffices to show that h is harmonic on each open disc Δ with $\overline{\Delta} \subset U$. Fix such a Δ, and define $k : \overline{\Delta} \to \mathbb{R}$ by

$$k := \begin{cases} h - P_\Delta h & \text{on } \Delta \\ 0 & \text{on } \partial\Delta \end{cases}$$

Then k is continuous on $\overline{\Delta}$ and has local mean-value property on Δ. Since $\overline{\Delta}$ is compact, k attains a maximum value M at some point of $\overline{\Delta}$. Define

$$A := \left\{z \in \Delta : k(z) < M\right\} \text{ and } B := \left\{z \in \Delta : k(z) = M\right\}.$$

Then A is open since k is continuous. B is also open, for if $k(w) = M$ then the local mean-value property forces k to be equal to M on all sufficiently small circles around w.

Now A and B partition the connected set Δ, either $A = \Delta$, in which case k attains its maximum on ∂D and so $M = 0$; or $B = \Delta$, in which case $k \equiv M$ and $M = 0$. Thus $k \leq 0$ and a similar argument tells us that $k \geq 0$. Hence $h = P_\Delta h$ on Δ and since $P_\Delta h$ is harmonic by **Theorem 1.6** (i), so is h.

□

The technique we used in proving **Theorem 1.8** by defining A and B will be used in proving the maximum principle for subharmonic functions. Ineed, the reason that this technique works is by our assumption that D is simply connected.

Combining **Theorem 1.2** and **Theorem 1.8** we obtain the following result.

Corollary 1.8.1: Harmonicity As Local Uniform Limit of Harmonic Functions

If $\{h_n\}_{n\geq 1}$ is a sequence of harmonic functions on D converging locally uniformly to a function h, then h is also harmonic on D.

A useful feature of **Theorem 1.8** is that one only needs to check that the mean-value property holds locally (that is, the value of ρ can depend on w). As an application of this, we derive a form of the reflection principle for holomorphic functions.

Theorem 1.9: Reflection Principle for Holomorphic Functions

Let $\Delta := \Delta(0,R)$ and write

$$\Delta^+ := \{z \in \Delta : \operatorname{Im} z > 0\} \text{ and } I := \{z \in \Delta : \operatorname{Im} z = 0\}.$$

Suppose that f is a holomorphic function on Δ^+ such that $\operatorname{Re} f$ extends continuously to $\Delta^+ \cup I$ with $\operatorname{Re} f = 0$ on I. Then f extends holomorphically to the whole of Δ.

Note that no assumption is made about the continuity of $\operatorname{Im} f$ on I, this comes free.

Proof of **Theorem 1.9**:

Define $h : \Delta \to \mathbb{R}$ by

$$h(z) := \begin{cases} \operatorname{Re} f(z), & z \in \Delta^+ \\ 0, & z \in I \\ -\operatorname{Re} f(\bar{z}), & \bar{z} \in \Delta^+ \end{cases}$$

Then h is continuous on Δ and has local mean-value property on Δ. Thus by **Theorem 1.8**, h is harmonic on Δ. Using **Theorem 1.1** (ii), there exists a holomorphic function $\tilde{f}$ on Δ such that $h = \operatorname{Re} \tilde{f}$. Now $f - \tilde{f}$ is holomorphic on Δ^+ by **Remark 1.3** (i) and takes only imaginary values, so it is constant there. Adjusting $\tilde{f}$ appropriately, we can make this constant to be zero. Then $\tilde{f}$ is the desired extension of f to the whole of Δ.

□

1.3 Positive Harmonic Functions

In this section we shall exploit the Poisson integral formula **Corollary 1.6.1** to derive some useful inequalities for positive harmonic functions. By "positive" here is meant "non-negative", although in this context there is hardly any difference since, by **Theorem 1.4** (i), any harmonic function which attains a minimum value zero on a domain must be identically zero throughout the domain.

Theorem 1.10: Harnack's Inequality

Let h be a positive harmonic function on the disc $\Delta(w,\rho)$. Then for $r < \rho$ and $0 \le t < 2\pi$,

$$\frac{\rho - r}{\rho + r} h(w) \le h(w + re^{it}) \le \frac{\rho + r}{\rho - r} h(w).$$

Proof:

Choose s with $r < s < \rho$. By **Corollary 1.6.1** applied to h on $\overline{\Delta}(w,s)$ in the first equality one has

$$h(w+e^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s^2-r^2}{s^2-2rs\cos(\theta-t)+r^2} h(w+se^{i\theta})d\theta$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{s+r}{s-r} h(w+se^{i\theta})d\theta$$
$$= \frac{s+r}{s-r} h(w),$$
where the last inequality holds by the mean-value property of h. Now sending $s \to \rho$ gives

$$h(e+re^{it}) \leq \frac{\rho+r}{\rho-r} h(w),$$

thus the desired upper bound is obtained. A similar argument gives the desired lower bound. Thus the proof is complete.

□

Corollary 1.10.1: Liouville Theorem

Every harmonic function on $\mathbb{C}$ which is bounded above or below is constant.

Proof:

It suffices to show that every positive harmonic function h on $\mathbb{C}$ is constant. Given $z \in \mathbb{C}$, put $r := |z|$ and let $\rho > r$. Applying **Theorem 1.10** to h on the disc $\Delta(0,\rho)$ gives

$$h(z) \leq \frac{\rho+r}{\rho-r} h(0).$$

Sending $\rho \to \infty$ yields $h(z) \leq h(0)$. Thus h attains a maximum value at 0 and by **Theorem 1.4** (i) h is constant on $\mathbb{C}$.

□

Harnack's inequality on discs implies an analogous result for general domains.

Corollary 1.10.2: Harnack's Inequality on General Domains

Let D be a domain in $\mathbb{C}_\infty$ and let $z, w \in D$. Then there exists a number τ such that for every positive harmonic function h on D,

$$\tau^{-1}h(w) \leq h(z) \leq \tau h(w), \tag{1.2}$$

Proof:

Given $z, w \in D$, write $z \sim w$ if there exists a number τ such that **(1.2)** holds for every positive harmonic function h on D. Then $\sim$ is an equivalence relation on D, and Harnack's inequality **Theorem 1.10** shows that the equivalent classes are open sets. As D is connected, there can only be one such an equivalent class, and this proves **(1.2)**.

□

Prompted by the last result, we make the following definition.

Definition: Harnack Distance

Let D be a domain in $\mathbb{C}_\infty$. Given $z, w \in D$, the Harnack distance between z and w is the smallest number $\tau_D(z,w)$ such that for every positive harmonic function h on D,

$$\tau_D(z,w)^{-1}h(w) \leq h(z) \leq \tau_D(z,w)h(w). \tag{1.3}$$

There is one case for which τ_D can be computed straightaway.

Theorem 1.11: Harnack Distance Inside Discs

If $\Delta = \Delta(w, \rho)$. Then
$$\tau_D(z, w) = \frac{\rho + |z - w|}{\rho - |z - w|} \text{ for } z \in \Delta.$$

Proof:

From Harnack's inequality **Theorem 1.10**, it follows that
$$\tau_\Delta(z, w) \leq \frac{\rho + |z - w|}{\rho - |z - w|} \text{ for } z \in \Delta.$$
On the other hand, by considering the positive harmonic function h on Δ given by
$$h(z) := P\Big(\frac{z - w}{\rho}, \zeta\Big) := \text{Re}\left(\frac{\rho\zeta + (z - w)}{\rho\zeta - (z - w)}\right)$$
for $|\zeta| = 1$, the equality follows immediately.

□

From this, one can compute or estimate τ_D for other domains D by means of the following subordination principle. Before stating it we first recall some terminology in complex analysis.

Definition: Memomorphic Function

A function on a domain Ω is said to be meromorphic if there exists a sequence of points $p_1, p_2, \cdots$ with no limit points in Ω such that if we denote
$$\Omega^* := \Omega \backslash \{p_1, p_2, \cdots\}$$
such that $f : \Omega^* \to \mathbb{C}$ is holomorphic and f has holes at $p_1, p_2, \cdots$.

Remark 1.4: Properties of Meromorphic Functions

Let f, g, and h be meromorphic functions on the same domain. Then

(i) $f \pm g$ is meromorphic.

(ii) fg is meromorphic.

(iii) $f(g + h) = fg + fh$.

(iv) $f \pm 0 = f$ and $f \cdot 1 = f$.

(v) $1/f$ is meromorphic. ⋄

Definition: Conformal Map

A map $f(z) := w$ is said to be conformal if it preserves angles between oriented curves in magnitude as well as in orientation.

Theorem 1.12: Subordination Principle

Let $f : D_1 \to D_2$ be a meromorphic map between domains D_1 and D_2 in $\mathbb{C}^\infty$. Then for $z, w \in D_1$,
$$\tau_{D_2}\big(f(z), f(w)\big) \leq \tau_{D_1}(z, w),$$
with equality holds if f is a conformal mapping of D_1 and D_2.

Proof:

Let $z, w \in D_1$. Given a positive harmonic function h on D_2, if f is holomorphic then by **Corollary 1.1.3** $h \circ f$ is harmonic on D_1. If f is meromorphic but not holomorphic, then $h \circ f$ agrees with a harmonic function on $D_1 \backslash \{p_1, p_2, \cdots\}$, which is a non-empty open set. Thus by **Theorem 1.3** $h \circ f$ is harmonic on D_1. In particular, $h \circ f$ is a positive harmonic function on D_1. So by **(1.3)** one has

$$\tau_{D_1}(z,w)^{-1} h\big(f(w)\big) \le h\big(f(z)\big) \le \tau_{D_1}(z,w) h\big(f(w)\big).$$

As this holds for arbitrary such a function h, the inequality is verified. Suppose in addition that f is a conformal map of D_1 onto D_2, then we can apply the same argument to f^{-1} and the equality follows.

□

Corollary 1.12.1: Inverse Monotonicity for Harnack Distance under Domain

If $D_1 \subset D_2$ then

$$\tau_{D_2}(z,w) \le \tau_{D_1}(z,w), \text{ where } z, w \in D_1.$$

Proof:

Take $f : D_1 \to D_2$ to be the inclusion map.

□

We can use this to study the continuity properties of τ_D.

Theorem 1.13: Log Harnack Distance Over is a Continuous Semimetric

If D is a subdomain of $\mathbb{C}^\infty$ then $\log \tau_D$ is a continuous semimetric on D.

Proof:

Step I: $\log \tau_D$ is a semimetric on D.

To show that $\log \tau_D$ is a semimetric, we need to verify that for $z, w \in D$,

- $\tau_D(z,w) \ge 1$ and $\tau_D(z,z) = 1$. (Non-“Negative”)
- $\tau_D(z,w) = \tau_D(w,z)$. (Symmetric)
- $\tau_D(z,w) \le \tau_D(z,z')\tau_D(z',w)$ for $z' \in D$. (Triangle Inequality)

All of these follows from the definition of τ_D.

Step II: $\log \tau_D$ is continuous on D.

To show that $\log \tau_D$ is continuous, it suffices to prove that

$$\log \tau_D(z,w) \to 0 \text{ as } z \to w,$$

because the general result then follows by the triangle inequality for $\log \tau_D$. To this end, let $w \in D$, and choose $\rho > 0$ so that $\Delta := \Delta(w,\rho) \subset D$. Then for $z \in \Delta$ we have

$$0 \le \log \tau_D(z,w) \le \log \tau_\Delta(z,w) = \log\left(\frac{\rho + |z-w|}{\rho - |z-w|}\right),$$

where the first inequality holds since $\tau_D(z,w) \ge 1$, the second inequality holds since $\Delta \subset D$ and by **Corollary 1.12.1**, and the last equality holds by the definition of τ_Δ. Since $\rho > 0$ is arbitrary, sending $\rho \downarrow 0$ yields $\log \tau_D(z,w) \to 0$ as $z \to w$, as desired.

□

Remark 1.5: Reason for $\log \tau_D$ Being Semimetric Instead of Metric In General

It may happen that $\log \tau_D(z,w) = 0$ even when $z \neq w$, so that $\log \tau_D$ is not quite a metric. For example, since the only positive harmonic function on $\mathbb{C}$ are constants, it follows that $\log \tau_{\mathbb{C}}(z,w) = 0$ $\forall z, w \in \mathbb{C}$. However, $\log \tau_D$ is a metric for many domains D. ⋄

It is now a short step to the following important theorem.

Theorem 1.14: Harnack's Theorem

Let $\{h_n\}_{n\geq 1}$ be harmonic functions on a domain D in $\mathbb{C}^\infty$ and suppose that
$$h_1 \leq h_2 \leq \cdots \text{ on } D.$$
Then either $h_n \to \infty$ locally uniformly or $h_n \to h$ locally uniformly where h is a harmonic function on D.

Proof:

Fix $w \in D$. Given a compact subset K of D, the quantity
$$C_K := \sup_{z\in K} \tau_D(z, w)$$
is finite since τ_D is continuous (a continuous function has finite supremum over compacts). Hence whenever $n \geq m \geq 1$, we have, for $z \in K$,
$$h_n(w) - h_1(w) \leq C_K\big(h_n(z) - h_1(z)\big)$$
$$h_n(w) - h_m(w) \leq C_K\big(h_n(w) - h_m(w)\big)$$
because $h_n - h_1$ and $h_n - h_m$ are positive harmonic functions on D by assumption. Now if $h_n(w) \to \infty$ as $n \to \infty$ then $h_n \to \infty$ uniformly on K. As K can be any compact subset of D, we conclude that $h_n \to \infty$ locally uniformly on D. On the other hand, if $h_n(w)$ tends to a finite limit, then $\{h_n\}_{n\geq 1}$ is uniformly Cauchy on K. Again, as K is an arbitrary compact susbet of D, it follows that h_n converges locally uniformly on D to a finite function h, by **Corollary 1.8.1** h is necessarily harmonic on D.

□

There is also a very useful variant of Harnack's theorem in which, instead of assuming that the sequence $\{h_n\}_{n\geq 1}$ is increasing, we suppose merely that it is positive. The price we pay is that, in general, only a subsequence will converge.

Theorem 1.15: Harnack's Theorem for Positive Harmonic Functions

Let $\{h_n\}_{n\geq 1}$ be positive harmonic functions on a domain D in $\mathbb{C}^\infty$. Then either $h_n \to \infty$ locally uniformly, or else some subsequence $h_{n_j} \to h$ locally uniformly, where h is a harmonic function on D.

Proof:

We proceed the proof with three steps.

Step I: Reduce assumption to a bounded sequence $\big\{\log h_n(w)\big\}_{n\geq 1}$

Fix $w \in D$. From the inequalities where $z \in D$ and $n \geq 1$,
$$\tau_D(z, w)^{-1} h_n(w) \leq h_n(z) \leq \tau_D(z, w) h_n(w), \tag{1.4}$$
it follows that if $h_n(w) \to \infty$ then also $h_n \to \infty$ locally uniformly on D; and if $h_n(w) \to 0$ then also $h_n \to 0$ locally uniformly on D. Therefore, replacing $\{h_n\}_{n\geq 1}$ by a subsequence if necessary, we can reduce to the case where the sequence $\big\{\log h_n(w)\big\}_{n\geq 1}$ is bounded. The inequality **(1.4)** then implies that $\{\log h_n\}_{n\geq 1}$ is locally uniformly bounded on D, and so it suffices to prove that there is a subsequence $\{h_{n_j}\}_{j\geq 1}$ such that $\{\log h_{n_j}\}_{j\geq 1}$ is locally uniformly convergent on D.

Step II: $\exists \{h_{n_j}\}_{j\geq 1}$ such that $\{\log h_{n_j}\}_{j\geq 1}$ is locally uniformly convergent on D.

Let S be a countable dense subset of D. The sequence $\left\{\log h_n(\zeta)\right\}_{n\geq 1}$ is bounded for each $\zeta \in S$. So by a diagonal argument we may find a subsequence $\{h_{n_j}\}_{j\geq 1}$ such that $\left\{\log h_{n_j}(\zeta)\right\}_{j\geq 1}$ is convergent $\forall \zeta \in S$. We shall show that, for this subsequence, $\{\log h_{n_j}\}_{j\geq 1}$ is locally uniformly convergent on D.

Step III: $\{\log h_{n_j}\}_{j\geq 1}$ converges locally uniformly on D

Let K be a compact subset of D, and let $\varepsilon > 0$. For each $z \in K$, let

$$V_z := \left\{z' \in D : \log \tau_D(z, z') < \varepsilon\right\},$$

and let $V_{z_1}, \cdots, V_{z_m}$ be a finite subcover o K. Since S is dense in D, for each ℓ we can pick a point $\zeta_\ell \in V_{z_\ell} \cap S$. Then there exists $N \geq 1$ such that for $n_j, n_k \geq N$, $\ell = 1, \cdots, m$,

$$\left|\log h_{n_j}(\zeta_\ell) - \log h_{n_k}(\zeta_\ell)\right| \leq \varepsilon.$$

Now by the definition of Harnack distance, for $z \in V_{z_\ell}$,

$$\left|\log h_{n_j}(z) - \log h_{n_j}(\zeta_\ell)\right| \leq \log \tau_D(z, \zeta_\ell) < 2\varepsilon$$

with a similar argument applied to h_{n_k}, one gets

$$\left|\log h_{n_j}(z) - \log h_{n_k}(\zeta_\ell)\right| < 5\varepsilon$$

for $n_j, n_k \geq N$ and $z \in K$. Thus $\{\log h_{n_j}\}_{j\geq 1}$ is uniformly Cauchy on K and thus uniformly convergent on K. Since this holds for any compact subset K, the locally uniformly convergence is verified.

□

We conclude this chapter by applying some of the ideas developed in it to give a beautiful recent proof of Picard's theorem due to John Lewis.

Theorem 1.16: Picard's Theorem

If $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire function. Then $\mathbb{C} \backslash f(\mathbb{C})$ contains at most one point.

The proof requires a lemma on harmonic functions which is of some interest in its own right. We shall use the notation

$$M_h(w, r) := \sup_{\Delta(w,r)} h = \sup_{\partial\Delta(w,r)} h.$$

Lemma 1.17: Sup of Harmonic Function Is Bouned Away From Zero on Discs

Let h be harmonic on a neighbourhood of $\overline{\Delta}(0,2R)$ with $h(0) = 0$. Then there exists a disc $\Delta(w, r) \subset \Delta(0,2R)$ such that $h(w) = 0$ and

(i) $M_h(w, r) \geq 3^{-11} M_h(0,R)$.

(ii) $M_h(w, r/2) \geq 3^{-11} M_h(w, r)$.

Of course the exact value of the constant 3^{-11} is unimportant here. The point is that it is positive!

Proof of Lemma 1.17:

For $z \in \Delta(0,2R)$ write $\delta(z) := \operatorname{dist}\big(z, \partial\Delta(0,2R)\big)$, and define

$$Z := \left\{z \in \Delta(0,2R) : h(z) = 0\right\}$$

$$U := \bigcup_{z\in Z} \Delta\big(z, \delta(z)/4\big),$$
$$\gamma := \sup_U h = \sup_{z\in Z} M_h\big(z, \delta(z)/4\big).$$
Choose $w \in Z$ such that $M_h\big(w, \delta(w)/4\big) \geq \gamma/3$ and set $r := \delta(w)/2$. We shall show that $\Delta(w, r)$ satisfies the conclusion of the lemma.
Clearly $\Delta(w, r) \subset \Delta(0,2R)$ and $h(w) = 0$. Also $M_h(w, r/2) \geq \gamma/3$, so to complete the proof it suffices to show that
$$\text{(a)} \quad M_h(0,R) \leq 3^{10}\gamma$$
$$\text{(b)} \quad M_h(w, r) \leq 3^{10}\gamma.$$
Step I: (a)
Take $z \in \Delta(0,R)$ with $h(z) \geq 0$. If $z \in \overline{U}$ then by continuity $h(z) \leq \gamma$. Now suppose that $z \notin \overline{U}$. Then (using the obvious notation for line segments in $\mathbb{C}$) there exists $z' \in (z,0) \cap \overline{U}$ such that $[z, z') \cap \overline{U} = \varnothing$. It follows that $h > 0$ on $[z, z')$. In fact, for each $\zeta \in [z, z')$ we have $h > 0$ on $\Delta(\zeta, R/5)$. For if not, then there exists $\zeta' \in \Delta(\zeta, R/5)$ with $h(\zeta') = 0$. But then $\zeta' \in Z$ and
$$\delta(\zeta') \geq \delta(\zeta) - |\zeta' - \zeta| \geq R - R/5 = 4R/5 > 4|\zeta' - \zeta|,$$
where the first inequality holds by triangle inequality, the second holds by assumption, and the last holds since $\zeta' \in \Delta(\zeta, R/5)$. This display implies that $\zeta \in U$, which is impossible.
Thus indeed for each $\zeta \in [z, z')$ we have $h > 0$ on $\Delta(\zeta, R/5)$. It follows from Harnack's inequality **Theorem 1.10** that for such ζ,
$$\sup_{\Delta(\zeta,R/10)} h \leq 3^2 \inf_{\Delta(\zeta,R/10)} h.$$
Since $[z, z']$ has length less than R, it can be covered by 5 overlapping discs of radius $R/10$ with centers in $[z, z')$. Therefore,
$$h(z) \leq (3^2)^5 h(z') \leq 3^{10}\gamma,$$
where the last inequality holds since $h \leq \gamma$ on $\overline{U}$.
Step II: (b)
This is virtually identical. Take $z \in \Delta(w, r)$ with $h(z) \geq 0$. If $z \in \overline{U}$ then by continuity $h(z) \leq \gamma$. Now suppose that $z \notin U$. Then there exists $z' \in (z, w) \cap \overline{U}$ such that $[z, z') \cap \overline{U} = \varnothing$. It follows that $h > 0$ on $[z, z')$. In fact, for each $\zeta \in [z, z')$ we have $h > 0$ on $\Delta(\zeta, r/5)$. For if not, then there exists $\zeta' \in \Delta(\zeta, r/5)$ with $h(\zeta') = 0$. But then $\zeta' \in Z$ and
$$\delta(\zeta') \geq \delta(w) - |\zeta' - \zeta| - |\zeta - w| \geq 2r - r/5 - r = 4r/5 > 4|\zeta' - \zeta|,$$
implying that $\zeta \in U$, which is impossible.
Thus indeed for each $\zeta \in [z, z')$ we have $h > 0$ on $\Delta(\zeta, r/5)$. It follows from Harnack's theorem **Theorem 1.10** that for such ζ,
$$\sup_{\Delta(\zeta,r/10)} h \leq 3^2 \inf_{\Delta(\zeta,r/10)} h.$$
Since $[z, z']$ has length less than r, it can be covered by 5 overlapping discs of radius $r/10$ with centers in $[z, z')$. Therefore,
$$h(z) \leq (3^2)^5 h(z') \leq 3^{10}\gamma,$$

where the last inequality holds since $h \leq \gamma$ on $\overline{U}$.

□

Proof of **Theorem 1.16**:

Suppose, for a contradiction, that $\mathbb{C} \backslash f(\mathbb{C})$ contains at least two points α and β. Then $h := \log|f-\alpha|$ and $k := \log|f-\beta|$ are both harmonic functions on $\mathbb{C}$ and they satisfy

$$\text{(i)} \qquad |h^+ - k^+| \leq |\alpha - \beta|$$
$$\text{(ii)} \qquad \max(h,k) \geq \log\Big|\, |\alpha-\beta|/2 \,\Big|$$

everywhere on $\mathbb{C}$. Since f is non-constant, so is h, and so by **Corollary 1.10.1** h is unbounded above and below. In particular, there exists $z_0 \in \mathbb{C}$ with $h(z_0) = 0$, and replacing $f(z)$ by $f(z+z_0)$ we can without loss of generality assume that $z_0 = 0$.

Now applying **Lemma 1.17** to h on each of the discs $\Delta(0,2^{j+1})$ to produce new discs $\Delta(w_j, r_j)$ such that $h(w_j) = 0$ and

$$M_h(w_j, r_j) \geq 3^{-11} M_h(0,2^j)$$
$$M_h(w_j, r_j/2) \geq 3^{-11} M_h(w_h, r_j).$$

For each $j \geq 1$ set $M + j := M_h(w_j, r_j)$. Since h is unbounded,

$$\lim_{j\to\infty} M_h \geq 3^{-11} \lim_{j\to\infty} M_h(0,2^j) = \infty.$$

Define two sequences of harmonic functions $\{h_j\}_{j\geq 1}$ and $\{k_j\}_{j\geq 1}$ on $\Delta(0,1)$ by

$$h_j(z) := \frac{h(w_j + r_j z)}{M_j} \text{ and } k_j(z) := \frac{k(w_j + r_j z)}{M_j}$$

for $|z| < 1$. Then h_j and k_j have the following properties:

(a) $\quad h_j(0) = 0.$

(b) $\quad M_{h_j}(0,1/2) \geq 3^{-11}.$

(c) $\quad |h_j^+ - k_j^+| \leq \dfrac{|\alpha-\beta|}{M_j}.$

(d) $\quad \max(h_j, k_j) \geq \dfrac{\log\big(|\alpha-\beta|/2\big)}{M_j}.$

Evidently $h_j \leq 1 \; \forall j \geq 1$. Using **Theorem 1.15** to $\{1-h_j\}_{j\geq 1}$ to deduce that a subsequence of the $\{h_j\}_{j\geq 1}$ converges locally uniformly to a function $\widetilde{h}$ on $\Delta(0,1)$.

The functions $\{k_j\}_{j\geq 1}$ are uniformly bounded above (for example by $1 + |\alpha-\beta|/M_1$), and so a further subsequence of these converges locally uniformly to a function $\widetilde{k}$ on $\Delta(0,1)$. Both $\widetilde{h}$ and $\widetilde{k}$ are harmonic (or possibly identically $-\infty$) and they have the following properties:

(a') $\quad \widetilde{h}(0) = 0.$

(b') $\quad M_{\widetilde{h}}(0,1/2) \geq 3^{-11}.$

(c') $\quad \widetilde{h^+} = \widetilde{k^+}.$

(d’) $\max(\widetilde{h}, \widetilde{k}) \geq 0$.

Property (b’) implies that $\widetilde{h}(\zeta) > 0$ for some ζ, and (c’) then tells us that $\widetilde{h} = \widetilde{k}$ in a neighbourhood of ζ. By the identity principle **Theorem 1.3** it follows that $\widetilde{h} = \widetilde{k}$ everywhere on $\Delta(0,1)$. From (d’) we then deduce that $\widetilde{h} \geq 0$ on $\Delta(0,1)$, and combining this with (a’) and the maximum principle **Theorem 1.4** (i), we conclude that $\widetilde{h} \equiv 0$ on $\Delta(0,1)$. But this is inconsistent with (b’), contradiction.

□

2. Subharmonic Functions

2.1 Upper Semicontinuous Functions

As part of their definition, subharmonic functions are going to be upper semicontinuous, so before making this definition, we take a brief look at upper semicontinuous functions in abstract.

Definition: Upper Semicontinuous

Let X be a topological space. We say that a function $u : X \to [-\infty, \infty)$ is upper semicontinuous if the set $\{x \in X : u(x) < \alpha\}$ is open in X $\forall \alpha \in \mathbb{R}$.

Definition: Lower Semicontinuous

Let X be a topological space. We say that a function $u : X \to (-\infty, \infty]$ is lower semicontinuous if $-u$ is upper semicontinuous.

A straightforward check shows that u is upper semicontinuous if and only if

$$\limsup_{y \to x} u(y) \leq u(x) \text{ for each } x \in X.$$

In particular, u is continuous if and only if it is lower semicontinuous and upper semicontinuous at the same time.

We shall make frequent use of the following basic compactness theorem. For the s-akeness of simplicity we shall denote upper semicontinuity as u.s.c. and lower semicontinuity as l.s.c. whenever necessary.

Theorem 2.1: USC Is Bounded Above and Attains Upper Bound on Compacts

Let U be an u.s.c. function on a topological space X and let K be a compact subset of X. Then u is bounded above on K and attains its bound.

Proof:

The sets $\left\{\{x \in X : u(x) < n\}\right\}_{n \geq 1}$ form an open cover of K, so have a finite subcover. Hence u is bounded above on K. Let $M := \sup_K u$. Then the open sets $\left\{\left\{x \in X : u(x) < M - \frac{1}{n}\right\}\right\}_{n \geq 1}$ cannot cover K since it has no finite subcover and thus $u(x) = M$ for at least one $x \in K$.

□

The other result we shall need is an approximation theorem.

Theorem 2.2: Continuous Approximation to Bounded Above USC Functions

Let u be an u.s.c. function on a metric space (X, d) and suppose that u is bounded above on X. Then there exist continuous functions

$$\left\{\varphi_n : X \to \mathbb{R}\right\}_{n \geq 1}, \text{ where } \varphi_1 \geq \varphi_2 \geq \cdots \geq u \text{ on } X$$

and $\lim_{n\to\infty} \varphi_n = u$.

Proof:

Without loss of generality, we may assume that $u \neq -\infty$ as otherwise $\varphi_n \equiv -n$. For $n \geq 1$, define $\varphi_n : X \to \mathbb{R}$ by

$$\varphi_n(x) := \sup_{y\in X} \big(u(y) - nd(x,y)\big), x \in X.$$

Then for each n one has

$$\Big|\varphi_n(x) - \varphi_n(x')\Big| \leq nd(x,x'), \text{ where } x, x' \in X,$$

so φ_n is continuous on D.

Moreover, $\varphi_1 \geq \varphi_2 \geq \cdots \geq u$ and so in particular $\lim_{n\to\infty} \varphi_n = u$. On the other hand, writing $\Delta(x,\rho)$ for the ball $\{y \in X : d(x,y) < \rho\}$, we have

$$\varphi_n(x) \leq \max\big(\sup_{\Delta(x,\rho)} u, \sup_X (n - n\rho)\big)$$

for $x \in X$ and $\rho > 0$. Thus

$$\lim_{n\to\infty} \varphi_n(x) \leq \sup_{\Delta(x,\rho)} u, \text{ for } x \in X, \rho > 0.$$

Since u is u.s.c., sending $\rho \uparrow \infty$ yields $\lim_{n\to\infty} \varphi_n \leq u$.

□

2.2 Subharmonic Function

In spirit, at least, a function u is subharmonic if its Laplacian satisfies $\Delta u \geq 0$. However, we shall not define subharmonicity this way. As we shall see later, one of the greatest virtues of subharmonic functions is their flexibility, and this would be lost if we were to assume that they are smooth.

Instead, we proceed by analogy with convex functions on $\mathbb{R}$ (indeed, this is a good analogy to keep in mind throughout this book). If $\psi \in C^2(\mathbb{R})$, then it is convex if and only if $\psi'' \geq 0$, but the convexity is actually defined via a submean property, which also allows non-smooth functions such as $\psi(t) := |t|$ to be convex. Taking this as our model, we shall define subharmonicity using an analogous submean property in the plane.

There is, however, one more technicality. Convex functions on open intervals are automatically continuous, but there is no such a result for subharmonic functions. We could demand continuity as part of our definition, but, for reasons that will become apparent later, it is advantageous merely to ask for u.s.c..

After this preamble, we are at last ready to make the definition.

Definition: Subharmonic Function

Let U be an open subset of $\mathbb{C}$. A function $u : U \to [-\infty, \infty)$ is said to be subharmonic if

(i) u is u.s.c..

(ii) u satisfies the local submean inequality, that is, given $w \in U$ there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{it})dt, 0 \leq r < \rho. \tag{2.1}$$

Definition: Superharmonic Function

A function $u : U \to (-\infty, \infty]$ is superharmonic if $-u$ is subharmonic.

Remark 2.1: Intepretation for Definition of Subharmonic Functions

The definition merits some comment:

(i) The integral in (2.1) is to be interpreted as the difference of the corresponding integrals of u^+ and u^-. By **Theorem 2.1**, u^+ is bounded on $\partial\Delta(w, r)$, so its integral is certainly finite. Thus the difference of the two integrals makes sense, even though the integral of u^- may be infinite. We shall see later that the latter only happens when $u \equiv -\infty$ on the whole component of U containing x. (Note that, according to our definition, $u \equiv -\infty$ is a subharmonic function, though many authors exclude it). (Infinity and Convention)

(ii) Since the subharmonicity is defined via the local submean inequality (that is, ρ may depend on w), it is a local property. This means that if $\{U_\alpha\}_{\alpha \in I}$ is an open cover of U where I is an arbitrary index set, then u is subharmonic on U if and only if it is subharmonic on each U_α. (Subharmonicity Is a Local Property)

(iii) We observe that a function is harmonic if and only if it is at the same time subharmonic and superharmonic. (Characterizes Harmonicity) ⋄

Theorem 2.3: Construct Subharmonic Function via Holomorphic Function

If f is holomorphic on an open set U in $\mathbb{C}$. Then $\log|f|$ is subharmonic on U.

Proof:

Evidently $u := \log|f|$ is u.s.c.. Also it satisfies the local submean property at each $w \in U$ for which $u(w) > -\infty$, because near such a point $\log|f|$ is actually harmonic. On the other hand, if $u(w) = -\infty$ then (2.1) is immediate.

□

Further examples can be generated using the following elementary result, which is an immediate result from the definition for subharmonicity.

Theorem 2.4: Some Elementary Properties for Subharmonic Functions

Let u and v be subharmonic functions on an open set U in $\mathbb{C}$. Then

(i) $\max(u, v)$ is subharmonic on U.

(ii) $\alpha u + \beta v$ is subharmonic on U $\forall \alpha, \beta \geq 0$.

From (i) it follows that a subharmonic function needs not to be smooth. One might reasonably guess that they do have to be continuous, but actually this is not true neither. An example is given below, another will be given in section 5 of this chapter.

Example 2.1: Subharmonic Functions Need Not To Be Continuous

Consider $\zeta \in \mathbb{C}$ and $r > 0$. One has

$$\frac{1}{2\pi}\int_0^{2\pi} \log|re^{it} - \zeta|dt = \begin{cases} \log|\zeta|, & \text{if } r \leq |\zeta| \\ \log r, & \text{if } r > |\zeta| \end{cases}$$

Thus the function

$$u(z) := \sum_{n\geq 1} 2^{-n} \log|z - 2^{-n}|$$

is subharmonic on $\mathbb{C}$ but u is discontinuous at 0. ⋄

2.3 The Maximum Principle

As a result of **Theorem 1.2** and **Theorem 1.8**, the local mean-value property implies the (global) mean-value property. To make much further progress with subharmonic functions, we need a corresponding result for the submean inequality. As with harmonic functions, we shall deduce this result with maximum principle. The importance of the maximum principle lies in the fact that from local assumptions it derives global conclusions. Indeed, many principles in potential theory involve extending a property of the potential of a measure from a set which the measure is concentrated to the whole space.

Such results are usually very powerful, and the maximum principle is no exception. Since it will feature prominently in what follows, we shall digress slightly in order to study it in a little more detail, returning to the submean inequality in the next section.

Theorem 2.5: Maximum Principle for Subharmonic Functions

Let u be a subharmonic function on a domain D in $\mathbb{C}$.

(i) If u attains a global maximum on D then u is constant.

(ii) If $\limsup_{z\to\zeta} u(z) \leq 0 \;\forall \zeta \in \partial D$ then $u \leq 0$ on D.

Remark 2.2: Max Principle for Subharmonic Fails with Global Min or Local Max

Note that in (i), u can attain a local maximum or a global minimum without being constant on D. For example, the non-constant subharmonic function

$$u(z) := \max(\operatorname{Re} z, 0)$$

does both in $\mathbb{C}$. Moreover, just as in **Theorem 1.4**, the validity of (ii) depends on our convention that $\infty \in \partial D$ whenever D is unbounded. ⋄

Proof of **Theorem 2.5**:

Step I: Assertion (i)

Suppose that u attains a maximum value M on D. Define

$$A := \{z \in D : u(z) < M\} \text{ and } B := \{z \in D : u(z) = M\}.$$

Then A is open by the u.s.c. of u. Moreover, B is also open because if $u(w) = M$ then the local submean inequality **(2.1)** forces u to be equal to M on all sufficiently small circles around w. Clearly A and B partitions D and since D is connected either $A = D$ or $B = D$. By our assumption $B \neq \emptyset$ thus $B = D$ and (i) follows.

Step II: Assertion (ii)

Extend u to ∂D by defining

$$u(\zeta) := \limsup_{z\to\zeta} u(z), \; \zeta \in \partial D.$$

Then u is u.s.c. on $\overline{D}$, which is compact, so by **Theorem 2.1** u attains a maximum at some $w \in \overline{D}$. If $w \in \partial D$, then by assumption $u(w) \leq 0$ thus $u \leq 0$ on D. On the other hand, if $w \in D$ then by (i) u is constant on D, hence on $\overline{D}$, thus $u \leq 0$ on D as desired.

□

Remark 2.3: (i) in **Theorem 2.5** Replaced by $\partial D \backslash \{\infty\}$ with Mild Growth at Infinity

In fact, it is possible to replace ∂D by $\partial D \backslash \{\infty\}$ in (ii) if u does not grow too rapid at infinity. Hence it is a rather general result that makes this statement precise. ◇

To this spirit, the following result guarantees the mild growth at infinity and thus the assumption in (i) of **Theorem 2.5** can be replaced by $\partial D \backslash \{\infty\}$.

Theorem 2.6: Phragmén-Lindelöf Principle

Let u be a subharmonic function on an unbounded domain D in $\mathbb{C}$ such that

$$\limsup_{z \to \zeta} u(z) \leq 0, \zeta \in \partial D \backslash \{\infty\}.$$

Suppose also that there exists a finite-valued superharmonic function v on D such that

$$\liminf_{z \to \infty} v(z) > 0 \text{ and } \limsup_{z \to \infty} \frac{u(z)}{v(z)} \leq 0.$$

Then $u \leq 0$ on D.

Proof:

Step I: Case when $v > 0$ on D.

Assume first that $v > 0$ on D. Let $\varepsilon > 0$ and set

$$u_\varepsilon := u - \varepsilon v.$$

Then u_ε is subharmonic on D, and

$$\limsup_{z \to \zeta} u_\varepsilon(z) \leq 0 \ \forall \zeta \in \partial D \text{ (even } \infty),$$

so by **Theorem 2.5** (ii) $u_\varepsilon \leq 0$ on D. Sending $\varepsilon \to 0$ we get $u \leq 0$ on D.

Step II: General case

Now consider a general v. Let $\eta > 0$ and set

$$F_\eta := \{z \in D : u(z) \geq \eta\}.$$

Since v is l.s.c. and $\liminf_{z \to \infty} v(z) > 0$, it follows that v is bounded below on F_η. Adding a constant to v if necessary, we can without loss of generality assume that $v > 0$ on F_η. Set

$$V := \{z \in D : v(z) > 0\}.$$

Then for $\eta \in \partial V \backslash \{\infty\}$ we have

$$\limsup_{z \to \infty} \left(u(z) - \eta\right) \leq \begin{cases} \limsup_{z \to \zeta} u(z), & \text{if } \zeta \in \partial D \backslash \{\infty\} \\ u(\zeta) - \eta, & \text{if } \zeta \in D \cap \partial V \end{cases} \leq 0.$$

Applying result in the first step to $u - \eta$ on each component of V, we get $u - \eta \leq 0$ on V. As $F_\eta \subset V$ it follows that $u \leq \eta$ on F_η, and plainly $u \leq \eta$ on $D \backslash F_\eta$, so in fact $u \leq \eta$ on D. Finally, since $\eta > 0$ is arbitrary, sending $\eta \downarrow 0$ yields the desired result.

□

Corollary 2.6.1: Maximum Principle for Subharmonic on Unbounded Domain

Let u be a subharmonic function on an unbounded proper subdomain D of $\mathbb{C}$ such that

$$\limsup_{z\to\zeta} \leq 0 \text{ for } \zeta \in \partial D\backslash\{\infty\} \text{ and } \limsup_{z\to\infty} \frac{u(z)}{\log|z|} \leq 0.$$

Then $u \leq 0$ on D.

Proof:

Take $w \in \partial D$ and apply **Theorem 2.6** with $v(z) := \log|z - w|$. ☐

Corollary 2.6.2: Liouville Theorem for Subharmonic Functions

Let u be a subharmonic function on $\mathbb{C}$ such that

$$\limsup_{z\to\infty} \frac{u(z)}{\log|z|} \leq 0.$$

Then u is constant on $\mathbb{C}$. In particular, every subharmonic function on $\mathbb{C}$ which is bounded above must be constant.

Proof:

If $u \equiv -\infty$ then this is clear. Suppose $u \not\equiv -\infty$, choose $w \in \mathbb{C}$ with $u(w) > -\infty$ and apply **Corollary 2.6.1** to $u - u(w)$ on $\mathbb{C}\backslash\{w\}$. Thus $u \leq u(w)$ on $\mathbb{C}$ and now by the maximum principle **Theorem 2.5** (i) u is constant on $\mathbb{C}$.

☐

For domains of a particular shape one needs to assume less about the growth near infinity. We consider two examples: strips and sectors These give rise to the classical forms of the Phragmén-Lindelöf principle.

Theorem 2.7: Phragmén-Lindelöf Principle for Strips

Let S_γ be the strip $\left\{z \in \mathbb{C} : |\text{Re } z| < \frac{\pi}{2\gamma}\right\}$, where $\gamma > 0$, and let u be a subharmonic function on S_γ such that for some constants $A < \infty$ and $\alpha < \gamma$,

$$u(x + iy) \leq Ae^{\alpha|y|}, x + iy \in S_\gamma.$$

If $\limsup_{z\to\zeta} u(z) \leq 0 \ \forall \zeta \in \partial S_\gamma\backslash\{\infty\}$ then $u \leq 0$ on S_γ.

Example 2.2: $\alpha < \gamma$ Is Necessary in **Theorem 2.7**

The function $u(z) : \text{Re}\left(\cos(\gamma z)\right) = \cos(\gamma x)\cosh(\gamma y)$ shows that the conclusion in **Theorem 2.7** fails if $\alpha = \gamma$. ⋄

Proof of **Theorem 2.7**:

Choose β such that $\alpha < \beta < \gamma$, and define $v : S_\gamma \to \mathbb{R}$ by

$$v(z) := \text{Re}\left(\cos(\beta z)\right) = \cos(\beta x)\cosh(\beta y)$$

for $z = x + iy \in S_\gamma$. Then v is subharmonic on S_γ. Moreover,

$$\liminf_{z\to\infty} v(z) \geq \liminf_{|y|>\infty} \cos\left(\frac{\beta\pi}{2\gamma}\right)\cosh(\beta y) = \infty$$

and

$$\limsup_{z\to\infty} \frac{u(z)}{v(z)} \leq \limsup_{|y|>\infty} \frac{Ae^{\alpha|y|}}{\cos(\beta\pi/2\gamma)\cosh(\beta y)} = 0.$$

The desired result follows from **Theorem 2.6**.

□

Corollary 2.7.1: Three-Lines Theorem

Let u be a subharmonic function on the strip $S := \{z : 0 < \text{Re } z < 1\}$ such that for some constants $A < \infty$ and $\alpha < \pi$,

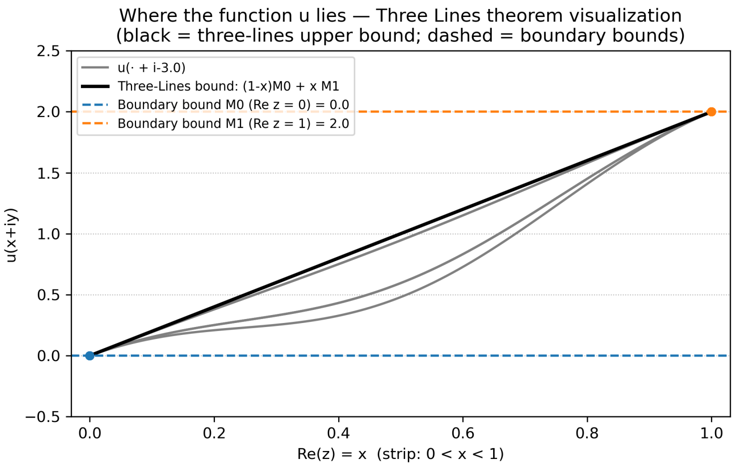
$$u(x + iy) \le Ae^{\alpha|y|}, x + iy \in S.$$

If

$$\limsup_{z \to \zeta} u(z) \le \begin{cases} M_0, & \text{Re } \zeta = 0 \\ M_1, & \text{Re } \zeta = 1 \end{cases}$$

then

$$u(x + iy) \le M_0(1 - x) + M_1 x, x + iy \in S.$$



(Figure 2.1: Demonstration for the three-line theorem)

Proof:

Define $\tilde{u} : S \to [-\infty, \infty)$ by

$$\tilde{u}(z) := u(z) - \text{Re}\left(M_0(1 - z) + M_1 z\right), z \in S.$$

Then applying (a translation of) **Theorem 2.7** with $\gamma = \pi$ yields $\tilde{u} \le 0$ on S.

□

Theorem 2.8: Phragmén-Lindelöf Principle for Sectors

Let T_γ be the sector $\left\{z \in \mathbb{C} \backslash \{0\} : |\arg(z)| < \dfrac{\pi}{2\gamma}\right\}$, where $\gamma > \dfrac{1}{2}$, and let u be a subharmonic function on T_γ such that for some constants $A, B < \infty$ and $\alpha < \pi$,

$$u(z) \le A + B|z|^\alpha, z \in T_\gamma.$$

Proof:

Choose β with $\alpha < \beta < \gamma$, and define $v : T_\gamma \to \mathbb{R}$ by

$$v(z) = \text{Re}\,(z^\beta) = r^\beta \cos(\beta t), z = re^{it} \in T_\gamma.$$

Then v is harmonic on T_γ by **Theorem 1.1** (i) and

$$\liminf_{z \to \infty} v(z) \ge \liminf_{r \to \infty} r^\beta \cos\left(\frac{\beta\pi}{2\gamma}\right) = \infty$$

and

$$\limsup_{z\to\infty}\frac{u(z)}{v(z)} \le \limsup_{r\to\infty}\frac{A+Br^\alpha}{r^\beta\cos\left(\frac{\beta}{2\gamma}\right)} = 0.$$

Hence again the result follows from **Theorem 2.6**.

□

As we mentioned earlier, the function $u(z) := \text{Re}\,(z^\gamma)$ shows that the theorem is no longer true if $\alpha = \gamma$, but we do have the following partial result, in which, for simplicity, we take $\gamma = 1$.

Corollary 2.8.1: Phragmén-Lindelöf Principle for Half Plane

Let u be a subharmonic function on the half-plane $H := \{z : \text{Re}\, z > 0\}$ such that for some constants $A, B < \infty$,

$$u(z) \le A + B|z|, z \in H.$$

If

$$\limsup_{z\to\zeta} u(z) \le 0 \ \forall \zeta \in \partial H \backslash \{\infty\} \text{ and } \limsup_{x\to\infty}\frac{u(x)}{x} = L.$$

Then $u(z) \le L(\text{Re}\, z)$ for $z \in H$.

Proof:

Given $L' > L$, define $\tilde{u} : H \to [-\infty, \infty)$ by

$$\tilde{u}(z) := u(z) - L'(\text{Re}\, z) \text{ for } z \in H.$$

Then applying (a rotated version of) **Theorem 2.8** with $\gamma = 2$ on each of the two sectors

$$\frac{-\pi}{2} < \arg(z) < 0 \text{ and } 0 < \arg(z) < \frac{\pi}{2},$$

we deduce that $\tilde{u}$ is bounded above on H. Applying **Theorem 2.8** once more with $\gamma = 1$, we have $\tilde{u} \le 0$ on H. Hence

$$u(z) \le L'(\text{Re}\, z),$$

since $L' > L$ is arbitrary, sending $L' \downarrow L$ yields the desired result.

□

2.4 Criteria for Subharmnicity

Now that the necessary tools are available, we can prove that subharmonic functions satisfy the global submean inequality. In fact, more is true: they also obey an inequality corresponding to the Poisson integral formula, as is shown in the following theorem.

Theorem 2.9: Criterion for U.S.C. Function to Be Subharmonic

Let U be an open subset of $\mathbb{C}$, and let $u : U \to [-\infty, \infty)$ be an u.s.c. function. Then the followings are equivalent:

(a) The function u is subharmonic.

(b) Whenever $\overline{\Delta}(w, \rho) \subset U$, for $r < \rho$ and $0 \le t < 2\pi$,

$$u(w + re^{it}) \le \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2}u(w + \rho e^{i\theta})d\theta.$$

(c) Whenever D is a relatively compact subdomain of U and h is a harmonic function on D satisfying

$$\limsup_{z\to\zeta}(u-h)(z)\le 0,\ \zeta\in\partial D.$$

Then $u\le h$ on D.

Proof:

(a) $\Rightarrow$ (c):

Suppose u is subharmonic on U. Given D and h as assumed in (c), the function $u-h$ is subharmonic on D, so the result follows from the maximum principle **Theorem 2.5** (ii).

(c) $\Rightarrow$ (b):

Suppose that $\overline{\Delta}:=\overline{\Delta}(w,\rho)\subset U$. By **Theorem 2.2** there exist continuous functions

$$\{\varphi_n\}_{n\ge 1},\varphi_n:\partial\Delta\to\mathbb{R},\text{ and }\varphi_n\downarrow u\text{ on }\partial\Delta.$$

By **Theorem 1.6** (i), each $P_\Delta\varphi_n$ is harmonic on Δ. Moreover by **Theorem 1.6** (ii) we have

$$\lim_{z\to\zeta}P_\Delta\varphi_n(z)=\varphi_n(\zeta)\ \forall\zeta\in\partial\Delta.$$

Therefore using u.s.c. in the first inequality and the fact $\varphi_n\downarrow u$ in the second,

$$\limsup_{z\to\zeta}\left(u-P_\Delta\varphi_n\right)(z)\le u(\zeta)-\varphi_n(\zeta)\le 0\ \forall\zeta\in\partial\Delta.$$

From (c) it follows that $u\le P_\Delta\varphi_n$ on Δ. Sending $n\to\infty$ and using Monotone convergence theorem gives the desired inequality.

(b) $\Rightarrow$ (a) is clear.

□

Putting $r=0$ in **Theorem 2.9** (b) yields the following result.

Corollary 2.9.1: Global Submean Inequality

If u is a subharmonic function on an open set U in $\mathbb{C}$, and if $\overline{\Delta}(w,\rho)\subset U$, then

$$u(w)\le\frac{1}{2\pi}\int_0^{2\pi}u(w+\rho e^{i\theta})d\theta.$$

The criterion (c) in **Theorem 2.9**, as well as explaining the name “subharmonic”, is also useful in its own right. For example, since it remains invariant under conformal mapping by the subordination principle **Theorem 1.12**, we immediately deduce the following result.

Corollary 2.9.2: Subharmonicity Is Closed Under Conformal Mapping

If $f:U_1\to U_2$ is a conformal mapping between open subsets U_1 and U_2 of $\mathbb{C}$, and if u is subharmonic on U_2, then $u\circ f$ is subharmonic on U_1.

Using this result, we can extend the definition of subharmonicity to the Riemann sphere in just the same way as we did for harmonicity in **Example 1.2**. It is easily checked that all the results in **Section 2.2** remain valid for subharmonic functions defined on an open subset of $\mathbb{C}^\infty$, as does the maximum principle **Theorem 2.5**.

Remark 2.4: Subharmonicity Is Closed Under General Holomorphic Functions

Corollary 2.9.2 remains true for a general holomorphic function f. One proof is outlined in **Exercise 2**, and the other will be given in **Theorem 2.23**. ◇

As an application of **Theorem 2.9**, we can characterize those C^2 functions which are subharmonic as those with positive Laplacian. This result vindicates what we said at the beginning of **Section 2.2**.

Theorem 2.10: Criterion for Subharmonicity via Positive Laplacian

Let U be an open subset of $\mathbb{C}$, and let $u \in C^2(U)$. Then

$$u \text{ is subharmonic } \Leftrightarrow \Delta u \geq 0 \text{ on } U.$$

Proof:

Step I: $\Leftarrow$

Assume first that $\Delta u \geq 0$ on U. We shall use **Theorem 2.9** (c) to prove that u is subharmonic. Let D be a relatively compact subdomain of U, and suppose that h is a harmonic function on D such that

$$\limsup_{z \to \zeta}(u - h)(z) \leq 0 \ \forall \zeta \in \partial D.$$

We need to show that $u \leq h$ on D.

Let $\varepsilon > 0$ and define

$$v_\varepsilon(z) := \begin{cases} u(z) - h(z) + \varepsilon |z|^2, & \text{if } z \in D \\ \varepsilon |z|^2, & \text{if } \partial D \end{cases}$$

Then v_ε is u.s.c. on $\overline{D}$, so it attains a maximum there by **Theorem 2.1**. But v_ε cannot attain a local maximum on D because

$$\Delta v_\varepsilon = \Delta u + 4\varepsilon > 0 \text{ on } D.$$

Therefore the maximum is attained on ∂D and hence

$$u - h \leq \sup_{\partial D} \varepsilon |z|^2 \text{ on } D.$$

Since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ yields $u - h \leq 0$ on D as desired.

Step II: $\Rightarrow$

Conversely, suppose that u is subharmonic on U. We prove by contradiction. Suppose $\Delta u(w) < 0$ for some $w \in U$. Then by continuity there exists $\rho > 0$ such that

$$\Delta u \leq 0 \text{ on } \Delta(w, \rho).$$

But what we have just proved in the first step, this implies that u is superharmonic on $\Delta(w, \rho)$, and hence harmonic there. In particular $\Delta u(w) = 0$, which contradicts with the original assumption.

□

The next result, which nicely illustrates the flexibility of subharmonic functions, shows that they can be “glued” together.

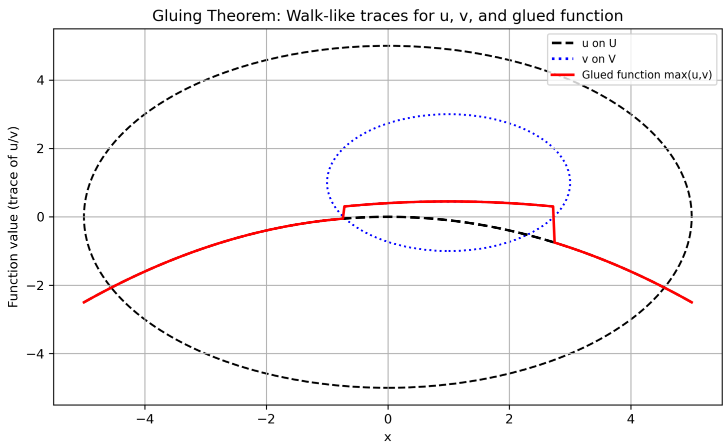
Theorem 2.11: Gluing Theorem

Let u be a subharmonic function on an open set U in $\mathbb{C}$, and let v be a subharmonic function on an open subset V of U such that

$$\limsup_{z \to \zeta} v(z) \leq u(\zeta), \zeta \in U \cap \partial V.$$

Then $\widetilde{u}$ is subharmonic on U, where

$$\widetilde{u} := \begin{cases} \max(u, v), & \text{on } V \\ u, & \text{on } U \backslash V \end{cases}$$



(Figure 2.2: Demonstration of Gluing two subharmonic functions)

Proof:

The boundary condition on v ensures that $\widetilde{u}$ is u.s.c. on U. By **Theorem 2.4** (i) $\widetilde{u}$ satisfies the local submean inequality at each $w \in V$, and it also does so when $w \in U \backslash V$ since $\widetilde{u} \geq u$ on U.

□

We conclude this section with three theorems about infinite families of subharmonic functions. The first of these, for decreasing sequences, is simply but important. It would no longer be true if we were to restrict subharmonic functions to be continuous, and ineed is one of the principal reasons for not doing so.

Theorem 2.12: Monotone Decreasing Limit Preserves Subharmonicity

Let $\{u_n\}_{n \geq 1}$ be subharmonic functions on an open set U in $\mathbb{C}$, and suppose that $u_1 \geq u_2 \geq \cdots$ on U. Then $u := \lim_{n \to \infty} u_n$ is subharmonic on U.

Proof:

The set $\{z : u(z) < \alpha\}$ is the union of the open sets $\{z : u_n(z) < \alpha\}$ for each $\alpha \in \mathbb{R}$, so it is open and thus u is u.s.c..

Moreover, if $\overline{\Delta}(w, \rho) \subset U$ then for each $n \geq 1$ one has

$$u_n(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u_n(w + \rho e^{i\theta}) d\theta.$$

Sending $n \to \infty$ and applying monotone convergence theorem we deduce that u satisfies the submean inequality and is therefore subharmonic on U.

□

The corresponding result for an increasing sequence $\{u_n\}_{n \geq 1}$ is false because, even it is finite, the limit u may fail to be u.s.c..

Example 2.3: Monotone Increasing Limit Does Not Preserve Subharmonicity

Let $u_n(z) := \dfrac{\log|z|}{n}$ on $\Delta(0,1)$, then

$$u(z) = \begin{cases} 0, & \text{if } 0 < |z| < 1 \\ -\infty, & \text{if } z = 0 \end{cases}$$

We will return to this topic in **Section 3.4**. ◇

The remaining two results generalize **Theorem 2.9** (a) and (b) respectively.

Theorem 2.13: Sup for Subharmonic Part of U.S.C. Functions Is Subharmonic

Let T be a compact topological space, let U be an open subset of $\mathbb{C}$, and let $v : U \times T \to [-\infty, \infty)$ be a function such that

(a) v is u.s.c. on $U \times T$.

(b) $z \mapsto v(z,t)$ is subharmonic on U $\forall t \in T$.

Then $u(z) := \sup_{t \in T} v(z,t)$ is subharmonic on U.

Proof:

Let $w \in U$ and suppose that $u(w) < \alpha$ for some $\alpha \in \mathbb{R}$. Then for each $t \in T$, $v(w,t) < \alpha$, so as v is u.s.c., there exists a neighbourhood N_t of t and $\rho_t > 0$ such that

$$v < \alpha \text{ on } \Delta(r, \rho_t) \times N_t.$$

As T is compact, it has a finite subcover $N_{t_1}, \cdots, N_{t_n}$. Then $u < \alpha$ on $\Delta(w, \rho')$, where $\rho' = \min(\rho_{t_1}, \cdots, \rho_{t_n})$. This shows that u is u.s.c. by **Theorem 2.1**.

Now suppose that $\overline{\Delta}(w, \rho) \subset U$. Then $\forall t \in T$,

$$v(w,t) \leq \frac{1}{2\pi}\int_0^{2\pi} v(w + \rho e^{i\theta}, t) \qquad \text{(by submean inequality)}$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} u(w + \rho e^{i\theta}) d\theta \qquad \text{(since } u := \sup v\text{)}$$

Taking the supremum over $t \in T$ yields the desired submean inequality.

□

Theorem 2.14: Integral Mean of Subharmonic Functions Is Subharmonic

Let (Ω, μ) be a measure space with $\mu(\Omega) < \infty$, let U be an open subset of $\mathbb{C}$, and let $v : U \times \Omega \to [-\infty, \infty)$ be a function such that

(a) v is measurable on $U \times \Omega$.

(b) $z \mapsto v(z, \omega)$ is subharmonic on U $\forall \omega \in \Omega$.

(c) $z \mapsto \sup_{\omega \in \Omega} v(z, \omega)$ is locally bounded above on U.

Then $u(z) := \int_\Omega v(z, \omega) d\mu(\omega)$ is subharmonic on U.

Proof:

It suffices to prove that u is subharmonic on each relatively compact subdomain D of U and then **Remark 2.1** concludes the proof.

Fix such a D. Then (c) implies that $\sup_\omega v(z, \omega)$ is bounded above on D, so by subtracting a constant if necessary, we can without loss of generality assume that $v \leq 0$ on $D \times \Omega$. This enables us to use Fatou's lemma and Fubini's Theorem. Whenever $w_n \to w$ in D, we have

$$\begin{aligned}\limsup_{n\to\infty} u(w_n) &\le \int_\Omega \limsup_{n\to\infty} v(w_n,\omega)d\mu(\omega) && \text{(Fatou's Lemma)}\\ &\le \int_\Omega v(w,\omega)d\mu(\omega) && (w_n \to w)\\ &=: u(w)\end{aligned}$$

It follows that u is u.s.c. on D.

Now we prove the submean inequality, suppose $\overline{\Delta}(w,\rho) \subset D$ then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} u(w+\rho e^{i\theta})d\theta &= \int_\Omega \left(\frac{1}{2\pi}\int_0^{2\pi} v(w+\rho e^{i\theta},\omega)d\theta\right)d\mu(\omega)\\ &\ge \int_\Omega v(w,\omega)d\mu(\omega) =: u(w),\end{aligned}$$

where the first equality holds by Fubini's theorem and the inequality holds by the submean inequality for v. Therefore u satisfies the submean inequality and it follows that u is subharmonic on D.

□

2.5 Integrability for Subharmonic Functions

As a subharmonic function is u.s.c., it is automaticaly bounded above on compact sets by **Theorem 2.1**. More subtle is the fact that also it cannot be 'too bounded below'.

Theorem 2.15: Subharmonic Function Is Locally Integrable

Let u be a subharmonic function on a domain D in $\mathbb{C}$ with $u \not\equiv -\infty$ on D. Then u is locally integrable on D, that is, for each compact subset K of D,

$$\int_K |u|dA < \infty,$$

where dA denotes the 2-dimensional Lebesgue measure.

Proof:

By a simple completeness argument, it suffices to show that for each $w \in D$, there exists $\rho > 0$ such that

$$\int_{\Delta(w,\rho)} |u|dA < \infty. \tag{2.2}$$

Denote

$$A := \left\{ w \in D : \exists \rho > 0 \text{ such that } \int_{\Delta(w,\rho)} |u|dA < \infty \right\}$$

and

$$B := \left\{ w \in D : \nexists \rho > 0 \text{ such that } \int_{\Delta(w,\rho)} |u|dA < \infty \right\}.$$

We shall show that both A and B are open, and that $u = -\infty$ on B, from which the result follows from the connectedness of D.

Step I: A is open

Let $w \in A$, choose $\rho > 0$ such that (2.2) holds. Given $w' \in \Delta(w, \rho)$ and set $\rho' := \rho - |w' - w|$. Then $\Delta(w', \rho') \subset \Delta(w, \rho)$, so

$$\int_{\Delta(w',\rho')} |u| dA < \infty$$

and it follows that $\Delta(w, \rho) \subset A$ and hence A is open.

Step II: B is open and $u = -\infty$ on B

Let $w \in B$, choose $\rho > 0$ such that $\overline{\Delta}(w, 2\rho) \subset D$. Then by the definition of B,

$$\int_{\Delta(w,\rho)} |u| dA = \infty.$$

Given $w' \in \Delta(w, \rho)$, set $\rho' := \rho + |w' - w|$. Then

$$\Delta(w', \rho') \supset \Delta(w, \rho)$$

and u is bounded above on $\overline{\Delta}(w', \rho')$ by **Theorem 2.1**. Therefore

$$\int_{\Delta(w',\rho')} u dA = -\infty.$$

Now u satisfies the submean inequality

$$u(w') \leq \frac{1}{2\pi} \int_0^{2\pi} u(w' + re^{i\theta}) d\theta, \ 0 \leq r \leq \rho'.$$

Multiplying $2\pi r$ and integrating over $r = 0$ and $r = \rho'$ yields

$$\pi(\rho')^2 u(w') \leq \int_{\Delta(w',\rho')} u dA = -\infty.$$

Hence $u = -\infty$ on $\Delta(w, \rho)$. Thus B is open and $u = -\infty$ on D.

□

From this, it follows that subharmonic functions are also integrable on circles.

Corollary 2.15.1: Subharmonic Function Is Integrable on Circles

Let u be a subharmonic function on a domain D in $\mathbb{C}$ with $u \not\equiv -\infty$. If $\overline{\Delta}(w, \rho) \subset D$ then

$$\frac{1}{2\pi} \int_0^{2\pi} u(w + \rho e^{i\theta}) d\theta > -\infty.$$

Proof:

Let $\overline{\Delta}(w, \rho) \subset D$. Since u is bounded above on $\overline{\Delta}(r, \rho)$, by subtracting a constant if necessary, we may without loss of generality assume that $u \leq 0$ on $\overline{\Delta}(w, \rho)$. Using **Theorem 2.9** (b) in the first inequality, if $r < \rho$ and $0 \leq t < 2\pi$ then one has

$$u(w + re^{it}) \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - t) + r^2} u(w + \rho e^{i\theta}) d\theta$$

$$\leq \left(\frac{\rho + r}{\rho - r}\right) \frac{1}{2\pi} \int_0^{2\pi} u(w + \rho e^{i\theta}) d\theta.$$

Hence, if the last integral were $-\infty$ then $u \equiv -\infty$ on $\Delta(w, \rho)$, contradicting **Theorem 2.15**. Therefore the integral is necessarily finite.

□

Another consequence of **Theorem 2.15** is that subharmonic functions can only equal to $-\infty$ on relatively small sets.

Corollary 2.15.2: Subharmonic Functions Are Locally Integrable Leb-A.E.

Let u be a subharmonic function on a domain D in $\mathbb{C}$ with $u \not\equiv -\infty$ on D. Then $E := \{z \in D : u(z) = -\infty\}$ is a set of Lebesgue measure zero.

Proof:

Let $\{K_n\}_{n\geq 1}$ be compact sets with $\bigcup_{n\geq 1} K_n = D$. For each n one has

$$\int_{K_n} |u| dA < \infty$$

by **Theorem 2.15**. Thus $E \cap K_n$ has measure zero. Since $E = \bigcup_{n\geq 1}(E \cap K_n)$ it also has Lebesgue measure zero.

□

The set E above is also small in other ways: one is outlined in Exercise 1 and others will be given in **Chapter 3**. Of course, if $u := \log|f|$ where f is holomorphic, then E is just the zero set of f, and is therefore countable. But as the following theorem shows, there are subharmonic functions which are $-\infty$ on uncountable sets.

Theorem 2.16: Uncountable Set Where Subharmonics Are Not Integrable

Let K be a compact subset of $\mathbb{C}$ with no isolated points, let $\{w_n\}_{n\geq 1}$ be a countable dense subset of K, and let $\{a_n\}_{n\geq 1}$ be strictly increasing positive numbers such that $\sum_{n\geq 1} a_n < \infty$. Define $u : \mathbb{C} : [-\infty, \infty)$ by

$$u(z) := \sum_{n\geq 1} a_n \log|z - w_n|, z \in \mathbb{C}.$$

Then the followings are true:

(a) u is subharmonc on $\mathbb{C}$ and u is not identically $-\infty$.

(b) $u = -\infty$ on an uncountable dense subset of K.

(c) u is discontinuous (Lebesgue) almost everywhere on K.

Proof:

Step I: (a)

Let μ be the fininte measure on $\mathbb{N}$ given by $\mu(\{n\}) = a_n$ for $n \geq 1$ and define

$$v : \mathbb{C} \times \mathbb{N} \to [-\infty, \infty)$$

by $v(z, n) := \log|z - w_n|$. Then $\forall z \in \mathbb{C}$,

$$\int_{\mathbb{N}} v(z, n) d\mu(n) = \sum_{n\geq 1} a_n \log|z - w_n| =: u(z),$$

where the first inequality holds by the definition of μ. Now by **Theorem 2.14** u is subharmonic on $\mathbb{C}$ and $u(z) > -\infty$ $\forall z \in \mathbb{C}\backslash K$ by **Theorem 2.15** and therefore $u \not\equiv -\infty$ as desired.

Step II: (b)

Set $E := \{z \in \mathbb{C} : u(z) = -\infty\}$. Clearly $E \subset K$ by (a), and each $w_n \in E$ so E

is dense in K. Since

$$K \backslash E = \bigcup_{n \geq 1} \{z \in K : u(z) \geq -n\}$$

is a countable union of closed **nowhere dense**[1] sets, it follows that $K \backslash E$ is **meager**[2] in K. If E were countable then K would be meager itself, contradicting the Baire category theorem, thus E is uncountable.

Step III: (c)

The function u is discontinuous at empty point of $\overline{E} \backslash E$. Since E is dense in K, and by **Corollary 2.15.2** E has Lebesgue measure zero. It follows that u is discontinuous Lebesgue-almost everywhere on K.

□

We shall return to the study of the sets where subharmonic functions are $-\infty$ in more details in **Section 3.5**.

2.6 Convexity for Subharmonic Functions

As we have already remarked, there are strong similarities between subharmonic functions on $\mathbb{C}$ and convex functions on $\mathbb{R}$. In this section we examine in more detail the relationship between two classes.

Definition: Convex Functions

Let $-\infty \leq a < b < \infty$. A function $\psi : (a, b) \to \mathbb{R}$ is said to be convex if for all $t_1, t_2 \in (a, b)$,

$$\psi\big((1-\lambda)t_1 + \lambda t_2\big) \leq (1-\lambda)\psi(t_1) + \lambda\psi(t_2).$$

It is well-known that the convex functions are continuous. Moreover, given that $\psi \in C^2\big((a, b)\big)$ then ψ is convex if and only if $\psi'' \geq 0$ on (a, b). We shall need a basic inequality for convex functions.

Theorem 2.17: Jensen's Inequality

Let $-\infty \leq a < b \leq \infty$ and let $\psi : (a, b) \to \mathbb{R}$ be a convex function. Suppose that (Ω, μ) is a measure space with total measure $\mu(\Omega) = 1$, and suppose that $f : \Omega \to (a, b)$ is μ-integrable. Then

$$\psi\left(\int_\Omega f d\mu\right) \leq \int_\Omega \psi \circ f d\mu.$$

Proof:

Set $c := \int_\Omega f d\mu$ so that $c \in (a, b)$. By convexity, if $a < t_1 < c < t_2 < b$, then

$$\psi(c) \leq \frac{t_2 - c}{t_2 - t_1}\psi(t_1) + \frac{c - t_1}{t_2 - t_1}\psi(t_2).$$

After arrangement, this implies that

$$\sup_{t_1 \in (a,c)} \frac{\psi(c) - \psi(t_1)}{c - t_1} \leq \inf_{t_2 \in (c,b)} \frac{\psi(t_2) - \psi(c)}{t_2 - c}.$$

---

[1] A set $A \subset X$ is said to be nowhere dense if its closure has empty interior.

[2] A set $A \subset X$ is said to be meager if it is the countable union of nowhere dense sets.

Hence there exists a constant M such that

$$\psi(t) \geq \psi(c) + M(t - c), t \in (a, b).$$

Setting $t := f(w)$ and integrating with respect to μ gives

$$\int_\Omega \psi\big(f(w)\big)d\mu(w) \geq \int_\Omega \psi(c)d\mu(w) + M \cdot \int_\Omega \big(f(w) - c\big)d\mu(w) = \psi(c)$$

since the total measure $\mu(\Omega) = 1$. The proof is completed.

□

This enables us to generate new examples of subharmonic functions.

Theorem 2.18: Increasing Convex Composition Preserves Subharmonicity

Let $-\infty \leq a < b \leq \infty$, let $u : U \to [a, b)$ be subharmonic function on an open set U in $\mathbb{C}$, and let $\psi : (a, b) \to \mathbb{R}$ be an increasing convex function. Denote $\psi(a) := \lim_{t \to a} \psi(t)$, then $\psi \circ u$ is subharmonic on U.

Proof:

Choose $\{a_n\}_{n \geq 1} \subset (a, b)$ with $a_n \downarrow a$, and for eaach n set $u_n := \max(u, a_n)$, so u_n is subharmonic by **Theorem 2.4** (i). Then certainly $\psi \circ u_n$ is upper semicontinuous on U. Moreover, if $\overline{\Delta}(w, \rho) \subset U$ then

$$\psi \circ u_n(w) \leq \psi\left(\frac{1}{2\pi}\int_0^{2\pi} u_n(w + \rho e^{i\theta})d\theta\right) \qquad \text{(Submean Inequality)}$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \psi \circ u_n(w + \rho e^{i\theta})d\theta$$

where the second inequality holds by Jensen's inequality **Theorem 2.17** applied to the measure $\dfrac{d\theta}{2\pi}$ on $[0,2\pi)$. Hence $\psi \circ u_n$ is subharmonic on U. Since $\psi \circ u_n \downarrow u$ as $n \to \infty$, it follows from **Theorem 2.12** that $\psi \circ u$ is subharmonic on U as desired.

□

Corollary 2.18.1: Exponential of Subharmonic Function Is Subharmonic

If u is subharmonic on an open subset U of $\mathbb{C}$ then so is $\exp u$.

Example 2.4: Subharmonic Functions Whose Exponential Is Subharmonic

Applying **Corollary 2.18.1** to $u := \alpha \log|f|$, where f is holomorphic and $\alpha > 0$, then $|f|^\alpha$ is subharmonic. ◇

It is of special interest to know under what conditions $\log u$ is subharmonic.

Theorem 2.19: Criterion for Log Functions to Be Subharmonic

Let $u : U \to [0,\infty)$ be a function on an open set U in $\mathbb{C}$. Then the following statements are equivalent:

(i) $\log u$ is subharmonic on U.

(ii) $u|e^q|$ is subharmonic on U for every (complex) polynomial q.

Proof:

(i) ⇒ (ii):

Suppose first that $\log u$ is subharmonic on U. Then by **Theorem 2.4** (ii) and **Theorem 1.1** (ii), $\log u + \operatorname{Re} q$ is subharmonic on U for each polynomial q,

taking exponentials, **Corollary 2.18.1** implies that $u|e^q|$ is subharmonic.
(ii) $\Rightarrow$ (i):
Conversely, suppose (ii) holds. Taking $q=0$, we see straightaway that u is subharmonic, and in particular upper semicontinuous. Hence $\log u$ is also upper semicontinuous by convexity.
It remains to check the submean inequality. Let $\Delta := \Delta(w,\rho)$ be a disc with $\overline{\Delta} \subset U$, and choose continuous functions $\varphi_n : \partial\Delta \to \mathbb{R}$ such that $\varphi_n \downarrow \log n$ on $\partial\Delta$. For each $n \geq 1$ we can find a polynomial $\{q_n\}_{n\geq 1}$ such that

$$0 = \mathrm{Re}\ q_n - \varphi_n \leq \frac{1}{n} \text{ on } \partial\Delta.$$

This follows from Stone-Wierstrass Theorem (which states that any continuous complex function over a compact interval can be approximated by an arbitrary degree of accuracy with a sequence of polynomials).
Then we have, for $\zeta \in \partial\Delta$, using submean inequality in the first display,

$$\limsup_{z\to\zeta} u(z)|e^{-q_n(z)}| \leq e^{\varphi_n(\zeta)}e^{-\mathrm{Re}\ q_n(\zeta)} \leq 1.$$

Since $u|e^{-q_n}|$ is assumed to be subharmonic, it follows from the maximum principle **Theorem 2.5** that $u|e^{-q_n}| \leq 1$ on Δ. Hence

$$\begin{aligned}\log u(w) &\leq \mathrm{Re}\ q_n(w)\\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\ q_n(w+\rho e^{i\theta})d\theta\\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \varphi_n(w+\rho e^{i\theta})d\theta + \frac{1}{n}.\end{aligned}$$

Sending $n\to\infty$ and applying Monotone convergence theorem yield

$$\log u(w) \leq \frac{1}{2\pi}\int_0^{2\pi} \log u(w+\rho e^{i\theta})d\theta,$$

which verifies the submean inequality and (i) follows.

□

Theorem 2.19 allows us to characterize radial subharmonic functions.

Theorem 2.20: Criterion for Radial Functions to Be Subharmonic

Let $v : \Delta(0,\rho) \to [-\infty,\infty)$ be a function which is radial $\big($i.e., $v(z)=v(|z|)$ for all $z\big)$, and assume that $v \not\equiv -\infty$. Then the followings are equivalent:

(i) v is subharmonic on $\Delta(0,\rho)$.

(ii) $v(r)$ is an increasing convex function of $\log r$, $0<r<\rho$ with $\lim_{r\to 0} v(r) = v(0)$.

Proof:

(ii) $\Rightarrow$ (i):
Applying **Theorem 2.18** with $u(z) := \log|z|$ and $\psi(t) = v(e^t)$ gives this direction.
(i) $\Rightarrow$ (ii):
Assume (i). Given $r_1, r_2 \in [0,\rho)$ with $r_1 < r_2$, then the maximum principle

Theorem 2.5 (ii) applied to v on $\Delta(0,r_2)$ yields
$$v(r_1) \leq \sup_{\partial\Delta(0,r_2)} v = v(r_2).$$
Hence v is increasing on $[0,\rho)$. Moreover, it follows that
$$\liminf_{r\to 0} v(r) \geq v(0).$$
On the other hand, upper semicontinuity implies that $\limsup_{r\to 0} v(r) \leq v(0)$ and hence $\lim_{r\to 0} v(r) = v(0)$.

It remains to show that $v(r)$ is a convex function of $\log r$. Observe first that by **Corollary 2.15.1** $v(r) > -\infty$ for $r > 0$. Given $r_1, r_2 \in (0,\rho)$ with $r_1 < r_2$, choose constants α, β such that
$$\alpha + \beta \log r = v(r) \text{ for } r = r_1 + r_2.$$
Applying the maximum principle **Theorem 2.5** (ii) to $v(z) - \alpha - \beta \log|z|$ on $\{z : r_1 < |z| < r_2\}$, we get
$$v(r) \leq \alpha + \beta \log r, r_1 < r < r_2.$$
Hence if $0 \leq \lambda \leq 1$ and $\log r := (1-\lambda)\log r_1 + \lambda \log r_2$ then
$$\begin{aligned} v(r) &\leq \alpha + \beta \log r \\ &= (1-\lambda)(\alpha + \beta \log r_1) + \lambda(\alpha + \beta \log r_2) \\ &= (1-\lambda)v(r_1) + \lambda v(r_2), \end{aligned}$$
which verifies the convexity.

□

Theorem 2.20 can be used to study various integral means of subharmonic functions.

Definition: Max, Circle Mean, and Area Mean

Let u be a subharmonic function on the disc $\Delta(0,\rho)$ with $u \not\equiv -\infty$. For $0 < r < \rho$, we define

(i) Maximum of u over $\Delta(0,r)$ as $M_u(r) := \sup_{|z|=r} u(z)$.

(ii) Circular mean of u over $\Delta(0,r)$ as $C_u(r) := \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(re^{it})dt$.

(iii) Area mean of u over $\Delta(0,r)$ as $B_u(r) := \dfrac{1}{\pi r^2}\displaystyle\int_{\Delta(0,r)} u dA$.

Note that by **Theorem 2.15** and **Corollary 2.15.1** (i), (ii), and (iii) are well-defined and are all finite. Moreover $C_u(r)$ and $B_u(r)$ are connected by the relation
$$B_u(r) = \frac{2}{r^2}\int_0^r C_u(s)sds. \tag{2.3}$$

Theorem 2.21: Properties for Modes of Mean Integrals for Subharmonic Functions

Let u be a subharmonic function on the disc $\Delta(0,\rho)$ with $u \not\equiv -\infty$. For $0 < r < \rho$, we have

(a) $M_u(r)$, $C_u(r)$, and $B_u(r)$ are all increasing convex functions of $\log r$.

(Log-Radius Convexity of Means)

(b) $M_u(r) \geq C_u(r) \geq B_u(r) \geq u(0)$ for $0 < r < \rho$.

(Ordering of Means)

(c) $\lim_{r\to 0} M_u(r) = \lim_{r\to 0} C_u(r) = \lim_{r\to 0} B_u(r) = u(0)$. (Continuity at Center)

Proof:

Step I: (a)

Observe that for $0 < r < \rho$,

$$M_u(r) = v(r) \text{ where } v(z) = \sup_{t\in[0,2\pi]} u(ze^{it}),$$

$$C_u(r) = v(r) \text{ where } v(z) = \frac{1}{2\pi}\int_0^{2\pi} u(ze^{it})dt,$$

$$B_u(r) = v(r) \text{ where } v(z) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 u(zse^{it})sdsdt.$$

In each case v is subharmonic on $\Delta(0,\rho)$: this is proved using **Theorem 2.13** in the first case and **Theorem 2.14** in the other two. Clearly each v is also radial, and so the result follows from **Theorem 2.20**.

Step II: (b)

The first inequality is trivial. To derive the others, we begin with the relation

$$C_u(r) \geq C_u(s) \geq u(0) \text{ for } r \geq s$$

proved in (a). Multiplying both sides by $\frac{2s}{r^2}$ and integrating from $s = 0$ to $s = r$ we get

$$C_u(r) \geq \frac{2}{r^2}\int_0^{2\pi} C_u(s)sds \geq u(0).$$

Combining this with **(2.3)** yields $C_u(r) \geq B_u(r) \geq u(0)$, as desired.

Step III: (c)

By (b), it suffices to show that $\limsup_{r\to 0} M_u(r) \leq u(0)$, and this is an immediate consequence from the upper semicontinuity of u.

□

2.7 Smoothing for Subharmonic Functions

Although subharmonic functions need not to be smooth, indeed sometimes far from it, they can nevertheless always be approximated by others which are smooth. A standard way to do this is to use convolutions.

Definition: Convolution

Let u be an open subset of $\mathbb{C}$, and for $r > 0$ define

$$U_r := \{z \in U : \text{dist}(z, \partial U) > r\}.$$

Let $u : U \to [-\infty, \infty)$ be a locally integrable function and let $\varphi : \mathbb{C} \to \mathbb{R}$ be a continuous function with $\text{supp}(\varphi) \subset \Delta(0,r)$. Then the convolution between u and φ is the function $u * \varphi : U_r \to \mathbb{R}$ given by

$$u * \varphi(z) = \int_{\mathbb{C}} u(z-w)\varphi(w)dA(w), z \in U_r.$$

After a change of variable, we also have

$$u * \varphi(z) = \int_{\mathbb{C}} u(w)\varphi(z-w)dA(w), z \in U_r.$$

This shows that if $\varphi \in C^\infty$ then $u * \varphi \in C^\infty$.

Theorem 2.22: Smoothing Theorem for Subharmonic Functions

Let u be a subharmonic function on a domain D in $\mathbb{C}$ with $u \not\equiv -\infty$. Let $\chi : \mathbb{C} \to \mathbb{R}$ be a function satisfying

(a) $\chi \in C^\infty$. (Continuous)

(b) $\chi \geq 0$. (Non-Negative)

(c) $\chi(z) = \chi(|z|)$. (Radial)

(d) $\text{supp}(\chi) \subset \Delta(0,1)$. (Concentrated on Unit Sphere)

(e) $\int_{\mathbb{C}} \chi dA = 1$. (Normalized)

For $r > 0$ define $\chi_r(z) := \frac{1}{r^2}\chi\left(\frac{z}{r}\right), z \in \mathbb{C}$. Then

(i) $u * \chi_r$ is a C^∞ subharmonic function on D for each $r > 0$.

(ii) $(u * \chi_r) \downarrow u$ on D as $r \downarrow 0$.

Example 2.5: Examples of Functions Described in **Theorem 2.22**

An example for a function χ satisfying (a)-(e) in **Theorem 2.22** is given by

$$\chi(z) := \begin{cases} C \cdot \exp\left\{\frac{-1}{1-4|z|^2}\right\}, & \text{if } |z| < \frac{1}{2} \\ 0, & \text{if } |z| \geq \frac{1}{2} \end{cases}$$

where C is a constant chosen so that $\int \chi d_A = 1$. ⋄

Proof of **Theorem 2.22**:

Step I: Assertion (i)

By **Theorem 2.15** u is locally integrable so $u * \chi_r$ makes sense and is C^∞ on D_r

To show it is subharmonic on D, applying **Theorem 2.14** with

$$(\Omega, \mu) = (\mathbb{C}, \chi_r dA) \text{ and } v(z,w) = u(z-w)$$

yields the desired result.

Step II: Assertion (ii)

Now fix $\zeta \in D$. For $0 < r < \text{dist}(\zeta, \partial D)$ we have

$$u * \chi_r(\zeta) = \int_0^{2\pi}\int_0^r u(\zeta - se^{it})r^{-2}\chi\left(\frac{s}{r}\right)sdsdt.$$

Making substitutions $\sigma := s/r$ and $v(z) := u(\zeta - z)$ yields

$$u * \chi_r(\zeta) = 2\pi\int_0^1 C_v(r\sigma)\chi(\sigma)\sigma d\sigma.$$

By **Theorem 2.21** (c) $C_v(r\sigma) \downarrow v(0)$ as $r \downarrow 0$. Hence by Monotone convergence theorem, $u * \chi_r(\zeta)$ decreases to

$$2\pi \int_0^1 v(0)\chi(\sigma)\sigma d\sigma = u(\zeta)\int_{\mathbb{C}} \chi dA = u(\zeta),$$

where the first equality holds by $v(z) := u(\zeta - z)$ and assuptions (c) and (d), the second equality holds by assumption (e). It follows that $(u * \chi_r) \downarrow u$ on D.

□

Corollary 2.22.1: Subharmonic Function Has Smoothing On Relatively Compacts

Let u be a subharmonic function on an open set U in $\mathbb{C}$, and let D be a relatively compact subdomain of U. Then there exist subharmonic functions $\{u_n\}_{n\geq 1} \subset C^\infty(D)$ such that $u_1 \geq u_2 \geq \cdots \geq u$ on D and $\lim_{n\to\infty} u_n = u$.

Proof:

If $u \equiv -\infty$ on D, take $u_n \equiv -n$. Otherwise, choose $r > 0$ such that $D \subset U_r$ and take $u_n := u * \chi_{r/n}$.

□

As an application of this result, we can extend **Corollary 2.9.2** to general holomorphic mappings as we promised in **Remark 2.4**.

Theorem 2.23: Subharmonicity Is Closed under Holomorphy

Let $f : U_1 \to U_2$ be a holomorphic map between open subsets U_1, U_2 of $\mathbb{C}$. If u is subharmonic on U_2 then $u \circ f$ is subharmonic on U_1.

Proof:

Let D_1 be a relatively compact subdomain of U_1. It suffices to show that $u \circ f$ is subharmonic on D_1.

Set $D_2 : f(D_1)$ and choose subharmonic functions

$$\{u_n\}_{n\geq 1} \subset C^\infty(D_2) \text{ such that } u_n \downarrow u \text{ on } D_2.$$

By **Theorem 2.10** $\Delta u_n \geq 0$ on D_2 $\forall n \geq 1$. Now an easy computation gives

$$\Delta(u_n \circ f) = \big((\Delta u_n) \circ f\big)\,|f'|^2 \text{ on } D_1.$$

Hence $\Delta(u_n \circ f) \geq 0$ on D_1, and using **Theorem 2.10** once more we conclude that $u_n \circ f$ is subharmonic there. Finally, sending $n \uparrow \infty$ and by **Theorem 2.12** $u \circ f$ is subharmonic on D_1.

□

Theorem 2.23 can also be used to prove a form of identity principle for subharmonic functions which, although rather weak, is still useful. In particular it extends the almost everywhere property to everywhere property.

Theorem 2.24: Weak Identity Principle for Subharmonic Functions

Suppose that u and v are subharmonic functions on an open set U in $\mathbb{C}$ such that $u = v$ almost everywhere on U then $u \equiv v$ on U.

Proof:

Step I: Bounded below case

Suppose first that u and v are bounded below on U. Taking χ as stated in **Theorem 2.22**, we then have $u * \chi_r = v * \chi_r$ on U_r. Sending $r \to 0$ we conclude that $u = v$ on U.

Step II: General case

The general case follows by applying the first step to $u_n := \max(u, -n)$ and

$v_n := \max(v, -n)$ and then sending $n \to \infty$.

□

We cannot hope for an identity principle as strong as that for harmonic functions we proved back in **Theorem 1.3**. In other words, the almost everywhere condition cannot be removed in **Theorem 2.24**.

Example 2.6: Almost Everywhere Condition Cannot Be Removed in **Theorem 2.24**

Consider $u(z) := \max(\operatorname{Re} z, 0)$ and $v(z) := 0$. They agree on an open subset of $\mathbb{C}$ without being equal on the whole of $\mathbb{C}$. ◇

In fact, as we shall see, it is this very lack of rigidity that makes subharmonic functions such a useful tool.

3. Potential Theory
3.1 Potentials

Potentials play at least two roles. Firstly they provide an important source of examples of subharmonic functions, giving us the means, for instance, of constructing such functions with various prescribed properties. Secondly, despite their apparently rather special nature, which makes them comparatively easy to study, we shall see that potentials turn out to be almost as general as arbitrary subharmonic functions, and for many purposes the two classes are equivalent.

We shall define potentials only for finite measures with compact support.

Definition: Potentials (of Measures)

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Its potential is the function $p_\mu : \mathbb{C} \to [-\infty, \infty)$ defined by

$$p_\mu(z) := \int \log|z-w|\, d\mu(w), \; z \in \mathbb{C}.$$

Since $p_\mu(z)$ is defined in this way, it is also known as the logarithmic potentials.

Theorem 3.1: Basic Properties of Potentials

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Then

(i) p_μ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} \backslash \operatorname{supp}(\mu)$.

(ii) $p_\mu(z) = \mu(\mathbb{C})\log|z| + O\left(|z|^{-1}\right)$ as $z \to \infty$.

Proof:

Step I: Assertion (i)

Set $K := \operatorname{supp}(\mu)$, so μ can be regarded as a measure on K. By **Theorem 2.14** applied with $v(z, w) := \log|z-w|$ on $\mathbb{C} \times K$, we see that p_μ is subharmonic on $\mathbb{C}$. Applying **Theorem 2.14** once more but with $v(z, w) := -\log|z-w|$ on $(\mathbb{C} \backslash K) \times K$, we also find that p_μ is superharmonic on $\mathbb{C} \backslash K$ and hence harmonic there, this proves (i).

Step II: Assertion (ii)

Observe that for $z \neq 0$, by change of variables

$$p_\mu(z) = \mu(\mathbb{C})\log|z| + \int \log\left|1 - \frac{w}{z}\right| d\mu(w).$$

As μ has compact support, the final term is $O\left(|z|^{-1}\right)$ as $n \to \infty$.

□

Potentials enjoy several properties over and above those displayed by general subharmonic functions. We now prove two of these: the continuity principle and the minimum principle.

Theorem 3.2: Continuity Principle for Potentials

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support K.

(a) If $\zeta_0 \in K$ then $\liminf_{z\to\zeta_0} p_\mu(z) = \liminf_{\zeta\to\zeta_0,\zeta\in K} p_\mu(\zeta)$. (Lower Bound)

(b) If in addition $\lim_{\zeta\to\zeta_0,\zeta\in K} p_\mu(\zeta) = p_\mu(\zeta_0)$ then $\lim_{z\to\zeta_0} p_\mu(z) = p_\mu(\zeta_0)$. (Continuity)

Proof:

Step I: Assertion (a)

If $p_\mu(\zeta_0) = -\infty$ then by upper semicontinuity $\lim_{z\to\zeta_0} p_\mu(z) = -\infty$ and the result is trivial. Thus, without loss of generality we may assume that $p_\mu(\zeta_0) > -\infty$. Then necessarily one has $\mu\big(\{\zeta_0)\}\big) = 0$ and so, given $\varepsilon > 0$, there exists $r > 0$ such that $\mu\big(\Delta(\zeta_0, r)\big) < \varepsilon$. Given $z \in \mathbb{C}$, choose $\zeta \in K$ minimizing $|\zeta - z|$. Then $\forall w \in K$, by triangle inequality one has

$$\frac{|\zeta - w|}{|z - w|} \le \frac{|\zeta - z| + |z - w|}{|z - w|} \le 2.$$

Therefore, using change of variables in the equality and fundamental theorem of calculus in the inequality yields

$$p_\mu(z) = p_\mu(\zeta) - \int_K \log\left|\frac{\zeta - w}{z - w}\right| d\mu(w)$$

$$\ge p_\mu(\zeta) - \varepsilon \log 2 - \int_{K\setminus\Delta(\zeta_0,r)} \log\left|\frac{\zeta - w}{z - w}\right| d\mu(w)\,.$$

As $z \to \zeta_0$ in $\mathbb{C}$, the corresponding $\zeta \to \zeta_0$ in K, and hence

$$\liminf_{z\to\zeta_0} p_\mu(z) \ge \liminf_{\zeta\to\zeta_0,\zeta\in K} p_\mu(\zeta) - \varepsilon \log 2 - 0.$$

Finally, since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ yields (a).

Step II: Assertion (b)

Suppose that in addition we have $\lim_{\zeta\to\zeta_0,\zeta\in K} p_\mu(\zeta) = p_\mu(\zeta_0)$. Then by (a) one has

$$\lim_{z\to\zeta_0} p_\mu(z) = p_\mu(\zeta_0).$$

Moreover, since p_μ is upper semicontinuous by **Theorem 3.1**, one has

$$\limsup_{z\to\zeta_0} p_\mu(z) \le p_\mu(\zeta_0).$$

Combining these two displays gives the assertion (b).

□

Theorem 3.3: Minimum Principle for Potentials

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support K. If $p_\mu \ge M$ on K then $p_\mu \ge M$ on $\mathbb{C}$.

Proof:

Denote $u := -p_\mu$ on $\mathbb{C}\backslash K$. Then u is subharmonic on $\mathbb{C}\backslash K$ and (assuming that $u \neq 0$) $u(z) \to -\infty$ as $n \to \infty$. Moreover if $\zeta_0 \in \partial K$, then

$$\limsup_{z\to\zeta_0, z\in\mathbb{C}\backslash K} u(z) \leq -\liminf_{z\to\zeta_0} p_\mu(z) = -\liminf_{\zeta\to\zeta_0,\zeta\in K} p_\mu(\zeta) \leq -M,$$

where the first inequality holds by the definition of u, the middle equality holds by **Theorem 3.2** (a), and the last inequality holds by assumption.

Finally, applying the maximum principle **Theorem 2.5** to u on each component of $\mathbb{C}\backslash K$ we get $u \leq -M$ there. Thus $p_\mu \geq M$ on $\mathbb{C}$.

□

3.2 Polar Sets

Polar sets play the role of negligible sets in potential theory, much as sets of measure zero do in measure theory. To define them, we first need to introduce the notion of energy.

Definition: Energy (of Measures)

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Its energy $I(\mu)$ is defined by

$$I(\mu) := \iint \log|z-w|\, d\mu(z)d\mu(w) = \int p_\mu(z)d\mu(z).$$

To explain this terminology, think of μ as being a charge distribution on $\mathbb{C}$. Then $p_\mu(z)$ represents the potential energy at z due to μ, and so the total energy of μ is just $\int p_\mu(z)d\mu(z)$, in other words, $I(\mu)$. In fact, since like charges repel, most physicists would define the energy as $-I(\mu)$, but our definition would be more convenient.

It is possible that $I(\mu) = -\infty$. Indeed some sets only support measures of infinite energy. These sets are so important and deserve a name.

Definition: Polar Set

A subset E of $\mathbb{C}$ is said to be polar if $I(\mu) = -\infty$ for every finite Borel measure $\mu \neq 0$ for which $\text{supp}(\mu)$ is a compact subset of E.

Definition: Non-Polar Set

A subset E of $\mathbb{C}$ is said to be non-polar if it is not polar.

Definition: Nearly Everywhere Property

A property is said to hold nearly everywhere (n.e.) on a subset S of $\mathbb{C}$ if it holds everywhere on $S\backslash E$, where E is some Borel polar set.

As we mentioned earlier, the polar sets serve as the "measure zero sets in measure theory", thus, as almost everywhere being translated to almost surely from measure theory to probability theory, same thing happens here as we translate almost everywhere to nearly everywhere. Some authors also call nearly everywhere property as quasi-everywhere property.

Remark 3.1: Some Properties of Polar Sets, Non-Polar Sets, and N.E. Properties

(i) Singletons are polar (when $d \geq 2$).

(ii) Every subset of polar set is polar.

(iii) Every non-polar set contains a compact non-polar subset, namely, $\text{supp}(\mu)$, for measure μ, such that $I(\mu) > -\infty$.

(iv) If a property $\mathscr{P}$ holds nearly everywhere then it holds (μ)-almost everywhere. The converse is not true.

(v) If a property $\mathscr{P}$ holds nearly everywhere on $\{E_n\}_{n\geq 1}$ then it holds nearly everywhere on $E := \bigcup_{n\geq 1} E_n$.

(vi) If a $f_1 \geq f_2$ nearly everywhere on E, $f_2 \geq f_3$ nearly everywhere on B, then $f_1 \geq f_3$ nearly everywhere on E.

(vii) Borel polar sets have Hausdorff dimension zero and for all $\alpha > 0$ Borel polar sets have α-Hausdorff measure zero. ⋄

It is easy to see that measures of finite energy can have no atoms. In fact, more generally, they do not charge any polar sets.

Theorem 3.4: Borel Measures with Finite Energy Do NOT Charge Any Polar Sets

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support, and suppose that $I(\mu) > -\infty$. Then $\mu(E) = 0$ for every Borel polar sets E.

Proof:

Let E be a Borel set such that $\mu(E) > 0$. We shall show that E is non-polar. By the regularity of μ, we can choose a compact subset K of E with $\mu(K) > 0$. Set $\widetilde{\mu} := \mu|_K$ and $d := \text{diam}\big(\text{supp}(\mu)\big)$. Then $\widetilde{\mu}$ is a finite non-zero measure whose support is a compact subset of E and

$$
\begin{aligned}
I(\widetilde{\mu}) &= \int_K \int_K \log\left|\frac{z-w}{d}\right| d\mu(z)d\mu(w) + \mu(K)^2 \log d \\
&\geq \int_{\mathbb{C}} \int_{\mathbb{C}} \log\left|\frac{z-w}{d}\right| d\mu(z)d\mu(w) + \mu(K)^2 \log d \\
&= I(\mu) - \mu(\mathbb{C})^2 \log d + \mu(K)^2 \log d \\
&> -\infty,
\end{aligned}
$$

where the first equality holds by change of variables and **Theorem 3.1** (ii), the first inequality holds since integrating over $\mathbb{C}$ results in some negative terms, and the last equality holds by using **Theorem 3.1** (ii) again. This proves the claim that E is non-polar.

□

Corollary 3.4.1: Borel Polar Set Has Lebesgue Measure Zero

Every Borel polar set has Lebesgue measure zero.

Proof:

It suffices to show that, for $\rho > 0$, the measure $d\mu := dA|_{\Delta(0,\rho)}$ has energy $I(\mu) > -\infty$. For then by **Theorem 3.4** every Borel polar set E has μ-measure zero, that is, $E \cap \Delta(0,\rho)$ has Lebesgue measure zero, and the result follows by letting $\rho \to \infty$.

To this end, fix $\rho > 0$ and let $d\mu := dA|_{\Delta(0,\rho)}$. Then for $z \in \Delta(0,\rho)$ one has

$$\begin{aligned}
p_\mu(z) &= \int_{\Delta(0,\rho)} \log\left|\frac{z-w}{2\rho}\right| dA(w) + \pi\rho^2 \log(2\rho) \\
&\geq \int_{t=0}^{2\pi}\int_{r=0}^{2\rho} \log\left(\frac{r}{2\rho}\right) r dr dt + \pi\rho^2(2\rho) \\
&= -2\pi\rho^2 + \pi\rho^2 \log(2\rho)
\end{aligned}$$

where the first equality holds by **Theorem 3.1** (ii) and the inequality holds since $|r| \leq |z-w|$. It follows from this display that

$$I(\mu) := \int_{\Delta(0,\rho)} p_\mu(z) d\mu(z) \geq \left(-2\pi\rho^2 + \pi\rho^2 \log(2\rho)\right)\pi\rho^2 > -\infty,$$

as desired.

□

Thus nearly everywhere property implies almost everywhere property and the converse is not true, as we claimed in **Remark 3.1** (iv). In fact an argument similar to the proof of **Corollary 3.4.1**, but rather more technical, shows that Borel polar sets actually have α-dimensional Hausdorff measure zero for each $\alpha > 0$, and thus are all of Hausdorff dimension zero. We shall not perform the details here.

Corollary 3.4.2: Borel Polar Set Is Stable Under Countable Union

A countable union of Borel polar sets is polar. In particular, every countable subset of $\mathbb{C}$ is polar.

Proof:

Suppose that $\{E_n\}_{n\geq 1}$ are Borel polar sets and $E := \bigcup_{n\geq 1} E_n$. Let μ be a finite Borel measure on $\mathbb{C}$ whose support is a compact subset of E. If $I(\mu) > -\infty$ then by **Theorem 3.4** $\mu(E_n) = 0$ $\forall n \geq 1$ thus $\mu(E) = 0$ and hence $\mu = 0$. It follows that E is polar.

□

The conclusion in **Corollary 3.4.2** fails if the sets are not Borel.

Example 3.1: Countable Union of Non-Borel Polar Sets May NOT Be Polar

Let S be a set and let $\mathscr{T}$ be a collection of infinite subsets of S such that the cardinality of T is greater or equal to the cardinality of $\mathscr{T}$ for all $T \in \mathscr{T}$. Then S can be partitioned into subsets P and Q and neither of them contains any element of $\mathscr{T}$.

In particular, if $\mathscr{T}_0$ is the collection of all uncountable compact subsets of $\mathbb{C}$ then $\mathbb{C}$ can be partitioned into subsets P and Q such that each compact subset of P or Q is countable. In this case, the union of two non-Borel polar sets needs not to be polar. ◇

Remark 3.2: Polar Sets Need Not to Be Countable

We conclude this section by remark that, though every countable set is polar, not every polar set is countable. This will be demonstrated in **Section 3.5**, and more concrete examples of uncountable polar sets will be given in **Section 5.3**.

◇

3.3 Equilibrium Measures

In physics, a charge placed upon a conductor will distribute itself so as to minimize energy. In our context, this suggests lookging at probability measures μ on a compact set K which minimize $I(\mu)$ (that is to say, when one puts charge on a conductor, the charges spread out until they stop being able to lower the system's electrical energy any further). Not only are they of physical relevance, but they turn out to be mathematically very useful too.

Definition: Equilibrium Measure (of Compacts)

Let K be a compact subset of $\mathbb{C}$, and denote $\mathscr{P}(K)$ the collection of all Borel probability measures on K. If there exists $\nu \in \mathscr{P}(K)$ such that

$$I(\nu) = \sup_{\mu \in \mathscr{P}(K)} I(\mu).$$

Then ν is called an equilibrium measure for K.

Theorem 3.5: Compact Sets Have Equilibrium Measure

Every compact set K in $\mathbb{C}$ has an equilibrium measure.

We shall see later in **Section 3.7** that in fact this equilibrium measure is unique, provided that K is non-polar. (Of course if K is polar then every $\mu \in \mathscr{P}(K)$ is an equilibrium measure since they all satisfy $I(\mu) = -\infty$.)

To prove **Theorem 3.5**, we shall need the notion of weak*-convergence of probability measures. Some of the authors, for example Sydney and Port, call this mode of convergence the vague convergence.

Definition: Weak* Convergence

A sequence $\{\mu_n\}_{n \geq 1} \subset \mathscr{P}(X)$, where X is a compact metric space, is said to be weak* convergent to $\mu \in \mathscr{P}(X)$, denoted as $\mu_n \xrightarrow[n \to \infty]{\text{weak*}} \mu$, if

$$\int_X \varphi d\mu_n \to \int_X \varphi d\mu \ \forall \varphi \in C(X),$$

where $C(X)$ is the space of continuous functions $\varphi : X \to \mathbb{R}$ equipped with the usual sup norm.

In fact, every sequence $\{\mu_n\}_{n \geq 1} \subset \mathscr{P}(K)$ has a weak* convergent subsequence via a classical diagonal argument.

Lemma 3.6: Weak* Convergence Implies Energy Upper Bound

If $\mu_n \xrightarrow[n \to \infty]{\text{weak*}} \mu$ in $\mathscr{P}(K)$ then $\limsup_{n \to \infty} I(\mu_n) \leq I(\mu)$.

Proof:

Given continuous functions φ and ψ on K, the definition of weak* convergence implies that, as $n \to \infty$,

$$\int_K \int_K \varphi(z)\psi(w) d\mu_n(z) d\mu_n(w) \to \int_K \int_K \varphi(z)\psi(w) d\mu(z) d\mu(w).$$

Now using the Stone-Weierstrass theorem (see the proof in **Theorem 2.19**), one can show that every continuous function $\chi(z, w)$ on $K \times K$ can be uniformly approximated by finite sums of the form $\sum_{j=1}^{n} \varphi_j(z)\psi_j(w)$, where φ_j, ψ_j, are continuous functions on K. It follows that for every such χ,

$$\int_K \int_K \chi(z,w)d\mu_n(z)d\mu_n(w) \to \int_K \int_K \chi(z,w)d\mu(z)d\mu(w)$$

as $n \to \infty$. Applying this with $\chi(z,w) := \max\left(\log|z-w|, -m\right)$, where $m \geq 1$, we obtain

$$\begin{aligned}\limsup_{n\to\infty} I(\mu_n) := \limsup_{n\to\infty} \int_K \int_K \log|z-w|\,d\mu_n(z)d\mu_n(w) \\ \leq \limsup_{n\to\infty} \int_K \int_K \max\left(\log|z-w|, -m\right)d\mu_n(z)d\mu_n(w) \\ = \int_K \int_K \max\left(\log|z-w|, -m\right)d\mu(z)d\mu(w),\end{aligned}$$

where the first equality holds by the definition of energy, the inequality holds by taking the maximum inside the integral, and the second equality holds by the weak* convergence. The desired result now follows upon sending $m \to \infty$ and using Monotone convergence theorem.

□

Proof of **Theorem 3.5**:

Let $M := \sup_{\mu \in \mathscr{P}(K)} I(\mu)$, and choose a sequence $\{\mu_n\}_{n\geq 1} \subset \mathscr{P}(K)$ such that $I(\mu_n) \to M$ as $n \to \infty$. It can be shown that there exists a sequence $\{\mu_{n_k}\}_{k\geq 1}$ which is weak* convergent to some $\nu \in \mathscr{P}(K)$. Now by **Lemma 3.6**,

$$I(\nu) \geq \limsup_{k\to\infty} I(\mu_{n_k}) =: M,$$

then ν is an equilibrium measure for K by definition.

□

Physical intuition would tend to suggest that if ν is an equilibrium measure for K then p_ν should be constant on K (for otherwise charge would flow from one part of K to another part, disturbing being equilibrium). This idea is confirmed by the next theorem, and even serves to motivate the proof.

Theorem 3.7: Frostman's Theorem

Let K be a compact set in $\mathbb{C}$, and let ν be an equilibrium measure for K. Then

(a) $p_\nu \geq I(\mu)$ on $\mathbb{C}$.

(b) $p_\nu = I(\mu)$ on $K \backslash E$, where E is an F_σ polar subset of ∂K.

It can happen that the exceptional set E is non-empty. An example is demonstrated below.

Example 3.2: Exceptional Set in Frostman's Theorem Can Be Empty

Let K be a compact set of the form $\overline{\Delta} \cup E$, where $\overline{\Delta}$ is a closed disc, and E is a polar subset of $\mathbb{C}\backslash\overline{\Delta}$. Let ν be an equilibrium measure for K. Then $\nu(E) = 0$ and p_ν is harmonic on $\mathbb{C}\backslash\overline{\Delta}$. ◇

Proof of **Theorem 3.7**:

If $I(\nu) = -\infty$ (that is, K is polar) then the result is trivial. Without loss of generality we may assume that $I(\nu) > -\infty$. It suffices to prove that

(i) $\quad K_n := \left\{ z \in K : p_\nu(z) \geq I(\nu) + \frac{1}{n} \right\}$ is polar $\forall n \geq 1$.

(ii) $\quad L_n := \left\{ z \in \operatorname{supp}(\nu) : p_\nu(z) < I(\nu) - \frac{1}{n} \right\}$ is empty $\forall n \geq 1$.

Step I: It suffices to prove (i) and (ii)

Indeed, for (ii) then implies that $p_\nu \geq I(\nu)$ on $\operatorname{supp}(\nu)$, and so by the minimum principle **Theorem 3.3** we get $p_\nu \geq I(\nu)$ on $\mathbb{C}$, which gives aseertion (a).

On the other hand, if we put $E := \bigcup_{n\geq 1} K_n$, then (i) and **Corollary 3.4.2** together imply that E is an F_σ polar set. Since $p_\nu \leq I(\nu)$ on $K \backslash E$, this gives the first part of assertion (b). As for the second part in (b), observe that as E is polar, it must have Lebesgue measure zero by **Corollary 3.4.1**, so $p_\nu = I(\nu)$ Lebesgue almost everywhere on K, and hence by the weak identity principle **Theorem 2.24**, $p_\nu = I(\nu)$ everywhere on $\mathring{K}$. This concludes assertion (b).

Step II: (i) holds

We will prove (i) by contradiction. Suppose, if possible, that some K_n is non-polar. Choose $\mu \in \mathscr{P}(K_n)$ with $I(\mu) > -\infty$. Since

$$I(\mu) = \int p_\nu d\nu,$$

there exists $z_0 \in \operatorname{supp}(\nu)$ such that $p_\nu(z_0) \leq I(\nu)$. By the upper semicontinuity there exists $r > 0$ such that

$$p_\nu < I(\nu) + \frac{1}{2n} \text{ on } \overline{\Delta}(z_0, r).$$

In particular,

$$\overline{\Delta}(z_0, r) \cap K_n = \varnothing.$$

As $z_0 \in \operatorname{supp}(\nu)$, the number $a := \nu\big(\overline{\Delta}(z_0, r)\big)$ is strictly positive. Define a signed measure σ on K by

$$\sigma := \begin{cases} \mu, & \text{on } K_n \\ -\frac{\nu}{a}, & \text{on } \overline{\Delta}(z_0, r) \\ 0, & \text{otherwise} \end{cases}$$

Then for each $t \in (0,a)$, the measure

$$\nu_t := \nu + t\sigma$$

is positive, and therefore $\nu_t \in \mathscr{P}(K)$. Moreover, noting that

$$I(\mu) > -\infty \Rightarrow I\big(|\sigma|\big) > -\infty$$

by the defintion of σ, we have

$$
\begin{aligned}
I(\nu_t) - I(\nu) &= 2t \iint \log|z-w|\,d\nu(w)d\sigma(z) + t^2 \iint \log|z-w|\,d\sigma(w)d\sigma(z) \\
&= 2t \int p_\nu(z)d\sigma(z) + O(t^2) \\
&= 2t\left(\int_{K_n} p_\nu(z)d\mu(z) - \int_{\overline{\Delta}(z_0,r)} p_\nu(z)\frac{d\nu(z)}{a} + O(t) \right) \\
&\geq 2t\left[\left(I(\nu) + \frac{1}{n} \right) - \left(I(\nu) + \frac{1}{2n} \right) + O(t) \right],
\end{aligned}
$$

where the first equality holds by change of measure and definition of σ, the second equality holds by **Theorem 3.1** (ii), the third equality holds by the definition of σ and integration by parts, and in the last inequality, the blue term comes from the inequality $p_\nu \geq I(\nu) + \dfrac{1}{n}$ on K_n and the red term comes from the inequality $p_n < I(\nu) + \dfrac{1}{2n}$ on $\overline{\Delta}(z_0, r)$. Therefore $I(\nu_t) > I(\nu)$ provided that t is sufficiently small, contradicting the assumption that ν is an equilibrium measure. Hence each K_n is necessarily polar, proving (i).

Step III: (ii) holds

We shall prove (ii) by contradiction. Suppose, if possible, that some L_n is non-empty. Pick $z_1 \in L_n$, by the upper semicontinuity, there exists $s > 0$ such that

$$p_\nu < I(\nu) - \frac{1}{n} \text{ on } \overline{\Delta}(z_1, s).$$

As $z_1 \in \operatorname{supp}(\nu)$, the number $b := \nu\big(\overline{\Delta}(z_1, s)\big)$ is strictly positive. Now by (i) and **Corollary 3.4.1**, $\nu(K_n) = 0\ \forall n \geq 1$, and so

$$p_\nu < I(\nu)\ \nu\text{-almost everywhere on } K.$$

Hence

$$
\begin{aligned}
I(\nu) := \int_K p_\nu d\nu &= \int_{\overline{\Delta}(z_1,s)} p_\nu d\nu + \int_{K\setminus\overline{\Delta}(z_1,s)} p_\nu d\nu \\
&\leq \left(I(\nu) - \frac{1}{n} \right) \cdot b + I(\nu) \cdot (1-b) \\
&< I(\nu)
\end{aligned}
$$

where the first equality holds by definition, the second equality holds by integration by parts, the first inequality holds since $p_\nu < I(\nu) - \dfrac{1}{n}$ on $\overline{\Delta}(z_1, s)$ and $p_\nu < I(\nu)$ ν-almost everywhere on K, and the last inequality holds since $b > 0$. This display is obviously a contradiction. Hence each L_n is empty, giving (b).

□

Frostman's theorem **Theorem 3.7** is very important, serving many different purposes. Indeed, it is sometimes referred to as "fundamental theorem of potential theory" - a grandiose title but, as we shall see, one that is fully justified.

3.4 Upper Semicontinuous Regularization

We saw in **Theorem 2.12** that the limit of a decreasing sequence of subharmonic functions is subharmonic. At the same time, we remarked that the corresponding result for an increasing sequence was false, because the limit might not be upper semicontinuous. One way round this problem is to make the limit upper semicontinuous by regularizing it.

Definition: Upper Semicontinuous Regularization

Let X be a topological space, and let $u : X \to [-\infty, \infty)$ be a function which is locally bounded above on X. Its upper semicontinuous regularization, denoted as $u^* : X \to [-\infty, \infty)$, is defined by

$$u^*(x) := \limsup_{y \to x} u(y) = \inf_N \Big(\sup_{y \in N} u(y) \Big),$$

where $x \in X$ and the infimum is taken over all neighbourhoods N of x.

It is easily check that u^* is an u.s.c. function on X such that $u^* \geq u$, and also that it is the least such a function.

Returning to our problem about an increasing sequence of subharmonic functions, it is perhaps not too surprising to learn that provided the limit u is locally bounded above, its u.s.c. regularization u^* is u.s.c.. What is much less obvious is that u^* is very nearly equal to u. It can be proved that $u = u^*$ almost everywhere on X and in fact mucm more than this is true.

Theorem 3.8: Brelot-Cartan Theorem

Let $\mathscr{V}$ be a collection of subharmonic functions on an open subset U of $\mathbb{C}$, and suppose that the function $u := \sup_{v \in \mathscr{V}} v$ is locally bounded above on U. Then

(a) u^* is subharmonic on U.

(b) $u^* = u$ n.e. on U.

Part (b) says that $u^* = u$ everywhere on U outside some Borel polar set. Note however that the set $\{z : u^*(z) \neq u(z)\}$ itself may not be Borel, since $\mathscr{V}$ can be uncountable.

Proof of **Theorem 3.8**:

Step I: (a)

The upper semicontinuity of u^* is trivial, it left us to prove the submean inequality. Suppose that $\overline{\Delta}(w, \rho) \subset U$. Then for each $v \in \mathscr{V}$,

$$v(w) \leq \frac{1}{2\pi} \int_0^{2\pi} v(w + \rho e^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} u^*(w + \rho e^{i\theta}) d\theta,$$

where the first inequality holds by submean inequality of v and the second inequality holds by the definition of u.s.c. regularization. Taking the supremum over all $v \in \mathscr{V}$, one has

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u^*(w + \rho e^{i\theta}) d\theta. \tag{3.1}$$

Now choose $w_n \to w$ such that $\lim_{n \to \infty} u(w_n) = u^*(w)$. If n is sufficiently large, then $\overline{\Delta}(w_n, \rho) \subset U$, so **(3.1)** holds with w replaced by w_n throughout. Thus

$$u^*(w) \leq \frac{1}{2\pi}\int_0^{2\pi} \limsup_{n\to\infty} u^*(w_n + \rho e^{i\theta})d\theta \qquad \text{(Fatou's Lemma)}$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} u^*(w + \rho e^{i\theta})d\theta \qquad \text{(Definition of } u^*\text{)}$$

Thus u^* satisfies the submean inequality as desired.
Step II: (b)
We first consider the case when $\mathscr{V}$ is countable so that u is Borel-measurable.
Step II.1: (b) when $\mathscr{V}$ is countable
Now the set $\{z \in U : u(z) \neq u^*(z)\}$ can be written as a countable union of Borel sets of the form

$$E := \big\{z \in \Delta : u(z) \leq \beta \leq u^*(z)\big\},$$

where Δ is a disc such that $\overline{\Delta} \subset U$ and $\beta \in \mathbb{Q}$ is a rational number. Thus it suffices to show that each such a set E is polar. We shall do this by contradc-tion.
Suppose, if possible, that for some Δ and β the set E is non-polar. Then E cont-ains a compact non-polar subset K by **Remark 3.1** (iii). Let ν be an equilibr-ium measure for K, and define $q : \mathbb{C} \to [-\infty, \infty)$ by

$$q := C \cdot \big(p_\nu - I(\nu)\big) + \beta,$$

where C is a positive constant chosen sufficiently large so that

$$\inf_{\partial\Delta} q > \sup_{\partial\Delta} u.$$

$\big($such a choice is possible since by Frostman's theorem **Theorem 3.7** and the maximum principle **Theorem 2.5** (ii), $p_\nu > I(\nu)$ on the unbounded component of $\mathbb{C}\backslash K.\big)$
Then for each $v \in \mathscr{V}$, the function $v - q$ is subharmonic on $\Delta\backslash K$, and if $\zeta \in \partial(\Delta\backslash K)$ then

$$\limsup_{z\to\zeta}(v - q)(z) \leq \begin{cases} u(\zeta) - \inf_{\partial\Delta} q, & \zeta \in \partial\Delta \\ u(\zeta) - \beta, & \zeta \in \partial K \end{cases} \leq 0.$$

Hence by the maximum principle **Theorem 2.5** (ii), $v \leq q$ on $\Delta\backslash K$. Therefore $u \leq q$ on $\Delta\backslash K$. Moreover, $u \leq \beta \leq q$ on K, in fact $u \leq q$ on the whole of Δ, and hence $u^* \leq q$ on Δ. This implies that $q > \beta$ on K, or in other words $p_\nu > I(\nu)$ on K, which contradicts **Theorem 3.7** (b). Thus E is polar, as desired.
Step II.2: (b) when $\mathscr{V}$ is not necessarily countable
We now turn to the case when $\mathscr{V}$ is uncountable. Choose a countable base $\{D_j\}_{j\geq 1}$ of relatively compact open subsets of U. For each pair $j, k \geq 1$, there exists $v_{jk} \in \mathscr{V}$ such that

$$\sup_{D_j} v_{jk} > \sup_{D_j} u^* - \frac{1}{k}.$$

If we set $u_0 := \sup_{j,k\geq 1} v_{jk}$, then $u_0 \leq u$ and $u_0^* = u^*$. By the countable base

$u_0^* = u_0$ nearly everywhere on U. Hence it follows that $u^* = u$ n.e. on U.

□

Of course, the Brelot-Cartan theroem **Theorem 3.8** applies in particular to limits of increasing sequences. There is also a corresponding result for more general sequences.

Theorem 3.9: Brelot-Cartan Theorem Applied to General Sequences

Let $\{u_n\}_{n\geq 1}$ be a sequence of subharmonic functions on an open set U, and suppose that $\sup_n u_n$ is locally bounded above on U. If $u := \limsup_{n\to\infty} u_n$. Then

(a) u^* is subharmonic on U.

(b) $u^* = u$ n.e. on U.

(c) If $\varphi : U \to \mathbb{R}$ is continuous and $\varphi \geq u$ then $\max(u_n, \varphi) \to \varphi$ locally uniformly on U as $n \to \infty$.

Proof:

Step I: (a)

If $\overline{\Delta}(w,\rho) \subset U$ then for each $n \geq 1$,

$$u_n(w) \leq \frac{1}{2\pi}\int_0^{2\pi} u_n(w+\rho e^{i\theta})d\theta.$$

Taking $\limsup_{n\to\infty}$ of both sides and using Fatou's lemma give

$$u(w) \leq \frac{1}{2\pi}\int_0^{2\pi} u(w+\rho e^{i\theta})d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} u^*(w+\rho e^{i\theta})d\theta.$$

The same argument as used in proving **Theorem 3.8** (a) now shows that u^* is subharmonic on U.

Step II: (b)

For each $n \geq 1$ denote $v_n := \sup_{\geq n} u_m$. Then $v_n \downarrow u$, and $v_n^* \downarrow v$, say, where $v \geq u^* \geq u$. Now by **Theorem 3.8** (b), $v_n^* = v_n$ n.e. for each $n \geq 1$ therefore $v = u$ n.e. and hence $u^* = u$ n.e. on U.

Step III: (c)

Since $\varphi \leq \max(u_n, \varphi) \leq \max(v_n^*, \varphi)$ for each $n \geq 1$, it suffices to prove that $\max(v_n^*, \varphi) \to \varphi$ uniformly on compacts. As $\{v_n^*\}_{n\geq 1}$ is a decreasing sequence of u.s.c. functions, by **Dini's theorem**[3] this will be true provided that $\lim_{n\to\infty} v_n^* \leq \varphi$, thus it left us to prove this inequality.

Step III.1: $\lim_{n\to\infty} v_n^* \leq \varphi$.

By **Theorem 3.8** (a), each v_n^* is subharmonic on U, and since $v_n^* \downarrow v$, it follows from **Theorem 2.12** that v is subharmonic on U. Moreover, by (a) we just proved, u^* is subharmonic on U and by (b) we just proved $b = u^*$ n.e. and

---

[3] **Dini's Theorem**: Let K be a compact metric space. Let $f : K \to \mathbb{R}$ be a continuous function and $f_n : K \to \mathbb{R}$, $n \geq 1$, be a sequence of continuous functions. If $\{f_n\}_{n\geq 1}$ converges pointwisely to f and if $f(x) \geq f_{n+1}(x)$ $\forall x \in K$ $\forall n \geq 1$ then $f_n \to f$ uniformly.

thus a.e. on U. Hence by the weak identity principle **Theorem 2.24** $v = u^*$ everywhere on U and so

$$\lim_{n\to\infty} v_n^* = v = u^* \le \varphi^* = \varphi,$$

as desired.

□

3.5 Minus Infimum Sets

Earlier we have proved in **Corollary 2.15.2** that if u is subharmonic on a domain and $u \not\equiv -\infty$ then the set where $u = -\infty$ has Lebesgue measure zero. We are now in a position to prove a much stronger result. Recall that a G_δ set is of the form of countable intersection of open sets.

Theorem 3.10: Subharmonic Function Is Minus Infinity On G_δ Polar Set

Let u be a subharmonic function on a domain D in $\mathbb{C}$ such that $u \not\equiv -\infty$. Then $E := \{z \in D : u(z) = -\infty\}$ is a G_δ polar set.

Proof:

Since $E := \bigcap_{n\ge 1} \{z : u(z) < -n\}$ is clearly a G_δ set. It left us to show that it is polar. Denote $v := \lim_{n\to\infty} \dfrac{u}{n}$ so that

$$v(z) = \begin{cases} 0, & z \in D\backslash E \\ -\infty, & z \in E \end{cases}$$

Now by **Theorem 3.9** (a) v^* is subharmonic on D, and since it evidently attains a maximum value 0 there, it follows that $v^* \equiv 0$ on D by **Theorem 2.5** (i). Moreover, by **Theorem 3.9** (b), $v^* = v$ n.e. on D. Therefore $v = 0$ n.e. on D and E is indeed polar by **Theorem 3.9** (b) and the definition of n.e. property.

□

This result allows us to demonstrate the existence of uncountable polar sets. For example, the set E occuring in the proof of **Theorem 2.16** (b) is uncountable and by **Theorem 3.10** it is polar. More concrete examples will appear in **Section 5.3**.

Theorem 3.10 is sharp in the sense that every G_δ polar set arises as the set where some subharmonic function $u = -\infty$. This converse, Deny's theorem, is too hard for us to prove here as the proof relies on the concept of condenser measure; instead we content ourselves with the following result which, though weaker, is good enough for most purposes.

Theorem 3.11: F_σ Polar Set Decomposition for Subharmonic Functions

Let E be an F_σ polar set, and let F be an F_σ set such that $E \cap F = \varnothing$. Then there exists a subharmonic function $u : \mathbb{C} \to [-\infty, \infty)$ such that

(i) $u = -\infty$ on E.

(ii) $u > -\infty$ on F.

We shall prove this via a lemma which is of interest in its own right.

Lemma 3.12: Existence of Borel Probability Measure Charging Compact Polar Sets

Let E be a compact polar set, and let F be a compact set disjoint from E. Then there exists a Borel probability measure μ on $\mathbb{C}$ with compact support such that

(i) $E = \{z \in \mathbb{C} : p_\mu(z) = -\infty\}$

(ii) $\operatorname{supp}(\mu) \cap F = \varnothing$.

Proof:

Step I: Assertion (ii)

Let $\{K_n\}_{n\geq 1}$ be a sequence of compact sets, with $K_{n+1} \subset \mathring{K}_n$ for all $n \geq 1$ such that

$$\bigcap_{n\geq 1} K_n = E \text{ and } K_1 \cap F = \varnothing.$$

For each n, let ν_n be an equilibrium measure for K_n. Note that $I(\nu_n) > -\infty$ since $\mathring{K}_n \neq \varnothing$ by Frostman's theorem **Theorem 3.7** (b).

Now $\nu_n \in \mathscr{P}(K_1)$ for all $n \geq 1$, so by a diagonal argument there exists a subsequence of $\{\nu_n\}_{n\geq 1}$ that is weak*-convergent to some $\nu \in \mathscr{P}(K_1)$. In fact, since $\operatorname{supp}(\nu_n) \subset K_n$ for all $n \geq 1$, we must have $\operatorname{supp}(\nu) \subset E$. As E is polar, it follows that $I(\nu) = -\infty$. Hence by **Lemma 3.6** $I(\nu_n) \to -\infty$ as $n \to \infty$, and so, replacing $\{\nu_n\}_{n\geq 1}$ by a further subsequence, we can suppose that

$$I(\nu_n) < -2^n \text{ for all } n \geq 1.$$

Put

$$\mu := \sum_{n=1}^{\infty} 2^{-n}\nu_n.$$

Then $\mu \in \mathscr{P}(K_1)$ so $\operatorname{supp}(\mu) \cap F = \varnothing$. Thus the measure we constructed satisfies assertion (ii).

Step II: Assertion (i)

First suppose that $z \in E$. Then $z \in \mathring{K}_n$ for each $n \geq 1$, so by **Theorem 3.7** (b),

$$p_{\nu_n}(z) = I(\nu_n) < -2^n.$$

Hence,

$$p_\mu(z) = \sum_{n=1}^{\infty} 2^{-n} p_{\nu_n}(z) \leq \sum_{n=1}^{\infty} 2^{-n}(-2^n) = -\infty,$$

where the first equality holds by the definition of μ and the first equality holds since $I(\nu_n) < -2^n$ for all $n \geq 1$.

Now suppose that $z \notin E$. Choose n_0 such that $z \notin K_{n_0}$ and put $\delta := \operatorname{dist}(z, K_{n_0})$ then for all $n \geq n_0$,

$$p_{\nu_n}(z) \geq \int \log \delta \, d\nu_n = \log \delta,$$

and also by **Theorem 3.7** (a), $p_{\nu_n}(z) \geq I(\nu_n) > -\infty$ for every $n \geq 1$. Hence

$$p_\mu(z) = \sum_{n=1}^{\infty} 2^{-n} p_{\nu_n}(z) \geq \sum_{n=1}^{n_0-1} 2^{-n} I(\nu_n) + \sum_{n=n_0}^{\infty} 2^{-n}\log\delta > -\infty,$$

where the first equality holds by the definition of μ and the first inequality by summing by parts. Thus $E = \{z \in \mathbb{C} : p_\mu(z) = -\infty\}$ as desired.

□

We remark in passing that it is not clear from the proof above whether μ can be chosen so that $\text{supp}(\mu) \subset E$. The fact that it can (Evan's theorem) will be proved in **Section 5.5**.

Proof of **Theorem 3.11**:

Denote $E := \bigcup_{n\geq 1} E_n$ and $F := \bigcup_{n\geq 1} F_n$, where $\{E_n\}_{n\geq 1}$ and $\{F_n\}_{n\geq 1}$ are increasing sequences of compact sets. By **Lemma 3.12**, for each $n \geq 1$, there exists a Borel probability measure μ_n with compact support such that

$$E_n = \left\{z \in \mathbb{C} : p_{\mu_n}(z) = -\infty\right\} \text{ and } \text{supp}(\mu_n) \cap F_n = \varnothing.$$

Then p_{μ_n} is bounded above on $\Delta(0,n)$ and below on F_n, so we can choose constants $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$ for each n such that $u_n := \alpha_n p_{\mu_n} + \beta_n$ satisfying

$$\sup_{\Delta(0,n)} u_n < 0 \text{ and } \inf_{F_n} u_n > -2^{-n}.$$

Denote $u := \sum_{n=1}^{\infty} u_n$. Then on any bounded set, the sequence of partial sums is eventually decreasing and so by **Theorem 2.12** u is subharmonic on $\mathbb{C}$. Moreover if $z \in E$ then $u_n(z) = -\infty$ for some n and so $u(z) = -\infty$. This proves the first assertion.

Finally, if $z \in F$ then $u_n(z) > -\infty$ for each $n \geq 1$ and $u_n(z) \geq -2^{-n}$ for all sufficiently large n, thus $u(z) > -\infty$ on F, proving assertion (ii).

□

We conclude by recording an important special case of **Theorem 3.11**.

Corollary 3.11.1: Characterization of Closed Polar Set via Subharmonic Functions

If E is a closed polar subset of $\mathbb{C}$. Then there exists a subharmonic function u on $\mathbb{C}$ such that $E = \{z \in \mathbb{C} : u(z) = -\infty\}$.

Proof:

Applying **Theorem 3.11** with $F := \mathbb{C} \backslash E$. □

3.6 Removable Singularities

In each of the last three sections we have encountered theorems asserting that certain exceptional sets are polar. It is thus of special interest to determine in what ways polar sets are "negligible". The key to this is the following removable singularity thoerem.

Theorem 3.13: Removable Singularity Theorem for Subharmonicity

Let U be an open subset of $\mathbb{C}$, let E be a closed polar set, and let u be a subharmonic function on $U \backslash E$. Suppose that each point of $U \cap E$ has a neighbourhood N such that u is bounded above on $N \backslash E$. Then u has a **unique** subharmonic extension to the whole of U.

Proof:

Uniqueness follows from the weak identity principle **Theorem 2.24** since E has (Lebesgue) measure zero by **Corollary 3.4.1**.

To construct the extension, we define u on $U \cap E$ by

$$u(w) := \limsup_{z \to w, z \in U \setminus E} u(z) \text{ where } w \in U \cap E.$$

The boundedness assumption ensures that $u < \infty$ everywhere, and so u is u.s.c. on U by **Theorem 2.2**. To check that u is subharmonic, we shall use (c) $\Leftrightarrow$ (a) in **Theorem 2.9**. Let D be a relatively compact subdomain of U, and let h be a harmonic function on D such that

$$\limsup_{z \to \zeta} (u - h)(z) \leq 0 \ \forall \zeta \in \partial D.$$

We need to show that $u \leq h$ on D.

Now by **Corollary 3.11.1** there exists a subharmonic function v on $\mathbb{C}$ such that $E = \{z : v(z) = -\infty\}$. For each $\varepsilon > 0$, the function $u - h + \varepsilon v$ is certainly subharmonic on $D \setminus E$ by **Theorem 2.4** (ii), and equals $-\infty$ on E, **Remark 2.1** (ii) tells us that $u - h + \varepsilon v$ is subharmonic on the whole of D. Therefore by the maximum principle **Theorem 2.5**,

$$u - h + \varepsilon \leq \sup_{\partial D} (\varepsilon v) \text{ on } D.$$

Sending $\varepsilon \downarrow 0$ we deduce that $u \leq h$ on $D \setminus E$. From the way that u is defined on $D \cap E$ it follows that $u \leq h$ on $D \cap E$ too. Therefore $u \leq h$ on D as desired.

□

Corollary 3.13.1: Removable Singularity Theorem for Harmonic Functions

Let U be an open subset of $\mathbb{C}$, let E be a closed polar set, and let h be a harmonic function on $U \setminus E$. Suppose that each point of $U \cap E$ has a neighbourhood N such that h is bounded on $N \setminus E$. Then h has a **unique** harmonic extension to the whole of U.

Proof:

The uniqueness is clear by **Theorem 1.3**. As for the existence, **Theorem 3.13** applying to $\pm h$ gives functions u and v which are subharmonic on U, and which agree respectively to h and $-h$ on $U \setminus E$. Then $u + v$ is subharmonic on U and $u + h = 0$ on $U \setminus E$, so by the weak identity principle **Theorem 2.24** $u + v = 0$ on the whole of U. Therefore u is superharmonic on U and as well as being subharmonic on U. Thus by **Remark 2.1** (iii) u is the desired harmonic extension of h.

□

The removable singularity theorem **Theorem 3.13** can be used to demonstrate a further sense in which polar sets are small.

Theorem 3.14: Removing Closed Polar Set Does Not Affect Connectivity

Let D be a domain in $\mathbb{C}$ and let E be a closed polar set. Then $D \setminus E$ is still connected.

Proof:

Suppose that $D \setminus E = A \cup B$, where A and B are disjoint non-empty open sets. Define $u : D \setminus E \to [-\infty, \infty)$ by

$$u = \begin{cases} 0, & \text{on } A \\ -\infty, & \text{on } B \end{cases}$$

By **Theorem 3.13**, u has a subharmonic extension to the whole of D. It then follows from **Corollary 2.15.2** that if $B \neq \varnothing$ then $u \equiv -\infty$ on D and so $A = \varnothing$, which contradicts our assuption that both A and B are non-empty. Thus $D \backslash E$ is connected.

□

A purely topological argument now yields the following argument.

Corollary 3.14.1: Closed Polar Set Is Totally Disconnected

Every closed polar set E is totally disconnected.

Proof:

We need to show that if $w \in E$, then its component in E is just $\{w\}$. Without loss of generality, we may assume that $w = 0$. Let $\varepsilon > 0$ and set

$$\Delta := \Delta(0,\varepsilon),\ \Delta^+ := \Delta\backslash[0,\varepsilon),\text{ and }\Delta^- := \Delta\backslash(-\varepsilon,0].$$

Choose $w_1, w_2 \in \Delta\backslash E$ with $\mathrm{Im}(w_1) > 0$ and $\mathrm{Im}(w_2) < 0$. By **Theorem 3.14** both $\Delta^+\backslash E$ and $\Delta^-\backslash E$ are connected, so we can join w_1 to w_2 by a path γ^+ in $\Delta^+\backslash E$, and w_2 to w_1 by a path γ^- in $\Delta^-\backslash E$. Then

$$\gamma := \gamma^+ \cup \gamma^-$$

is a closed path in $\Delta\backslash E$ which winds once aroung 0. It must therefore also wind once around every point in the same component of E as 0. Hence this component lies inside the disc $\Delta(0,\varepsilon)$, since ε is arbitrary, sending $\varepsilon \downarrow 0$ gives the component to be $\{0\}$, as desired.

□

Here is a beautiful application of these ideas to complex analysis.

Theorem 3.15: Rado-Stout Theorem

Let D be a domain of $\mathbb{C}$, let E be a closed polar set, and let $f : D \to \mathbb{C}$ be a continuous function which is holomorphic on $D\backslash f^{-1}(E)$. Then f is holomorphic on the whole of D.

Proof:

If $f(D) \subset E$, then, as $f(D)$ is connected and E by **Corollary 3.14.1** is totally disconnected, it follows that f is constant, in which case the result is trivial. Without loss of generality, we may assume that $f(D) \not\subset E$. **Corollary 3.11.1** tells us that there exists a subharmonic function u on $\mathbb{C}$ such that

$$E = \{z : u(z) = -\infty\}.$$

Then $u \circ f$ is subharmonic on $D\backslash f^{-1}(E)$ by **Theorem 2.23**, and equals $-\infty$ on $f^{-1}(E)$, so it is subharmonic on the whole of D by **Theorem 3.13**. Now

$$u \circ f \not\equiv -\infty \text{ on } D$$

thus by **Theorem 3.10** $f^{-1}(E)$ is a G_δ polar set. Using **Corollary 3.13.1** in conjunction with **Theorem 1.1** to $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ yields the fact that they are harmonic in D, and hence that $f \in C^\infty(D)$ by **Corollary 1.1.2**. Since f satisfies the Cauchy-Riemann equations on $D\backslash E$, by continuity it must also do so on E, and hence it is holomorphic on D.

□

Corollary 3.15.1: Preimage of Polar Set under Non-Constant Holomorphy Is Polar

Let D be a domain in $\mathbb{C}$, let f be a non-constant holomorphic function on D, and let E be a polar set. Then $f^{-1}(E)$ is also polar.

Proof:

If E is closed in $\mathbb{C}$, then this is an immediate consequence of the proof for **Theorem 3.15**. For the general case, it suffices to show that every compact subset of $f^{-1}(E)$ is polar, and this is easily deduced from the case already proved.

□

Note that the non-constant assumption is necessary in the above result.

Remark 3.3: Polarity Is Invariant under Conformal Mapping

In particular, it follows that the property of being a polar set is invariant under conformal mapping. ◇

Thus we can extend the notion of polarity to $\mathbb{C}^\infty$, by declaring

Definition: Polar Set in $\mathbb{C}^\infty$

A set E in $\mathbb{C}^\infty$ is polar if $\varphi(E)$ is polar for some conformal mapping φ of a neighbourhood of E into $\mathbb{C}$. It is easy to see that in fact E is polar in this sense if and only if $E \backslash \{\infty\}$ is polar in the standard case.

Both the Liouville theorem **Corollary 2.6.2** and the maximum principle **Theorem 2.5** have extended versions, which will later be proved to be very important.

Theorem 3.16: Extended Liouville Theorem for Subharmonic Functions

Let E be a closed polar subset of $\mathbb{C}$, and let u be a subharmonic function on $\mathbb{C} \backslash E$ which is bounded above. Then u is constant.

Proof:

By **Theorem 3.13**, u extends to be subharmonic on the whole of $\mathbb{C}$. Moreover, if $M := \sup_{\mathbb{C} \backslash E} u$ then $\max(u, M) = M$ on $\mathbb{C} \backslash E$ and hence everywhere on $\mathbb{C}$ by **Theorem 2.24**. Therefore u is bounded above on $\mathbb{C}$, and by **Corollary 2.6.2** we conclude that u is constant.

□

Remark 3.4: Converse of Extended Liouville Theorem Also Holds

Let E be a closed subset of $\mathbb{C}$ with the property that every subharmonic function bounded above on $\mathbb{C} \backslash E$ is constant, then E is polar. ◇

Corollary 3.16.1: Extended Liouville Theorem for Holomorphic Functions

Let E be a closed polar subset of $\mathbb{C}$, and let f be a holomorphic function on $\mathbb{C} \backslash E$ such that $\mathbb{C} \backslash f(\mathbb{C} \backslash E)$ is non-polar. Then f is constant.

Proof:

Choose a compact non-polar set K such that $f(\mathbb{C} \backslash K) \subset \mathbb{C} \backslash K$, and let ν be an equilibrium measure for K. Then p_ν is harmonic and bounded below on $\mathbb{C} \backslash K$ by **Theorem 3.7** (a), so $-p_\nu \circ f$ is harmonic and bounded above on $\mathbb{C} \backslash E$. Hence by **Theorem 3.16** $-p_\nu \circ f$ is constant. By **Theorem 3.1** (ii),

$$\lim_{z \to \infty} p_\nu(z) = \infty,$$

this implies that f is bounded on $\mathbb{C} \backslash E$. Applying **Theorem 3.16** once more, this time to $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$, we deduce that f is constant.

□

Theorem 3.17: Extended Maximum Principle for Subharmonic Functions

Let D be a domain in $\mathbb{C}$, and let u be a subharmonic function on D which is bounded above.

(a) If ∂D is polar then u is constant.

(b) If ∂D is non-polar and $\limsup_{z\to\zeta} u(z) \leq 0$ for n.e. $\zeta \in \partial D$ then $u \leq 0$ on D.

Proof:

Step I: (a)

Denote $E := \partial D \backslash \{\infty\}$. Then by **Remark 3.3** E is a closed polar subset of $\mathbb{C}$, so by **Theorem 3.14**, $\mathbb{C}\backslash E$ is connected. Since D is a component of $\mathbb{C}\backslash E$, it follows that $D = \mathbb{C}\backslash E$. Now assertion (a) follows from **Theorem 3.16**.

Step II: (b)

Given $\varepsilon > 0$, define

$$E_\varepsilon := \left\{\zeta \in \partial D\backslash\{\infty\} : \limsup_{z\to\zeta} u(z) \geq \varepsilon\right\}.$$

Then E_ε is a closed polar subset of $\mathbb{C}$. Define v on $\mathbb{C}\backslash E_\varepsilon$ by

$$v := \begin{cases} \max(u, \varepsilon), & \text{on } D \\ \varepsilon, & \text{on } \mathbb{C}\backslash(D \cup E_\varepsilon) \end{cases}$$

By gluing theorem **Theorem 2.11** v is subharmonic on $\mathbb{C}\backslash E_\varepsilon$, and it is clearly bounded above there, so by **Theorem 3.16** it is constant. Since $v = \varepsilon$ on $\partial D\backslash(E_\varepsilon \cup \{\infty\})$, which is non-empty, then by **Theorem 2.24** $v \equiv \varepsilon$. Hence $u \leq \varepsilon$ on D. Finally, since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ give assertion (b).

□

3.7 The Generalized Laplacian

By **Theorem 2.10**, a C^2 subharmonic function u satisfies that $\Delta u \geq 0$. In this section we shall develop an appropriate generalization of this fact to arbitrary subharmonic functions. This turns out to be an important idea, with many applications.

Definition: C_c^∞ Space

Let D be a domain in $\mathbb{C}$. The space $C_c^\infty(D)$ is defined to be the space of all C^∞-functions $\varphi : D \to \mathbb{R}$ whose support $\text{supp}(\varphi)$ is a compact subset of D.

If u is a C^2 subharmonic function on D, then, identifying Δu with positive measure $\Delta u\, dA$, it follows from Green's theorem that

$$\int_D \varphi \Delta u = \int_D u \Delta \varphi\, dA,\ \varphi \in C_c^\infty(D). \tag{3.2}$$

Now if u is an arbitrary subharmonic function on D with $u \not\equiv -\infty$, then by **Theorem 2.15**, u is locally integrable, and so the right hand side of **(3.2)** makes sense. We therefore use it to define the left hand side of **(3.2)**.

Definition: Radon Measure

A Borel measure μ on a topological space X is called a Radon measure if $\mu(K) < \infty$ for each compact subset of X.

Remark 3.5: Radon Measure and Riesz Representation

Each Radon Measure μ on the topological space X gives rise to a linear functional Λ on $C_c(X)$ via

$$\Lambda(\varphi) = \int_X \varphi d\mu, \varphi \in C_c(X).$$

This linear functional is positive in the sense that $\Lambda(\varphi) \geq 0 \ \forall \varphi \geq 0$. For certain spaces X there is an important converse called Riesz representation theorem:

Let X be a metric space possessing a compact exhaustion (increasing sequence of compact subsets $\{K_n\}_{n\geq 1}$ such that $K_n \subset \mathring{K}_{n+1}$, and their union is the whole space X). If Λ is a positive linear functional on $C_c(X)$ then there exists a **unique** Radon measure μ on X such that

$$\Lambda(\varphi) = \int_X \varphi d\mu \ \forall \varphi \in C_c(X). \qquad \diamond$$

Definition: Generalized Laplacian

Let u be a subharmonic function on a domain D in $\mathbb{C}$ with $u \not\equiv -\infty$. The generalized Laplacian of u is the Radon measure Δu on D such that (3.2) holds.

To justify this definition, we need to prove the following theorem.

Theorem 3.18: Existence and Uniqueness of the Generalized Laplacian

The generalized Laplacian for a subharmonic function u on a domain D in $\mathbb{C}$ such that $u \not\equiv -\infty$ exists and is unique.

The proof relies on a simple approximation lemma. We write $C_c(D)$ for the space of all continuous functions $\varphi : D \to \mathbb{R}$ whose support supp(φ) is a compact subset of D.

Definition: Sup Norm on $C_c(D)$

We define the sup-norm on $C_c(D)$ by

$$\|\varphi\|_\infty := \sup_D |\varphi|, \varphi \in C_c(D).$$

Lemma 3.19: Approximation Lemma for Element in $C_c(D)$

Let $\varphi \in C_c(D)$, and let U be a relatively compact open subset of D such that supp$(\varphi) \subset U$. Then

(i) There exists $\{\varphi_n\}_{n\geq 1} \subset C_c^\infty(D)$ such that supp$(\varphi_n) \subset U$ for all $n \geq 1$ and $\|\varphi_n - \varphi\|_\infty \to 0$ as $n \to \infty$.

(ii) If in addition that $\varphi \geq 0$, then $\{\varphi_n\}_{n\geq 1}$ in (i) can be chosen so that $\varphi_n \geq 0$ for all $n \geq 1$ as well.

Proof:

Extend φ to the whole of $\mathbb{C}$ by defining $\varphi \equiv 0$ on $\mathbb{C}\backslash D$. Then if $\{\chi_r\}_{r\geq 0}$ are the functions used in **Theorem 2.22**, we have, by **Theorem 2.22** (i) that

$$\varphi * \chi_r \in C^\infty(\mathbb{C}) \ \forall r > 0.$$

Moreover,

$$\text{supp}(\varphi * \chi_r) \subset \left\{z \in \mathbb{C} : \text{dist}\big(z, \text{supp}(\varphi)\big) \leq r\right\},$$

so that supp$(\varphi * \chi_r) \subset U$ provided that r is sufficiently small. Finally,

$$\|\varphi * \chi_r - \varphi\|_\infty := \sup_{z\in\mathbb{C}} \left| \int_{\Delta(0,r)} \big(\varphi(z-w) - \varphi(z)\big)\chi_r(w)dA(w) \right|$$
$$\leq \sup_{z\in\mathbb{C}, |w|<r} \left| \varphi(z-w) - \varphi(z) \right|$$
where the second equality holds by **Theorem 2.22** (d) and (e). Moreover, this $\|\varphi * \chi_r - \varphi\|_\infty \to 0$ as $r \to 0$ because φ is uniformly continuous on $\mathbb{C}$. Hence we may take
$$\varphi_n := \varphi * \chi_{\delta/n} \text{ for each } n \geq 1,$$
where $\delta > 0$ is chosen sufficiently small. Moreover, with this definition, it is clear that if $\varphi \geq 0$ then $\varphi_n \geq 0$ for each $n \geq 1$ as desired.

□

Proof of **Theorem 3.18**:

Step I: Uniqueness

We begin with the uniqueness. Suppose that μ_1 and μ_2 are two Radon measures on D such that
$$\int_D \varphi d\mu_1 = \int_D \varphi d\mu_2, \varphi \in C_c^\infty(D).$$
Then by **Lemma 3.19**, this equation also holds $\forall \varphi \in C_c(D)$. By the uniqueness part of the Riesz representation theorem in **Remark 3.5** we conclude that $\mu_1 = \mu_2$.

Step II: Existence

Now we turn to the question of existence. Define $\Lambda : C_c^\infty(D) \to \mathbb{R}$ by
$$\Lambda(\varphi) := \int_D u\Delta\varphi dA, \varphi \in C_c^\infty(D).$$
Clearly Λ is a linear functional, and our first step is to show that this linear functional is positive, that is

Step II.1: $\varphi \geq 0 \Rightarrow \Lambda(\varphi) \geq 0$

Suppose then that $\varphi \in C_c^\infty(D)$ with $\varphi \geq 0$. Choose a relatively compact open subset U of D such that $\mathrm{supp}(\varphi) \subset U$. By **Corollary 2.22.1** there exist C^∞ subharmonic functions $\{U_n\}_{n\geq 1}$ on U such that $u_n \downarrow u$ there. By **Theorem 2.10** $\Delta u_n \geq 0$ for each $n \geq 1$, and so using Green's theorem it follows that
$$\int_D u_n \Delta\varphi dA = \int_D \varphi\Delta u_n dA \geq 0.$$
Sending $n \to \infty$ and applying Lebesgue's dominated convergence theorem we conclude that
$$\int_D u\Delta\varphi dA \geq 0,$$
in other words $\Lambda(\varphi) \geq 0$. Thus Λ is indeed positive.

Step II.2: Boundedness of Λ

Next, we show that, given a relatively compact open subset V of D, there exists a constant C such that

$$|\Lambda(\varphi)| \leq C \cdot \|\varphi\|_\infty, \varphi \in C_c^\infty(D), \operatorname{supp}(\varphi) \subset V. \tag{3.3}$$

To do this, take $\psi \in C_c^\infty(D)$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on V. Then given $\varphi \in C_c^\infty(D)$ with $\operatorname{supp}(\varphi) \subset V$, we have

$$-\|\varphi\|_\infty \psi \leq \varphi \leq \|\varphi\|_\infty \psi \text{ on } D,$$

so, since Λ is positive, it follows that

$$-\|\varphi\|_\infty \Lambda(\psi) \leq \Lambda(\varphi) \leq \|\varphi\|_\infty \Lambda(\psi).$$

Thus (3.3) holds with $C = \Lambda(\psi)$.

Step II.3: Using Riesz representation theorem to conclude the existence

Now combining (3.3) and Lemma 3.19, we deduce that Λ extends to a positive linear functional on the whole of $C_c(D)$. Therefore, by the existence part of the Riesz representation theorem in Remark 3.5, there exists a unique Radon measure μ on D such that

$$\Lambda(\varphi) = \int_D \varphi \, d\mu, \varphi \in C_c(D).$$

In particular,

$$\int_D u \Delta \varphi \, dA = \int_D \varphi \, d\mu, \varphi \in C_c^\infty(D),$$

which completes the proof of the existence.

□

The reader familiar with the distribution theory will recognize the generalized Laplacian as being just the Laplacian interpreted in the distributional sense. Although no previous knowledge of distribution theory is assumed in this book, it is helpful in understanding several of the results. For example:

Remark 3.6: Interpreting Potential via Distribution Theory Perspective

The potential p_μ can be regarded as the distributional convolution of the measure μ with the locally integrable function $|\log z|$, and the latter is just (a multiple of) the fundamental solution of the Laplacian. One might therefore expect Δp_μ to be the convolution of μ with a delta-function, that is, a multiple of μ itself. That this is indeed the case is confirmed by the next result. ⋄

Theorem 3.20: Poisson's Equation in Complex Plane

Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Then

$$\Delta p_\mu = 2\pi\mu.$$

Proof:

Given $\varphi \in C_c^\infty(\mathbb{C})$, we have

$$\begin{aligned}\int_{\mathbb{C}} p_\mu \Delta\varphi \, dA &= \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \log|z-w| \, d\mu(w) \right) \Delta\varphi(z) \, dA(z) \\ &= \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \log|z-w| \, \Delta\varphi(z) \, dA(z) \right) d\mu(w)\end{aligned}$$

where the first equality holds by the definition of the potential p_μ and the second equality holds by Fubini's theorem (The use of Fubini's theorem is justified, because $\Delta\varphi$ is bounded with compact support and $\log|z|$ is locally integrable with respect to the Lebesgue measure on $\mathbb{C}$). Now if $w \in \mathbb{C}$, then

$$\begin{aligned}
&\int_{\mathbb{C}} \log|z-w|\,\Delta\varphi(z)dA(z) \\
&= \lim_{\varepsilon\to 0}\int_{|z-w|>\varepsilon} \log|z-w|\,\Delta\varphi(z)dA(z) \\
&= \lim_{\varepsilon\to 0}\int_0^{2\pi} \left(\varphi(w+re^{it}) - r\log r\frac{\partial\varphi}{\partial r}(w+re^{it})\right)\Big|_{r=\varepsilon} dt \\
&= 2\pi\varphi(w),
\end{aligned}$$

where the first equality holds since $\{z : |z-w| > \varepsilon\} \uparrow \mathbb{C}$ as $\varepsilon \downarrow 0$, the second equality holds by Green's theorem. Hence

$$\int_{\mathbb{C}} p_\mu \Delta\varphi dA = \int_{\mathbb{C}} 2\pi\varphi d\mu,\ \varphi \in C_c^\infty(\mathbb{C}),$$

as desired.

□

Corollary 3.20.1: Local Uniqueness of Log Potential Up to Hamonic Translation

Let μ_1 and μ_2 be finite Borel measures on $\mathbb{C}$ with compact support. If

$$p_{\mu_1} = p_{\mu_2} + h$$

on an open set U, where h is harmonic on U, then

$$\mu_1|_U = \mu_2|_U.$$

Proof:

Since h is harmonic on U, $\Delta h = 0$ on U. Therefore

$$(\Delta p_{\mu_1})|_U = (\Delta p_{\mu_2})|_U.$$

The desired result follows from **Theorem 3.20**.

□

As an application of this result, we can justify the statement made in **Section 3.3** concerning the uniqueness of equilibrium measures.

Theorem 3.21: Compact Non-Polar Set Has Unique Equilibrium Measure

Let K be a compact non-polar subset of $\mathbb{C}$. Then its equilibrium measure ν is unique, and $\operatorname{supp}(\nu) \subset \partial_e K$, the exterior boundary of K.

Proof:

Step I: $\partial_e K$ is non-polar

Suppose $\partial_e K$ is polar, then by **Theorem 3.14** $\mathbb{C}\backslash\partial_e K$ would be connected, and this would imply that $\partial_e K = K$, but K is non-polar by assumption, this is impossible and thus $\partial_e K$ is non-polar.

Step II: Uniqueness

Let ν and $\widetilde{\nu}$ be equilibrium measures on K and $\partial_e K$ respectively. It suffices to prove that $\nu = \widetilde{\nu}$. By Frostman's theorem **Theorem 3.7** one has

$$p_\nu \geq I(\nu) \text{ on } \mathbb{C} \text{ and } p_\nu = I(\nu) \text{ n.e. on } K.$$

Moreover, p_ν is bounded above on each bounded component of $\mathbb{C}\backslash K$, so applying the extended maximum principle **Theorem 3.17** (ii) we deduce that

$$p_\nu \equiv I(\nu) \text{ on } \mathbb{C}\backslash K.$$

Similarly,

$$p_{\widetilde{\nu}} \geq I(\widetilde{\nu}) \text{ on } \mathbb{C} \text{ and } p_{\widetilde{\nu}} = I(\widetilde{\nu}) \text{ n.e. on } \partial_e K,$$

and also

$$p_{\widetilde{\nu}} \equiv I(\widetilde{\nu}) \text{ on each bounded component of } \mathbb{C}\backslash\partial_e K.$$

Finally, on the unbounded component of $\mathbb{C}\backslash K$, which is the same as the unbounded component of $\mathbb{C}\backslash\partial_e K$, the difference $(p_\nu - p_{\widetilde{\nu}})$ is harmonic and bounded, and so by the extended maximum principle **Theorem 3.17** (ii) once again,

$$p_\nu - p_{\widetilde{\nu}} \equiv I(\nu) - I(\widetilde{\nu})$$

on each unbounded components of $\mathbb{C}\backslash K$ and $\mathbb{C}\backslash\partial_e K$. Moreover, since

$$p_\nu(z) - p_{\widetilde{\nu}}(z) = \big(\log|z| + o(1)\big) - \big(\log|z| + o(1)\big) = o(1)$$

as $z \to \infty$. It follows that $I(\nu) = I(\widetilde{\nu})$. Thus $p_\nu = p_{\widetilde{\nu}}$ n.e. on $\mathbb{C}$, and therefore everywhere on $\mathbb{C}$ by the weak identity principle **Theorem 2.24**. Finally, applying **Corollary 3.20.1** we deduce that $\nu = \widetilde{\nu}$.

□

Corollary 3.21.1: Equilibrium Measure of $\overline{\Delta}$ Is Lebesgue Measure on $\partial\Delta$

The equilibrium measure of a closed disc $\overline{\Delta}$ is the normalized Lebesgue measure on $\partial\Delta$.

Proof:

By **Theorem 3.21**, the equilibrium measure is supported on $\partial\Delta$, and since it is unique it must be rotational invariant. This implies that it is a multiple of the Lebesgue measure on $\partial\Delta$.

□

As a further application of **Theorem 3.20** we can compute $\Delta(\log|f|)$ when f is holomorphic.

Theorem 3.22: Solution to Generalized Laplacian via Holomorphic Zero Mass

Let f be a holomorphic function on a domain D such that $f \not\equiv 0$. Then $\Delta(\log|f|)$ consists of (2π)-masses at the zeros of f, counted according to multiplicity.

Proof:

Given a relatively compact open subset U of D, we can write

$$f(z) = (z - w_1)\cdots(z - w_n)g(z), \; z \in U,$$

where $w_1, \cdots, w_n$ are the zeros of f in U, and g is holomorphic and non-zero on U. Then for $z \in U$,

$$\log|f(z)| = \sum_{j=1}^{n} \log|z - w_j| + \log|g(z)| =: p_\mu(z) + h(z),$$

where μ consists of unit masses at $w_1, \cdots, w_n$ and h is harmonic on U. By **Theorem 3.20**,

$$\Delta(\log|f|) = 2\pi\mu \text{ on } U.$$

As this holds for each such U, the result holds by **Remark 3.1** (ii).

□

The proof above shows that $\log|f|$ can be expressed locally as the sum of a potential and a harmonic function. This is actually a special case of a quite general result.

Theorem 3.23: Riesz Decomposition Theorem

Let u be a subharmonic function on a domain D in $\mathbb{C}$ with $u \not\equiv -\infty$. Then, given a relatively compact open subset U of D, we can decompose u as
$$u = p_\mu + h \text{ on } U,$$
where $\mu = (2\pi)^{-1}\Delta u\Big|_U$ and h is harmonic on U.

This is a very powerful result. It means that many problems about general subharmonic functions can be reduced to questions about potentials. Most of the work for proving **Theorem 3.23** has already been done. What remains to be proved is the following lemma, which is a converse to **Corollary 3.20.1**.

Lemma 3.24: Weyl's Lemma

Let u and v be subharmonic functions on a domain D in $\mathbb{C}$ with $u, v \not\equiv -\infty$. If $\Delta u = \Delta v$ then $u = v + h$ for some harmonic function h on D.

Proof:

Let $\{\chi_r\}_{r\geq 0}$ be the functions we used in the smoothing theorem **Theorem 2.22**, and for $r > 0$ we write
$$D_r := \left\{z \in D : \operatorname{dist}(z, \partial D) > r\right\}.$$
Then $u * \chi_r \in C^\infty(D_r)$, and for $z \in D_r$ we have
$$\begin{aligned}\Delta(u * \chi_r)(z) :&= \int u(w)\Delta_z \chi_r(z - w)dA(w)\\ &= \int u(w)\Delta_w \chi_r(z - w)dA(w)\\ &= \int \varphi \Delta u\end{aligned}$$
where the first equality holds since Laplacian is commutative under convolution, the second equality holds since Laplacian is closed under translation, and the last equality holds by Green's theorem and $\varphi(w) := \chi_r(z - w) \in C_c^\infty(D)$. The same calculation works with u replaced by v. Since $\Delta u = \Delta v$, it follows that
$$\Delta(u * \chi_r) = \Delta(v * \chi_r) \text{ on } D_r.$$
Therefore there exists a harmonic function h_r on D_r such that
$$u * \chi_r = v * \chi_r + h_r \text{ on } D_r.$$
Now by **Theorem 2.22** applied to $\pm h_r$, we have $h_r * \chi_s = h_r$ on D_{r+s} for each $s > 0$, and hence
$$h_r = h_r * \chi_s = (u - v) * \chi_r * \chi_s = h_s * \chi_r = h_r \text{ on } D_{r+s},$$
where the third equality holds since $\{\chi_r\}_{r\geq 0}$ is commutative. Therefore there is a single harmonic function h on D such that for each $r > 0$,
$$u * \chi_r = v * \chi_r + h \text{ on } D_r.$$
Since $r > 0$ is arbitrary, sending $r \downarrow 0$ and using **Theorem 3.22** gives
$$u = v + h \text{ on } D,$$

as desired.

□

Proof of **Theorem 3.23**:

Put $\mu := (2\pi)^{-1}\Delta u\Big|_U$. Then by **Theorem 3.20**,

$$\Delta p_\mu = 2\pi\mu = \Delta u \text{ on } U.$$

Applying **Lemma 3.24** on each component of U, it follows that

$$u = p_\mu + h \text{ on } U,$$

where h is harmonic on U.

□

3.8 Thinness

Let u be a subharmonic function on a neighbourhood of $\zeta \in \mathbb{C}$. Even though u may be discontinuous at ζ, it is always true that

$$\limsup_{z\to\zeta, z\neq\zeta} u(z) = u(\zeta). \tag{3.4}$$

For by u.s.c. one certainly has $\limsup_{z\to\zeta} u(z) \leq u(\zeta)$, and if the inequality is strict, then u would violate the submean inequality on small circles around ζ. Thus the value of u at ζ is completely determined by its values on a punctured disc around ζ. It turns out to be useful to know to what extent the punctured disc may be replaced by a smaller set S.

Definition: Thin and Non-Thin

Let S be a subset of $\mathbb{C}$ and let $\zeta \in \mathbb{C}$. Then S is non-thin at ζ if

(i) $\zeta \in \overline{S\backslash\{\zeta\}}$.

(ii) For every subharmonic function u defined on a neighbourhood of ζ,

$$\limsup_{z\to\zeta, z\in S\backslash\{\zeta\}} u(z) = u(\zeta).$$

Otherwise S is said to be thin at θ.

A complete characterization of thinness is quite complicated, and must await developments in **Chapter 5**. However, for many purposes it is enough to be able to handle a few important special cases, which we shall study in this section. We begin with elementary remarks.

Remark 3.7: Elementary Properties of Thinness

(i) Thinness is obviously a local property, that is, S is non-thin at ζ if and only if $U \cap S$ is non-thin at ζ for each open neighbourhood U of ζ. (Thinness is a Local Property)

(ii) Thinness is invariant under conformal mapping, so that although we have defined thinness in the plane, we would equally well study it on the sphere. (Thinness is Invariant under Conformal Mapping)

(iii) If two sets are both thin at a particular point, then so is their union. (Union of finitely many thin sets is thin)

(iv) From (3.4) it follows that a set S is non-thin at each point of its interior. In particular, an open set is non-thin at every point of iteself.

(A set is non-thin at every point of its interior)

◇

Though S cannot be thin in its interior, it can be thin at some point on its boundary.

Example 3.3: A Set Can Be Thin at Its Boundary Points

Let u be a subharmonic function which is discontinuous at ζ, and choose α so that

$$\liminf_{z\to\zeta} u(z) < \alpha < u(\zeta).$$

Then $S := \{z : u(z) < \alpha\}$ is an open set with $\zeta \in \partial S$ and clearly S is thin at ζ.

◇

We shall look at special types of set S, beginning with the small ones.

Theorem 3.25: F_σ Polar Set Is Thin at All Points of $\mathbb{C}$

An F_σ polar set is thin at every point of $\mathbb{C}$.

Proof:

Let S be an F_σ polar set and let $\zeta \in \mathbb{C}$. Then $S\backslash\{\zeta\}$ is also an F_σ polar set and is obviously disjoint from $\{\zeta\}$, so by **Theorem 3.11** (i) there exists a subharmonic function u on $\mathbb{C}$ such that $u = -\infty$ on $S\backslash\{\zeta\}$ and $u(\zeta) > -\infty$ by **Theorem 3.11** (ii). Therefore S is thin at ζ.

□

As the other extreme we have the following theorem.

Theorem 3.26: Non-Trivial Connected Set Is Non-Thin at Its Closure Points

A connected set containing more than one point is non-thin at every point of its closure.

The proof is based on a lemma which is actually a special case of the main result.

Lemma 3.27: Subharmonic "Barier" on Boundary Points

Let u be a subharmonic function on $\Delta(0,1)$. If $u \le 0$ on the segment $(0,1)$ then $u(0) \le 0$.

Proof:

Replacing u by $\max(u,0)$, we can suppose that $u \ge 0$ on $\Delta(0,1)$ and $u = 0$ on $(0,1)$. It left us to show that $u(0) = 0$. Define v on $\Delta(0,1)\backslash\{0\}$ by

$$v(z) := \begin{cases} u(z^2), & \mathrm{Im}(z) > 0 \\ 0, & \mathrm{Im}(z) \le 0 \end{cases}$$

Then v is subharmonic on $\Delta(0,1)\backslash\{0\}$ by the gluing theorem **Theorem 2.11**. Moreover v is bounded above near 0, so by the removable singularity theorem **Theorem 3.13** it extends to a subharmonic function on the whole of $\Delta(0,1)$. Then by **Theorem 2.21** (c),

$$v(0) = \lim_{r\to 0} M_v(r) = \lim_{r\to 0} M_u(r^2) = u(0)$$

and also

$$v(0) = \lim_{r\to 0} C_v(r) = \lim_{r\to 0} \frac{1}{2} C_u(r^2) = \frac{1}{2} u(0)$$

where the middle equality holds as $r_1 = r_2$ implies $2\pi r_1 = 4\pi r_2$ for the circumference. Combining these two display yields $u(0) = 0$.

□

Proof of **Theorem 3.26**:

We argue by contradiction. Let S be a connected set with at least two points and suppose, if possible, that S is thin at some point ζ of its closure. Applying a conformal mapping (which does not change thinness by **Remark 3.7** (ii)), we may assume that $\zeta = 0$. Then there exists a subharmonic function u, defined on a neighbourhood of 0, such that

$$\limsup_{z\to 0, z\in S\setminus\{0\}} u(z) < u(0).$$

By the Riesz decomposition theorem **Theorem 3.23** we can decompose u on a neighbourhood of 0 as $u = p_\mu + h$, where p_μ is the potential of a finite Borel measure μ of compact support, and h is harmonic. Since h is continuous it follows that

$$\limsup_{z\to 0, z\in S\setminus\{0\}} p_\mu(z) < p_\mu(0).$$

Now define $T : \mathbb{C} \to \mathbb{R}$ by $T(z) := |z|$ and set

$$\mu_1(B) := \mu\left(T^{-1}(B)\right), B \subset \mathbb{C} \text{ Borel},$$

so that μ_1 is also a finite Borel measure with compact support. Then for $\in \mathbb{C}$

$$p_{\mu_1}(|z|) = \int \log\Big| |z| - |w| \Big| d\mu(w) \le p_\mu(z)$$

where the equality holds by the definition of log potential and the inequality holds by the triangle inequality $|z - w| \ge \Big| |z| - |w| \Big|$. The equality holds if $z = 0$. Therefore,

$$\limsup_{z\to 0, z\in S\setminus\{0\}} p_{\mu_1}(|z|) < p_{\mu_1}(0).$$

Since S is connected and contains a point other than 0, it follows that the set $\{|z| : z \in S\}$ includes an interval $(0,\alpha)$ for some $\alpha > 0$. Hence

$$\limsup_{z\to 0, z\in(0,\alpha)} p_{\mu_1}(z) < p_{\mu_1}(0).$$

It is therefore possible to choose constants r and s so that

$$u_1(z) := p_{\mu_1}(rz) + s$$

which by **Theorem 2.4** (ii) is subharmonic on $\Delta(0,1)$ and satisfies $u_1 \le 0$ on $(0,1)$ and $u_1(0) > 0$. This violates the conclusion of **Lemma 3.27**.

□

Combining the last two theorems immediately leads to a generalization to the fact that every closed polar set is totally disconnected, which we proved in **Corollary 3.14.1**.

Corollary 3.26.1: F_σ Polar Set Is Totally Disconnected

Every F_σ polar set is totally disconnected.

A set may be thin at "many" points. As an extreme example, a countable dense subset of $\mathbb{C}$ is thin everywhere. However, as our final theorem of this section shows, a set cannot be thin at too many points of itself.

Theorem 3.28: A Set Cannot Be Thin at Too Many Points of Itself

A subset S of $\mathbb{C}$ is non-thin at n.e. point of itself.

Proof:

Let $\{U_j\}_{j\geq 1}$ be a countable base of open sets for $\mathbb{C}$ with $\operatorname{diam}(U_j) < 1$. For each j, let $\mathscr{V}_j$ be the collection of all subharmonic functions v on U_j such that

$$v \leq \begin{cases} 0, & \text{on } U_j \\ -1, & \text{on } U_j \cap S \end{cases}$$

Set $u_j := \sup_{v\in\mathscr{V}_j} v$, and let u_j^* be its upper semicontinuous regularization. Then by Brelot-Cartan theorem **Theorem 3.8** (b) there exists a Borel polar set E_j such that $u_j^* = u_j$ on $U_j \backslash E_j$. Set $E := \bigcup_{j\geq 1} E_j$. Then by **Corollary 3.4.2** E is a Borel polar set, and we shall show that S is non-thin at each point of $S\backslash E$. Suppose that $\zeta \in S$ and that S is thin at ζ.

Case I: ζ is non-isolated point of S

If ζ is a non-isolated point of S, then there exists a subharmonic function u on a neighbourhood of ζ such that

$$\limsup_{z\to\zeta, z\in S\backslash\{\zeta\}} u(z) < -1 < u(\zeta) < 0.$$

Therefore there exists a neighbourhood of ζ, which we may take to be member U_j of the countable base such that

$$u \leq \begin{cases} 0, & \text{on } U_j \\ -1, & \text{on } U_j \cap (S\backslash\{\zeta\}) \end{cases}$$

Case II: ζ is an isolated point of S

If ζ is an isolated point of S, then we reach the same conclusion by choosing U_j so that $U_j \cap S = \{\zeta\}$, and setting $u \equiv 0$.

For each case, then, for each $\varepsilon > 0$, the function

$$v_\varepsilon(z) := u(z) + \varepsilon \log|z - \zeta|$$

belongs to the class $\mathscr{V}_j$, and so $u_j \geq u_\varepsilon$. Sending $\varepsilon \downarrow 0$ we deduce that $u_j \geq u$ on $U_j\backslash\{\zeta\}$, and hence that $u_j^* \geq u$ on U_j. In particular,

$$u_j^*(\zeta) \geq u(\zeta) > -1.$$

On the other hand, it is clear that $u_j(\zeta) \leq -1$ since $\zeta \in S$. Hence $\zeta \in E_j$. We have therefore shown that the only point of S can be thin are those that lie in E. This proves the desired result.

□

As a special case, we obtain a converse to **Theorem 3.25**.

Corollary 3.28.1: Set Thin at All Its Points Is Polar

A set which is thin at every point of itself must be polar.

Summary of Chapter 3

In this chapter we studied the potential theory and some elementary properties. The reason we study potentials is that potentials turn out to be almost as general as arbitrary functions and for many purposes the two classes are equivalent. In fact as we shall see in Riesz decomposition theorem that problem in subharmonic functions can be reduced to problems in potentials.

In the first section we defined the potential and prove some its properties. In particular, unlike the subharmonic functions, the potentials have continuity principle and minimum principle.

In the second section we introduced the concept of polar sets, which serves as the "measure" zero set in potential theory. To this end we defined the energy of finite Borel measures and based on this terminology we defined the polar sets to be subset having minus infinite energy. Similar to the a.e. property translated to probability theory as a.s. property we translate the a.e. property to potential theory as n.e. property. We proved that "Measures with Finite Energy do not Charge Polar Sets". As a consequence, "Borel Polar set Has Lebesgue Measure Zero". Thus the polar sets are small in the sense they have measure zero. Moreover, "Polarity Is Stable under Countable Union". In fact this holds only for countable union of Borel polar sets, it fails when the Borel condition is removed. Moreover, the polar sets are not necessarily countable as a consequnce.

In the third section we studied the equilibrium measures and proved that "Compact Sets Have Equilibrium Measure". For the proof we introduced the concept of weak* convergence, for which some authors call it the vague convergence. This motion of convergence helps one prove the lemma "Weak* Convergence Implies Energy Upper Bound". Then we proved the fundamental theorem of potential theory, namely the Frostman's theorem, which establishes the key relation between potentials and energies.

Motivated by the fact that limit of decreasing sequence of subharmonic functions is subharmonic but the same argument fails for increasing sequence. Thus we force it to be u.s.c. by introducing the u.s.c. regularization in section 3.4. We proved Brelot-Cartan theorem which justifies that the u.s.c. regularization agree with the original nearly everywhere. Then we proved our motivating questions and it follows that the u.s.c. regularization for the limit of increasing subharmonic function is again subharmonic and agree with the original one n.e..

It is of special interest to study the minus infinity set. In section 3.5 we first proved that "Subharmonic Function Is Minus Infinity On G_δ Polar Set", for whic allows us to demonstrate the existence of uncountable polar sets. In fact, the converse of this result also holds but relies on the application of condenser measure, for which we did not introduce but instead proved a weaker version good for most cases: "F_σ Polar Set Decomposition for Subharmonic Functions", for which the proof relies on"Existence of Borel Probability Measure Charging Compact Polar Sets". As a corollary, we proved "Characterization of Closed Polar Set via Subharmonic Functions".

In section 3.6, we discussed that the polar sets are small in the other sense. We first proved "Removable Singularity Theorem for Subharmonicity" and similarly "Removable Singularity Theorem for Harmonic Functions". Then the fact that the polar set are small as removing them does not affect connectedness is proved in "Removing Closed Polar Set Does Not Affect Connectivity", and thus "Closed Polar Set Is Totally Disconnected". For an application we proved the Radó-Stout theorem and from which the connection between polar set and holomorphic function is established, that is, "Preimage of Polar Set under Non-Constant Holomorphy Is Polar". We remarked that the polarity is invariant under conformal mapping and therefore we can extend this concept to Riemann sphere. For this we can extend the previous properties for subharmonic functions, namely "Extended Liouville Theorem for Subharmonic Functions" and "Extended Maximum Principle for Subharmonic Functions", as a corollary a version of extended Liouville theorem also holds for holomorphic function. Moreover the converse of the extended Liouville can serves as a way to identify polar set by constant subharmonic functions.

In section 3.7 we studied the generalized Laplacian as the Radon measure such that a certain equality holds. For the concept to be complete we first proved "Existence and Uniqueness of Generalized Laplacian" for which is proved by a lemma called "Approximation Lemma for Element in $C_c(D)$". An observation of these results enables us to view the potentials under the perspective of distribution theory and motivates "Poisson's Equation in Complex Plane". We are able to prove that "Local Uniqueness of Log Potential Up to Harmonic Translation" and as an application we showed that "Compact Non-Polar Set Has Unique Equilibrium Measure". In particular, "Equilibrium Measure of $\overline{\Delta}$ Is Lebesgue Measure on $\partial\Delta$". We can compute the "Solution to Generalized Laplacian via Holomorphic Zero Mass", for which the solution counts multiplicity. Finally we proved the Riesz decomposition theorem which enables us to solve problems concerning subharmonic functions by concerning potentials. This result is proved by Weyl's lemma.

In section 3.8, we studied thinness. The motivation is that subharmonic function, though may be discontinuous, the approximation always holds. Thus we defined thinness and non-thinness, which are local property, closed under conformal mapping and finite union, and a set is non-thin at its interior points. We proved that "F_σ Polar Set Is Thin at All Points of $\mathbb{C}$" and "Non-Trivial Connected Set Is Non-Thin at Its Closure Points". The proof for the latter one relies on "Subharmonic "Barier" on Boundary Points". By these two results we proved that "F_σ Polar Set Is Totally Disconnected". A set may be thin at many points but not too many. This is confirmed by "A Set Cannot Be Thin at Too Many Points of Itself", as a corollary, we proved that "Set Thin at All Its Points Is Polar".

4. The Dirichlet Problem

4.1 Solution of Dirichlet Problem

We recall the definition from **Section 1.2** that, given a domain D and a continuous function $\varphi : \partial D \to \mathbb{R}$, the Dirichlet problem is to find a harmonic function h on D such that

$$\lim_{z\to\zeta} h(z) = \varphi(\zeta)\ \forall\zeta \in \partial D.$$

By **Theorem 1.5**, if such a solution h exists, then it is unique. Moreover, if D is a disc, then a solution always does exist, and **Theorem 1.6** (iii) even gives a formula for it.

For a general domain D, the situation is more complicated. In this case, the Dirichlet problem, at least in the form stated above, may well have no solution.

Example 4.1: Example of Dirichlet Problem Fails to Have Solution

Let $D := \{z : 0 < |z| < 1\}$ and let $\varphi : \partial D \to \mathbb{R}$ be given by

$$\varphi(\zeta) = \begin{cases} 0, & |\zeta| = 1 \\ 1, & |\zeta| = 0 \end{cases}$$

Then by **Corollary 3.13.1**, any solution h would have a removable singularity at 0, and the maximum principle **Theorem 1.4** (ii) would then imply that $h(0) \le 0$, violating the condition that $\lim_{z\to 0} h(z) = \varphi(0) = 1$. ◇

In this section and the next, we shall consider conditions under which a solution does exist, and also, even more importantly, derive a natural reformulation of the Dirichlet problem which **always** has has a solution. To this end, it is convenient to extend the set-up described above in two ways:

(i) Firstly, we shall allow D to be any proper subdomain of $\mathbb{C}^\infty$. Of course, since the Dirichlet problem is invariant under conformal mapping of the sphere, there is really no more general than working on a subdomain of $\mathbb{C}$. However, the gain in flexibility does turn out to be useful. We shall exploit without further comment the fact that harmonicity, subharmonicity, and polarity all extend in a natural way to $\mathbb{C}^\infty$.

(ii) Secondly, we shall consider arbitrary bounded function $\varphi : \partial D \to \mathbb{R}$ instead of only the continuous ones. Although certainly no solution to the Dirichlet problem is possible if φ is discontinuous, it is nevertheless useful to allow this extra freedom, as will become clear later.

The key idea, sometimes called the Perrof method, is enshrined in the following definition:

Definition: Perron Function

Let D be a proper subdomain of $\mathbb{C}^\infty$ and let $\varphi : \partial D \to \mathbb{R}$ be a bounded function. The associated Perron function $H_D\varphi : D \to \mathbb{R}$ is defined by

$$H_D\varphi := \sup_{u\in\mathcal{U}} u,$$

where $\mathcal{U}$ (lower class class) denotes the family of all subharmonic functions u on D such that

$$\limsup_{z\to\zeta} u(z) \le \varphi(\zeta) \ \forall\zeta \in \partial D.$$

The motivation for this definition is that, if the Dirichlet problem has a solution at all, then $H_D\varphi$ is it! Indeed, if h is such a solution, then certainly $h \in \mathcal{U}$, and so $h \le H_D\varphi$. On the other hand, by the maximum principle **Theorem 1.4**, if $u \in \mathcal{U}$ then $u \le h$ on D, and so $H_D\varphi \le h$. Therefore $H_D\varphi = h$.

Our first result is that, regardless of whether the Dirichlet problem has a solution or note, $H_D\varphi$ is always a bounded harmonic function.

Theorem 4.1: Perron Function Is Always Bounded Harmonic

Let D be a proper subdomain of $\mathbb{C}^\infty$ and let $\varphi : \partial D \to \mathbb{R}$ be a bounded function. Then $H_D\varphi$ is a harmonic function on D and

$$\sup_D |H_D\varphi| \le \sup_{\partial D} |\varphi|. \tag{4.1}$$

The proof of **Theorem 4.1** hinges on the following lemma.

Lemma 4.2: Poisson Modification

Let D be a domain in $\mathbb{C}$, let Δ be an open disc with $\overline{\Delta} \subset D$, and let u be a subharmonic function on D with $u \not\equiv -\infty$. If we define $\widetilde{u}$ on D by

$$\widetilde{u} := \begin{cases} P_\Delta u, & \text{on } D \\ u, & \text{on } D \backslash \Delta \end{cases}$$

where $P_\Delta u$ is the Poisson integral. Then

(i) $\widetilde{u}$ is subharmonic on D.

(ii) $\widetilde{u}$ is harmonic on Δ.

(iii) $\widetilde{u} \geq u$ on D.

Proof:

Step I: Assertion (ii) and (iii)

First, note that **Corollary 2.15.1** guarantees that u is Lebesgue integrable on $\partial\Delta$, so $P_\Delta u$ makes sense. **Theorem 1.6** (i) tells us that $P_\Delta u$ is harmonic on Δ, and by **Theorem 2.9** (b) $P_\Delta u \geq u$ there.

Step II: Assertion (i)

It remains to show that $\widetilde{u}$ is subharmonic on D, and by the gluing theorem **Theorem 2.11** this will follow provided that

$$\limsup_{z \to \zeta} P_\Delta u(z) \leq u(\zeta) \ \forall \zeta \in \partial\Delta.$$

To prove this inequality, choose continuous functions ψ_n on $\partial\Delta$ such that $\psi_n \downarrow u$ there (the existence of such a choice is guaranteed by **Theorem 2.12**). Then by **Theorem 2.12** using in the inequality and **Theorem 1.6** (ii) using in the equality, one has

$$\limsup_{z \to \zeta} P_\Delta u(z) \leq \lim_{z \to \zeta} P_\Delta \psi_n(z) = \psi_n(\zeta), \zeta \in \partial\Delta,$$

and the desired conclusion follows by sending $n \to \infty$.

□

Proof of **Theorem 4.1**:

By applying a conformal mapping of the sphere, we can suppose that D is a subdomain of $\mathbb{C}$. Let $\mathscr{U}$ be as in the definition of Perron function.

Step I: **(4.1)** holds.

If we set $M := \sup_{\partial D} |\varphi|$ then certainly $-M \in \mathscr{U}$ so $H_D\varphi \geq -M$. Moreover, given $u \in \mathscr{U}$, it follows from the maximum principle **Theorem 2.5** (ii) that $u \leq M$ on D, and therefore $H_D\varphi \leq M$. This proves **(4.1)**.

Step II: $H_D\varphi$ is harmonic on D.

It suffices to prove harmonicity of $H_D\varphi$ on each open disc Δ with $\overline{\Delta} \subset D$. Fix such a Δ, and also a point $w_0 \in \Delta$. By the definition of $H_D\varphi$, we can find $\{u_n\}_{n\geq 1} \subset \mathscr{U}$ such that $u_n(w_0) \to H_D\varphi(w_0)$. Replacing u_n by $\max(u_1, \cdots, u_n)$, we can further suppose that $u_1 \leq u_2 \leq \cdots$ on D. Now for each n, let $\widetilde{u}_n$ denote the Poisson modification of u_n, as defined in **Lemma 4.2**. Then we also have $\widetilde{u}_1 \leq \widetilde{u}_2 \leq \cdots$ on D and we claim that $\widetilde{u} := \lim_{n\to\infty} \widetilde{u}_n$ satisfies the followings:

(a) $\widetilde{u} \leq H_D\varphi$ on D (b) $\widetilde{u}(w_0) = H_D\varphi(w_0)$ (c) $\widetilde{u}$ is harmonic on Δ

Step II.1: (a) holds

By **Lemma 4.2** (i) each $\widetilde{u}_n$ is subharmonic on D and evidently

$$\limsup_{z\to\zeta} \widetilde{u}_n(z) = \limsup_{z\to\zeta} u_n(z) \le \varphi(\zeta), \zeta \in \partial D,$$

where the equality holds since $\widetilde{u} := \lim_{n\to\infty} \widetilde{u}_n$ and the inequality holds by the definition of $\mathscr{U}$, so that $\widetilde{u}_n \in \mathscr{U}$. Hence $\widetilde{u}_n \le H_D\varphi$ for each $n \ge 1$ and therefore $\widetilde{u} \le H_D\varphi$, proving (a).

Step II.2: (b) holds

By (a) and **Lemma 4.2** (iii) $\widetilde{u}_n \ge u_n$ and thus

$$\widetilde{u}(w_0) = \lim_{n\to\infty} \widetilde{u}_n(w_0) \ge \lim_{n\to\infty} u_n(w_0) = H_D\varphi(w_0),$$

where the first equality holds by (a), the inequality holds since $\widetilde{u}_n \ge u_n$, and the last equality holds since $u_n(w_0) \to H_D\varphi(w_0)$. Thus (b) holds since the reversed inequality holds by (a).

Step II.3: (c) holds

Since each $\widetilde{u}_n$ is harmonic on Δ, so by Harnack's theorem **Theorem 1.14** the same is true for the increasing limit $\widetilde{u}$, thus (c) holds.

Step III: $\widetilde{u} \le H_D\varphi$ on Δ.

Take an arbitrary point $w \in \Delta$, and choose $\{v_n\}_{n\ge 1} \subset \mathscr{U}$ such that

$$v_n(w) \to H_D\varphi(w).$$

Replacing v_n be $\max(u_1, \cdots, u_n, v_1, \cdots, v_n)$, we can suppose that

$$v_1 \le v_2 \le v_3 \le \cdots \text{ and } v_n \ge u_n \text{ on } D.$$

Let $\widetilde{v}_n$ denote the Poisson modification of v_n. Then $\widetilde{v}_n \uparrow \widetilde{v}$ where

(a') $\widetilde{v} \le H_D\varphi$ on D (b') $\widetilde{v}(w) = H_D\varphi(w)$ (c') $\widetilde{v}$ is harmonic on Δ

In particular, (a') implies that

$$\widetilde{v}(w_0) \le H_D\varphi(w_0) = \widetilde{u}(w_0),$$

where the last equality holds by (b). On the other hand, $\widetilde{v}_n \ge \widetilde{u}_n$ for each $n \ge 1$ so $\widetilde{v} \ge \widetilde{u}$. Thus the function $\widetilde{u} - \widetilde{v}$, which is harmonic on Δ, attains maximum value 0 at w_0. By the maximum principle **Theorem 1.4**, this implies that

$$\widetilde{u} - \widetilde{v} \equiv 0 \text{ on } \Delta.$$

In particular, it follows that

$$\widetilde{u}(w) = \widetilde{v}(w) = H_D\varphi(w).$$

Since w is chosen arbitrary in Δ, it follows that $\widetilde{u} \equiv H_D\varphi$ on Δ.

□

From the definition of $H_D\varphi$, one might expect that

$$\lim_{z\to\zeta} H_D\varphi(z) = \varphi(\zeta) \ \forall \zeta \in \partial D.$$

But if $D := \{z : 0 < |z| < 1\}$ then this cannot be true, because, as we have seen in **Example 4.1**, the Dirichlet problem may have no solution. It is instrucive to see exactly what is going wrong.

Remark 4.1: Reason Dirichlet Problem Is Unsolvable in **Example 4.1**

First let

$$\varphi(\zeta) := \begin{cases} 0, & |\zeta| = 1 \\ 1, & |\zeta| = 0 \end{cases}$$

If $u \in \mathscr{U}$ then by the extended maximum principle **Theorem 3.17** (ii) $u \leq 0$ on D and so $H_D\varphi \leq 0$. Since $0 \in \mathscr{U}$, in fact $H_D\varphi \equiv 0$ on D.

Now let

$$\widetilde{\varphi}(\zeta) := \begin{cases} 0, & |\zeta| = 1 \\ -1, & |\zeta| = 0 \end{cases}$$

The same argument applies (even using the ordinary maximum principle **Theorem 2.5**) and thus $H_D\varphi \leq 0$. This time $0 \notin \mathscr{U}$. However it is true, according to **Corollary 1.1.1**, that

$$\varepsilon \log |z| \in \mathscr{U} \ \forall \varepsilon > 0.$$

Sending $\varepsilon \downarrow 0$ and again $H_D\varphi \equiv 0$ on D.

In both cases, the isolated boundary point 0 lacked sufficient "influence" on the subharmonic functions in $\mathscr{U}$, and the result was that $H_D\varphi$ had the wrong boundary limit there. ◇

To overcome this problem mentioned in **Remark 4.1**, we introduce a notion of barrier.

Defintion: Barrier

Let D be a proper subdomain of $\mathbb{C}^\infty$ and let $\zeta_0 \in \partial D$. A barrier at ζ_0 is a subharmonic function b defined on $D \cap N$, where N is an open neighbourhood of ζ_0, such that

(i) $b < 0$ on $D \cap N$.

(ii) $\lim\limits_{z \to \zeta_0} b(z) = 0$.

Definition: Regular Boundary Point

A boundary point at which a barrier exists is called regular.

Definition: Irregular Boundary Point

A boundary point at which a barrier does not exist is called irregular.

Definition: Regular Domain

Let D be a proper subdomain of $\mathbb{C}^\infty$ then D is called a regular domain if ζ is a regular boundary point $\forall \zeta \in \partial D$.

Theorem 4.3: Sufficiency for Perron Function Solving Dirichlet Problem

Let D be a proper subdomain of $\mathbb{C}^\infty$ and let ζ_0 be a regular boundary point of ∂D. If $\varphi : \partial D \to \mathbb{R}$ is a bounded function which is continuous at ζ_0 then

$$\lim_{z \to \zeta_0} H_D\varphi(z) = \varphi(\zeta_0).$$

This time we need two lemmas. The first is a simple consequence of the definition of Perron functions.

Lemma 4.4: Perron Function Is Antisymmetric

If D is a proper subdomain of $\mathbb{C}^\infty$ and $\varphi : \partial D \to \mathbb{R}$ is a bounded function then

$$H_D\varphi \leq -H_D(-\varphi) \text{ on } D.$$

Proof:

Let $\mathscr{U}$ be the family of subharmonic functions prescribed in the definition of Perron functions, and let $\mathscr{V}$ be the corresponding family for $-\varphi$. Then, given

$u \in \mathscr{U}$ and $v \in \mathscr{V}$, their sum is subharmonic on D and satisfies

$$\lim_{z \to \zeta}(u+v)(z) \leq \varphi(\zeta) - \varphi(\zeta) = 0, \zeta \in \partial D.$$

Hence by the maximum principle **Theorem 3.17** (ii), $u + v \leq 0$ on D. Taking supremum over all such u and v, we get

$$H_D\varphi + H_D(-\varphi) \leq 0 \text{ on } D,$$

thus $H_D\varphi \leq -H_D(-\varphi)$ on D as desired.

□

The second lemma enables us to 'globalize' barriers by allowing a bit more space.

Lemma 4.5: Bouligand's Lemma

Let ζ_0 be a regular boundary point of a domain D, and let N_0 be an open neighbourhood of ζ_0. Then, given $\varepsilon > 0$, there exists a subharmonic function b_ε on D such that

(i) $b_\varepsilon < 0$ on D.

(ii) $b_\varepsilon \leq -1$ on $D \backslash N_0$.

(iii) $\liminf_{z \to \zeta_0} b_\varepsilon(z) \geq -\varepsilon$.

Proof:

We may suppose that $\zeta_0 \not\equiv \infty$ (otherwise we may apply a conformal mapping). Since ζ_0 is regular, there exists a neighbourhood N of ζ_0 and a barrier b on $D \cap N$ by the definition of barrier.

Let $\Delta = \Delta(\zeta_0, \rho)$, where ρ is chosen sufficiently small so that $\overline{\Delta} \subset N \cap N_0$. Then the normalized Lebesgue measure on $\partial\Delta$ is a regular measure (since if μ is a finite Borel measure on a metric space X then μ is regular), so we can find a compact set $K \subset D \cap \partial\Delta$ such that

$$L := (D \cap \partial\Delta)\backslash K$$

has measure smaller than ε. Since L is open in $\partial\Delta$, using **Theorem 1.6** (ii) we get

$$\lim_{z \to \eta, z \in D} P_\Delta 1_L(z) = 1, \eta \in L.$$

Now put $m := -\sup_K b$ so that $m > 0$. Then for $\eta \in D \cap \partial\Delta$,

$$\limsup_{z \to \eta, z \in D \cap \Delta} \left(\frac{b(z)}{m} - P_\Delta 1_L(z)\right) \leq \begin{cases} \frac{b(\eta)}{m} - 0, & \eta \in K \\ 0 - 1, & \eta \in L \end{cases} \leq -1.$$

Hence if we define b_ε on D by

$$b_\varepsilon := \begin{cases} \max\left(-1, \frac{b}{m} - P_\Delta 1_L\right), & \text{on } D \cap \Delta \\ -1, & \text{on } D \backslash \Delta \end{cases}$$

then by the gluing theorem **Theorem 2.11** b_ε is subharmonic on D. Clearly

$$b_\varepsilon < 0 \text{ on } D \text{ and } b_\varepsilon \leq -1 \text{ on } D \backslash N_0,$$

proving (i) and (ii). Finally, using the definition of b_ε in the first inequality and the definition of barrier in the equality, one has

$$\liminf_{z\to\zeta_0} b_\varepsilon(z) \geq \lim_{z\to\zeta_0}\left(\frac{b(z)}{m} - P_\Delta 1_L(z)\right) = 0 - P_\Delta 1_L(\zeta_0) > -\varepsilon,$$

where the last inequality holds by the fact that, as ζ_0 is the center of Δ, the value of $P_\Delta 1_L(\zeta_0)$ is exactly the normalized Lebesgue measure of L, which is smaller than ε.

□

Proof of **Theorem 4.3**:

Let $\varepsilon > 0$. Since φ is continuous at ζ_0 by assumption, there exists an open neighbourhood N_0 of ζ_0 such that

$$\zeta \in \partial D \cap \overline{N}_0 \Rightarrow \left|\varphi(\zeta) - \varphi(\zeta_0)\right| < \varepsilon$$

by continuity. Construct b_ε as in the proof of **Lemma 4.5** and set

$$u := \varphi(\zeta_0) - \varepsilon + \big(M + \varphi(\zeta_0)\big)b_\varepsilon,$$

where $M := \sup_{\partial D} |\varphi|$. Then u is subharmonic on D, and if $\zeta \in \partial D$ then

$$\limsup_{z\to\zeta} u(z) \leq \begin{cases} \varphi(\zeta_0) - \varepsilon + 0, & \text{if } \zeta \in \partial D \cap \overline{N}_0 \\ \varphi(\zeta_0) - \varepsilon - \big(M + \varphi(\zeta_0)\big), & \text{if } \zeta \in \partial D \backslash \overline{N}_0 \end{cases} \leq \varphi(\zeta).$$

Hence by the definition of Perron function, $u \leq H_D\varphi$ on D. In particular,

$$\lim_{z\to\zeta_0} H_D\varphi(z) \geq \liminf_{z\to\zeta_0} u(z) \geq \varphi(\zeta_0) - \varepsilon\big(1 + M + \varphi(\zeta_0)\big),$$

where the first inequality holds since $u \leq H_D\varphi$ on D and the second inequality holds by rewritting u and using **Lemma 4.5** (iii). Since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ yields

$$\liminf_{z\to\zeta_0} H_D\varphi(z) \geq \varphi(\zeta_0). \tag{4.2}$$

Repeating the argument with φ replaced by $-\varphi$, we also have

$$\liminf_{z\to\zeta_0} H_D(-\varphi)(z) \geq -\varphi(\zeta_0).$$

By **Lemma 4.4**, $H_D\varphi \leq -H_D(-\varphi)$ and it follows that

$$\limsup_{z\to\zeta_0} H_D\varphi(z) \leq \varphi(\zeta_0). \tag{4.3}$$

Finally, combining (4.2) and (4.3) yields the desired result.

□

Putting together what we have learned, we obtain the following result.

Corollary 4.3.1: Existence and Unique Solution to the Dirichlet Problem

Let D be a regular domain and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there exists a **unique** harmonic function h on D such that

$$\lim_{z\to\zeta} h(z) = \varphi(\zeta)\ \forall \zeta \in \partial D.$$

Proof:

Uniqueness has been established in **Theorem 1.5**, existence follows from setting $h := H_D\varphi$ and applying **Theorem 4.1** and **Theorem 4.3**.

□

For the sakeness of simplicity, we shall denote DP the abbrivation of the Dirichlet problem whenever necessary.

Remark 4.2: Regularity Is Necessary and Sufficient for Solvability of DP

There is also a converse to **Theorem 4.3**, which means regularity is not only sufficient to guarantee the solvability of the Dirichlet problem, but also necessary. Thus **Corollary 4.3.1** is, in some sense, the best possible result. ⋄

4.2 Criteria for Regularity

Although the results of the previous section appear to solve the Dirichlet problem completely, they leave one important question unanswered, namely, how to tell whether a given boundary point of D is regular? In this section we examine some geometric criteria for the existence and non-existence of barriers.

Theorem 4.6: Simply Connected Domain Smaller than $\mathbb{C}^\infty$ Is Regular

If D is a simply connected domain such that $\mathbb{C}^\infty \backslash D$ contains at least two points then D is a regular domain.

Proof:

We need to show that every boundary point of D is regular. Given $\zeta \in \partial D$, pick $\zeta_1 \in \partial D \backslash \{\zeta_0\}$. Applying a conformal mapping to the sphere, we can suppose, without loss of generality, that $\zeta_0 = 0$ and $\zeta_1 = \infty$. Then D is a simply connected domain of $\mathbb{C}_1 \backslash \{0\}$, so by **Corollary 1.1.1** there exists a holomorphic branch of $\log z$ on D. Put $N := \Delta(0,1)$ and define b on $D \cap N$ by

$$b(z) := \mathrm{Re}\left(\frac{1}{\log z}\right), z \in D \cap N.$$

Then b clearly satisfies all the conditions of being a barrier at 0.

□

This result can be 'localized' to obtain a sufficient condition for regularity of a single point.

Theorem 4.7: Boundary Point in Non-Trivial Component Is Regular

Let D be a subdomain of $\mathbb{C}^\infty$, let $\zeta_0 \in \partial D$, and let C be a component of ∂D which contains ζ_0. If $C \neq \{\zeta_0\}$ then ζ_0 is regular.

Proof:

Choose $\zeta_1 \in C \backslash \{\zeta_0\}$. Again we can suppose that $\zeta_0 = 0$ and $\zeta_1 = \infty$. Then no closed curve in $\mathbb{C}^\infty \backslash C$ can wind around any point of C, otherwise it would disconnect C. Hence, Cauchy's theorem holds in $\mathbb{C}^\infty \backslash C$, and we can repeat the proof of **Theorem 1.1** and **Corollary 1.1.1** to obtain a holomorphic branch of $\log z$ there, and hence on D. Repeating the proof of **Theorem 4.6** yields the desired result.

□

At the other extreme, here is a condition for irregularity.

Theorem 4.8: Boundary Point with Polar Neighbourhood Is Irregular

Let D be a proper subdomain of $\mathbb{C}^\infty$ and let $\zeta_0 \in \partial D$. If there exists a neighbourhood N of ζ_0 such that $\partial D \cap N$ is polar, then ζ_0 is irregular.

Proof:

Suppose, if possible, that there exists a barrier b for ζ_0. We can assume that b is defined on $D \cap N$, where N is a connected open neighbourhood of ζ_0 such that $E := \partial D \cap \overline{N}$ is polar. Then by **Theorem 3.14** $N \backslash E$ is still connected, and so it follows that

$$D \cap N = N \backslash E.$$

Hence by the removable singularity theorem **Theorem 3.13** b has a subharmonic extension to the whole of N. Since $b < 0$ on $N \backslash E$, we have

$$\max(0,b) = 0 \text{ n.e. on } N,$$

so the same equality persists everywhere by the upper semicontinuity of b, and thus $b \leq 0$ on N. Moreover,

$$b(\zeta_0) \geq \limsup_{z \to \zeta_0} b(z) \geq 0,$$

where the first inequality holds by upper semicontinuity and the second holds by the definition of barrier (ii). By maximum principle **Theorem 3.17** $b \equiv 0$ on N, which contradicts the definition of $b < 0$ on $D \cap N$.

□

Theorem 4.7 and **Theorem 4.8** between them provide practical tests for regularity and irregularity which cover the most commonly occurring cases. The next result, though less easy to apply, actually gives a complete characterization of regularity.

Theorem 4.9: Criterion for Regularity

Let D be a proper subdomain of $\mathbb{C}^\infty$ and let $\zeta_0 \in \partial D$. Set $K := \mathbb{C}^\infty \backslash D$. Then the following statements are equivalent:

(a) ζ_0 is a regular boundary point of D.

(b) K is non-thin at ζ_0.

If in addition that $\infty \in D$ then (a) and (b) are also equivalent to

(c) K is non-polar and $p_\nu(\zeta_0) = I(\nu)$, where ν is the equilibrium measure for K.

Proof:

Since both (a) and (b) are invariant under conformal mapping (by **Remark 3.7**) we can suppose from the start that $\infty \in D$ so that $K \subset \mathbb{C}$. We shall prove the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

Step I: (a) $\Rightarrow$ (b)

Suppose that ζ_0 is a regular point for D with barrier b, let u be a function subharmonic on a neighbourhood of ζ_0, and take α such that

$$\limsup_{z \to \zeta_0, z \in K \backslash \{\zeta_0\}} u(z) < \alpha. \tag{4.4}$$

Then there exists $r > 0$ such that if $\Delta = \Delta(\zeta_0, r)$, then u is subharmonic on a neighbourhood of $\overline{\Delta}$ and $u < \alpha$ on $\overline{\Delta} \cap \left(K \backslash \{\zeta_0\}\right)$. Decreasing r if necessary, we can also suppose that b is defined on a neighbourhood of $\overline{\Delta} \backslash K$. Then

$$\left\{\zeta \in \partial\Delta \backslash K : u(\zeta) \geq \alpha\right\}$$

is a compact set on which $b < 0$, so there exists $t > 0$ such that

$$u + tb < \alpha \text{ on } \partial\Delta \backslash K.$$

Now for $\zeta \in \partial(\Delta \backslash K) \backslash \{\zeta_0\}$,

$$\limsup_{z\to\zeta, z\in\Delta\backslash K} (u+tb)(z) \le \begin{cases} (u+tb)(\zeta), & \zeta \in \partial\Delta\backslash K \\ u(\zeta), & \zeta \in (\Delta \cap K)\backslash\{\zeta_0\} \end{cases} \le \alpha.$$

Hence by the extended maximum principle **Theorem 3.17** (ii),

$$u + tb \le \alpha \text{ on } \Delta\backslash K.$$

Since $\lim_{z\to\zeta_0} b(z) = 0$, it follows that

$$\limsup_{z\to\zeta_0, z\in\Delta\backslash K} u(z) \le \alpha.$$

Combining this with **(4.4)** yields

$$\limsup_{z\to\zeta_0, z\neq\zeta_0} u(z) \le \alpha.$$

Hence, by the submean inequality, $u(\zeta_0) \le \alpha$. As this holds for all u and α satisfying **(4.4)**, we conclude that K is non-thin at ζ_0 from definition.

Step II: (b) $\Rightarrow$ (c)

Suppose now that K is non-thin at ζ_0. From **Theorem 3.25** it follws straightforward that K must be non-polar. Moreover, if ν denotes the equlibrium measure of K, then by Frostman's theorem **Theorem 3.7** (ii) the set

$$E := \{z \in K : p_\nu(z) > I(\nu)\}$$

is an F_σ polar set. Using **Theorem 3.25** once more, E is thin at ζ_0 and therefore $K\backslash E$ must be non-thin at ζ_0. Since $p_\nu = I(\nu)$ on $K\backslash E$ by **Theorem 3.7** (ii), it follows that $p_\nu(\zeta_0) = I(\nu)$.

Step III: (c) $\Rightarrow$ (a)

Assume that $p_\nu(\zeta_0) = I(\nu)$. Define $b : D \to [-\infty, \infty)$ by

$$b(z) := I(\nu) - p_\nu(z).$$

Then b is subharmonic on D, and by Frostman's theorem **Theorem 3.7** (i) $b \le 0$ there. Since $b(\infty) = -\infty$, the maximum principle **Theorem 3.17** (ii) implies that in fact $b < 0$ on D. Moreover,

$$\liminf_{z\to\zeta_0} b(z) \ge I(\nu) - p_\nu(\zeta_0) = 0$$

by assumption (c), thus b is a barrier and ζ_0 is regular for D.

□

This result will not be of much practical use until we have a general criterion for thinness (see **Section 5.4**). However, it does have some interesting theoretical consequences. The equivalence of (a) and (b), for example, explains the close correspondence between the earlier theorems in this section and the results about the thinness in **Section 3.8**. More importantly, the equivalence of (a) and (c) shows that the set of irregular points is always small.

Theorem 4.10: Kellogg's Theorem

Let D be a proper subdomain of $\mathbb{C}^\infty$. Then the set of irregular boundary points is an F_σ polar set.

Proof:

By first performing a conformal mapping. We can suppose that $\infty \in D$. Set $K := \mathbb{C}^\infty\backslash D$.

Case I: K is polar

If K is polar, then by **Theorem 4.8** every point of ∂D is irregular, and the result is clear.

Case II: K is non-polar

If K is non-polar, then by step II in the proof of **Theorem 4.9**, the set of irregular points is exactly

$$\left\{z \in K : p_\nu(z) > I(\nu)\right\},$$

where ν is the equilibrium measure for K, and this is an F_σ polar set by an application of Frostman's theorem **Theorem 3.7** (ii).

□

This result has a beautiful and important consequence.

Corollary 4.10.1: Solution of the Generalized Dirichlet Problem

Let D be a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar, and let $\varphi : \partial D \to \mathbb{R}$ be a bounded function which is continuous n.e. on ∂D. Then there exists a unique bounded harmonic function h on D such that

$$\lim_{z\to\zeta} h(z) = \varphi(\zeta) \text{ for n.e. } \zeta \in \partial D.$$

Remark 4.3: Non-Polarity Is Necessary but Is Not a Great Restriction

In order for this result to make sense, it is necessary to assume that ∂D is non-polar. However this is no great restriction, because if ∂D were polar, then by the extended maximum principle **Theorem 3.17** (a), every bounded harmonic function on D would be constant anyway. ⋄

Proof of **Corollary 4.10.1**:

Step I: Existence

Set $h := H_D\varphi$. Then by **Theorem 4.1** h is harmonic and bounded on D. Moreover, by **Theorem 4.3**,

$$\lim_{z\to\zeta} h(z) = \varphi(\zeta), \zeta \in \partial D\backslash(E_1 \cup E_2),$$

where E_1 is the set of irregular boundary points of D, and E_2 is the set of points of discontinuity of φ. Now E_1 is polar by **Theorem 4.10** and E_2 is polar by assumption. Moreover, both E_1 and E_2 are Borel sets, thus

$$\lim_{z\to\zeta} h(z) = \varphi(\zeta) \text{ for n.e. } \zeta \in \partial D.$$

Step II: Uniqueness

Suppose that h_1 and h_2 are two solutions. Then $h_1 - h_2$ is a bounded harmonic function on D satisfying

$$\lim_{z\to\zeta}(h_1 - h_2)(z) = 0 \text{ for n.e. } \zeta \in \partial D.$$

Applying the maximum principle **Theorem 3.17** (b) to $\pm(h_1 - h_2)$ we deduce that $h_2 = h_2$ on D.

□

The fact that this generalized form of the Dirichlet problem can always be solved makes it more suitable for many applications than the original form. Indeed, it will provide the basis for much of the rest of this chapter.

4.3 Harmonic Measure

When studying the Dirichlet problem on a disc Δ in Section 1.2, we not only proved that a unique solution exists, but also gave an explicit formula for it. In the notation we have now developed, this formula may be succinctly expressed by saying that, if $\varphi : \partial\Delta \to \mathbb{R}$ is a continuous function, then

$$H_\Delta\varphi = P_\Delta\varphi \text{ on } \Delta,$$

where $H_\Delta\varphi$ and $P_\Delta\varphi$ are respectively the Perron function and the Poisson integral of φ. We now seek to extend this to more general domains. While the Perron function has already been defined for an arbitrary domain, we currently lack an appropriate analogue for the Poisson integral. To help define this, we introduce the notion of harmonic measure.

Definition: Harmonic Measure and Generalized Poisson Integral

Let D be a proper subdomain of $\mathbb{C}^\infty$, and denote by $\mathscr{B}(\partial D)$ the σ-algebra of Borel subsets of ∂D. A harmonic measure for D is a function

$$\omega_D : D \times \mathscr{B}(\partial D) \to [0,1]$$

such that

(a) For each $z \in D$, the map $B \mapsto \omega_D(z, B)$ is a Borel probability measure on ∂D.

(b) If $\varphi : \partial D \to \mathbb{R}$ is a continuous function, then $H_D\varphi = P_D\varphi$ on D, where $P_D\varphi$ is the generalized Poisson integral of φ on D given by

$$P_D\varphi(z) := \int_{\partial D} \varphi(\zeta) d\omega_D(z,\zeta), \; z \in D.$$

To those who may be concerned, ω_D is a transition probability kernel. Moreover, as in the construction of the harmonic measure all one needs is the generalization of the Poisson integral from $P_\Delta u$ to $P_D u$, in later applications we will implicitly refer to the (generalized) Poisson kernel whenever we revoke the definition of harmonic measure (b).

Example 4.2: Example for Harmonic Measure

Consider $\Delta := \Delta(0,1)$. By **Theorem 1.6** (i),

$$d\omega_\Delta(z,\zeta) := \frac{1}{2\pi} P(z,\zeta)\,|d\zeta|$$

is a harmonic measure for Δ. This conciles the two definitions we have for the Poisson integral $P_\Delta\varphi$. ◇

Since the definition of harmonic measure has been concocted to fit the desired conclusion, it is really only justified once that we have proved the following theorem.

Theorem 4.11: Existence and Uniqueness for Harmonic Measure

Let D be a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar. Then there exists a **unique** harmonic measure ω_D for D.

The case when ∂D is polar is less interesting, see Exercise 1 for example.

Proof of **Theorem 4.11**:

Denote $C(\partial D)$ the space of continuous functions $\varphi : \partial D \to \mathbb{R}$. If $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in C(\partial D)$, then by linearity

$$\alpha_1 H_D\varphi_1 + \alpha_2 H_D\varphi_2$$

is a solution to the generalized Dirichlet problem with boundary value

$$\alpha_1\varphi_1 + \alpha_2\varphi_2$$

(see **Corollary 4.10.1**), so by the uniqueness it follows that

$$H_D(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 H_D\varphi_1 + \alpha_2 H_D\varphi_2 \text{ on } D.$$

Moreover, it is clear from the definition of Perron function that

$$\varphi \geq 0 \text{ on } \partial D \Rightarrow H_D\varphi \geq 0 \text{ on } D,$$
$$\varphi \equiv 1 \text{ on } \partial D \Rightarrow H_D\varphi \equiv 1 \text{ on } D.$$

Hence, for each $z \in D$, the map $\varphi \mapsto H_D\varphi(z)$ is a positive linear functional on $C(\partial D)$ sending the constant function 1 to 1, so by the Riesz representation theorem (see **Remark 3.5**), there exists a unique Borel probability measure μ_z on ∂D such that

$$H_D\varphi(z) = \int_{\partial D} \varphi \, d\mu_z, \; \varphi \in C(\partial D).$$

Setting

$$\omega_D(z, B) := \mu_z(B), \; z \in D, \; B \in \mathscr{B}(\partial D),$$

we see immediately that the definition of harmonic measure holds. This proves the existence of ω_D, the uniqueness follows from the uniqueness part of Riesz representation theorem (see **Remark 3.5**).

□

Harmonic measure is defined so that $H_D\varphi = P_D\varphi$ for all continuous functions

$$\varphi : \partial D \to \mathbb{R}.$$

The next result shows that, as a bonus, the same relation extends to a much wider class of functions φ.

Theorem 4.12: $H_D\varphi = P_D\varphi$ for All Bounded Borel Function φ On Non-Polar ∂D

Let D be a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar. Then

$$H_D\varphi = P_D\varphi \text{ on } D$$

for every bounded Borel function $\varphi : \partial D \to \mathbb{R}$.

This gives us new information, even when D is a disc.

Remark 4.4: $H_D\varphi$ is Linear on Bounded Borel Functions

As $H_D\varphi$ is always harmonic on D by **Theorem 4.1**, the same must be true for $P_D\varphi$. In the same direction, since the map $\varphi \mapsto P_D\varphi$ is clearly linear on bounded Borel functions, the same holds for $\varphi \mapsto H_D\varphi$, which was not obvious before. ◇

Proof of **Theorem 4.12**:

We first show that $H_D\varphi \geq P_D\varphi$ on D when φ is bounded u.s.c. on ∂D and then show that $H_D\varphi \leq P_D\varphi$ on D when φ is bounded l.s.c. on ∂D, and then we shall remove the u.s.c. and l.s.c. conditions.

Step I.1: $H_D\varphi \geq P_D\varphi$ on D when φ is bounded u.s.c. on ∂D

First suppose that φ is bounded and u.s.c. on ∂D. Choose continuous functions $\varphi_n : \partial D \to \mathbb{R}$ such that $\varphi_n \downarrow \varphi$. Then we know that

$$P_D\varphi_n = H_D\varphi_n$$

by **Theorem 4.11** in conjunction with the definition of harmonic measure.

Thus $P_D\varphi_n$ is harmonic on D for each $n \geq 1$ by **Theorem 4.1**. From the monotone convergence theorem we know that

$$P_D\varphi_n \downarrow P_D\varphi \text{ on } D$$

and so by Harnack's theorem **Theorem 1.14** $P_D\varphi$ is harmonic on D. Let $w \in D$ and $\varepsilon > 0$ be arbitrary. By the definition of Perron function, for each $n \geq 1$ we can find a subharmonic function u_n of D such that

$$\limsup_{z\to\zeta} u_n(z) \leq \varphi_n(\zeta),\ \zeta \in \partial D, \text{ and } u_n(w) > H_D\varphi_n(w) - \frac{\varepsilon}{2^n}.$$

Define u on D by

$$u := P_D\varphi + \sum_{n\geq 1}(u_n - H_D\varphi_n).$$

Since $P_D\varphi$ is a harmonic function and $(u_n - H_D\varphi_n)$ is a negative subharmonic function for each n, it follows that u is subharmonic on D. Moreover, if $\zeta \in \partial D$ then for each $n \geq 1$ one has

$$\begin{aligned}\limsup_{z\to\zeta} u(z) &\leq \limsup_{z\to\zeta} \left(P_D\varphi + u_n - H_D\varphi_n\right)(z)\\ &\leq \limsup_{z\to\zeta} u_n(z)\\ &\leq \varphi_n(\zeta)\end{aligned}$$

where the first inequality holds since $(u_n - H_D\varphi_n)$ is negative thus removing it results in a greater value, the second inequality holds since

$$H_D\varphi_n = P_D\varphi_n \downarrow P_D\varphi$$

and thus $P_D\varphi - H_D\varphi_n \leq 0$. Finally, the last inequality holds by the definition of Perron function.

An application of the monotone convergence theorem tells us that

$$\limsup_{z\to\zeta} u(z) \leq \varphi(\zeta).$$

Hence by the definition of Perron function, $H_D\varphi \geq u$ on D. In particular

$$H_D\varphi(w) \geq u(w) \geq P_D\varphi(w) - \sum_{n\geq 1}\frac{\varepsilon}{2^n} = P_D\varphi(w) - \varepsilon,$$

where the last equality holds by the sum of geometric series. Since ε and w are chosen arbitrarily, it follows that

$$H_D\varphi \geq P_D\varphi \text{ on } D.$$

Step I.2: $H_D\varphi \leq P_D\varphi$ when φ is bounded l.s.c. on ∂D

Now suppose that φ is bounded and l.s.c. on ∂D. Applying the argument we did in Step I.1 to $-\varphi$, we obtain

$$H_D(-\varphi) \geq P_D(-\varphi) \text{ on } D.$$

Hence, using **Lemma 4.4** in the first inequality and linearity of the generalized Poisson integral in the last we obtain that

$$H_D(\varphi) \leq -H_D(-\varphi) \leq -P_D(-\varphi) = P_D\varphi \text{ on } D.$$

Step II: $H_D\varphi = P_D\varphi$ when φ is an arbitrary bounded Borel function on ∂D

Finally, suppose that φ is an arbitrary bounded Borel function on ∂D. Let

$w \in D$ and $\varepsilon > 0$. Then, as the Borel probability measure $\omega_D(w, \cdot\,)$ is regular, we can appeal to the **Vitali-Carathéodory theorem**[4] to obtain an u.s.c. function ψ_u and a l.s.c. function ψ_ℓ on ∂D such that

$$\psi_u \le \varphi \le \psi_\ell \text{ and } \int_{\partial D} \big(\psi_u - \psi_\ell\big)(\zeta)\, d\omega_D(w, \zeta) < \varepsilon.$$

Replacing ψ_u by $\max\big(\psi_u, -\|\varphi\|_\infty\big)$ and ψ_ℓ by $\min\big(\psi_\ell, \|\varphi\|_\infty\big)$, we can further suppose that ψ_u and ψ_ℓ are bounded on ∂D. Then by Step I.1 and Step I.2,

$$H_D\psi_u \ge P_D\psi_u \text{ and } H_D\psi_\ell \le P_D\psi_\ell \text{ on } D.$$

Therefore,

$$\begin{aligned} H_D\varphi(w) &\le H_D\psi_\ell(w) && \big(\text{Vitali-Carathéodory Theorem (a)}\big)\\ &\le P_D\psi_\ell(w) && (\text{Step I.2})\\ &\le P_D\psi_u(w) + \varepsilon && \big(\text{Vitali-Carathéodory Theorem (b)}\big)\\ &\le P_D\varphi(w) + \varepsilon && \big(\text{Vitali-Carathéodory Theorem (a)}\big) \end{aligned}$$

and

$$\begin{aligned} H_D\varphi(w) &\ge H_D\psi_u(w) && \big(\text{Vitali-Carathéodory Theorem (a)}\big)\\ &\ge P_D\psi_u(w) && (\text{Step I.1})\\ &\ge P_D\psi_\ell(w) - \varepsilon && \big(\text{Vitali-Carathéodory Theorem (b)}\big)\\ &\ge P_D\varphi(w) - \varepsilon && \big(\text{Vitali-Carathéodory Theorem (a)}\big) \end{aligned}$$

Since w and ε are arbitrary, we conclude that $H_D\varphi = P_D\varphi$ on D.

□

From this result we can deduce a characterization of harmonic measure which explains its nomenclature.

Theorem 4.13: Characterization of Harmonic Measure

Let D be a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar and let B be a Borel subset of ∂D. Then

(a) The function $z \mapsto \omega_D(z, B)$ is harmonic and bounded on D.

(b) If ζ is a regular boundary point of D which lies outside the relative boundary of B in ∂D, then

$$\lim_{z \to \zeta} \omega_D(z, B) = 1_B(\zeta).$$

Moreover, if the relative boundary of B in ∂D is polar, then the function $\omega_D(\,\cdot\,, B)$ is **uniquely** determined by (a) and (b).

Proof:

---

[4] **Vitali-Carathéodory Theorem**: Suppose that μ is a regular Borel measure on a topological space X, and that $\varphi : X \to \mathbb{R}$ is an integrable function. Then, given $\varepsilon > 0$, there exists an u.s.c. function $\psi_u : X \to [-\infty, \infty)$ and a l.s.c. function $\psi_\ell : (-\infty, \infty]$ such that

(a) $\psi_u \le \varphi \le \psi_\ell$.

(b) $\displaystyle\int_X (\psi_u - \psi_\ell)\, d\mu < \varepsilon$.

By **Theorem 4.12** we have

$$\omega_D(z, B) = H_D 1_B(z),\ z \in D.$$

Therefore (a) follows immediately from **Theorem 4.1**. Moreover, if ζ satisfies assumptions in (b), then 1_B is continuous at ζ, and so the conclusion of (b) will follow from **Theorem 4.3**. Finally, the uniqueness part of the result is an immediate consequence of **Corollary 4.10.1**.

□

Remark 4.5: Harmonic Measure and Solution to Generalized Dirichlet Problem

This theorem says that, provided the relative boundary of B in ∂D is polar, the function $\omega_D(\cdot, B)$ is exactly the solution of the generalized Dirichlet problem with boundary date $\varphi = 1_B$. ◇

This provides a quick way of identifying the harmonic measure in a number of important special cases — one simply 'spots' a harmonic measure with the right boundary values, see the examples below.

Example 4.3: Some Examples of Harmonic Measure

Domain D	**Borel Subset** $B \subset \partial D$	Harmonic Measure $\omega_D(z, B)$
$\{\text{Im}(z) > 0\}$	$[a, b]$	$\frac{1}{\pi}\arg\left(\frac{z-b}{z-a}\right)$
$\{\lvert z\rvert < 1, \text{Im}(z) > 0\}$	$\{\lvert z\rvert = 1, \text{Im}(z) > 0\}$	$\frac{2}{\pi}\arg\left(\frac{1+z}{1-z}\right)$
$\{a < \text{Re}(z) < b\}$	$\{\text{Re}(z) = b\}$	$\frac{\text{Re}(z) - a}{b - a}$
$\{\alpha < \arg z < \beta\}$	$\{\arg z = \beta\}$	$\frac{\arg z - \alpha}{\beta - \alpha}$
$\{r < \lvert z\rvert < s\}$	$\{\lvert z\rvert = s\}$	$\frac{\log(\lvert z\rvert / r)}{\log(s/r)}$

Theorem 4.13 also has another interesting consequence.

Corollary 4.13.1: Mutual Absolute Continuity for Harmonic Functions

Let D be a domain in $\mathbb{C}_\infty$ such that ∂D is non-polar. Then the measures

$$\left\{\omega_D(z, \cdot)\right\}_{z \in D}$$

are mutually absolutely continuous. In fact, if $z, w \in D$ then for $B \in \mathscr{B}(\partial D)$,

$$\omega_D(z, B) \le \tau_D(z, w)\omega_D(w, B),$$

where $\tau_D(z, w)$ is the Harnack distance between z and w.

Proof:

We recall the definition for Harnack distance that

$$h(z) \le \tau_D(z, w)h(w)$$

for every positive harmonic function h on D. The result follows by applying this with $h := \omega_D(\cdot, B)$.

□

It thus makes sense to describe subsets of ∂D as having harmonic measure zero without referring to a particular base point $z \in D$. The next result gives some examples of these.

Theorem 4.14: Borel Polar Subset Has Harmonic Measure Zero

Let D be a domain in $\mathbb{C}_\infty$ such that ∂D is non-polar. Then every Borel polar subset of ∂D has harmonic measure zero.

Proof:

Let E be a Borel polar subset of ∂D. If u is a subharmonic function on D such that

$$\limsup_{z \to \zeta} u(z) \leq 1_E(\zeta), \zeta \in \partial D,$$

then by the extended maximum principle **Theorem 3.17** (b) $u \leq 0$ on D. It follows that $H_D 1_E \equiv 0$ on D, and thus by **Theorem 4.12** $P_D 1_E \equiv 0$ on D.

□

It is remarkable to ask whether, conversely, every set of harmonic measure zero must be polar. The answer is unfortunately NO, though this will only become apparent later.

We now prove two basic general inequalities involving harmonic measure, one for subharmonic functions and one for holomorphic functions. Under the perspective of **Theorem 4.14**, the first of these is a generalization of the extended maximum principle.

Theorem 4.15: Two Constant Theorem for Harmonic Measure

Let D be a domain in $\mathbb{C}_\infty$ such that ∂D is non-polar, and let B be a Borel subset of ∂D. If u is subharmonic on D and satisfies

$$u(z) \leq M, z \in D, \text{ and } \limsup_{z \to \zeta} u(z) \leq m, \zeta \in B,$$

where M and m are constants. Then

$$u(z) \leq m\omega_D(z, B) + M\big(1 - \omega_D(z, B)\big)$$

for $z \in D$.

Proof:

Set $\varphi := m1_B + M(1 - 1_B)$ on ∂D. Then

$$\limsup_{z \to \zeta} u(z) \leq \varphi(\zeta) \ \forall \zeta \in \partial D$$

by assumption. Thus by the definition of Perron function, $u \leq H_D\varphi$ on D. Using **Theorem 4.12** in the first inequality gives

$$u \leq P_D\varphi = mP_D 1_B + M(1 - P_D 1_B) \text{ on } D,$$

which yields the desired inequality.

□

Theorem 4.16: Subordination Principle for Harmonic Measure

Let D_1 and D_2 be domains in $\mathbb{C}_\infty$ with non-polar boundaries, and let B_1 and B_2 be Borel subsets of ∂D_1 and ∂D_2, respectively. Let

$$f : D_1 \cup B_1 \to D_2 \cup B_2$$

be a continuous map which is meromorphic on D_1, and suppose that

$$f(D_1) \subset D_2 \text{ and } f(B_1) \subset B_2.$$

Then

$$\omega_{D_2}\left(f(z), B_2\right) \geq \omega_{D_1}(z, B_1),\ z \in D_1,$$

with equality holds if f is also a homeomorphism of $D_1 \cup B_1$ onto $D_2 \cup B_2$.

Proof:

Set $\varphi_1 := 1 - 1_{B_1}$ and $\varphi_2 := 1 - 1_{B_2}$ on ∂D_1 and ∂D_2, respectively. Let u be a subharmonic function on D_2 such that

$$\limsup_{z \to \zeta} u(z) \leq \varphi_2(\zeta),\ \zeta \in \partial D_2.$$

Then by **Corollary 1.1.3** $u \circ f$ is subharmonic on D_1, thus

$$\limsup_{z \to \zeta} \left(u \circ f\right)(z) \leq \varphi_1(\zeta),\ \zeta \in \partial D_1$$

and therefore

$$u \circ f \leq H_{D_1}\varphi_1 \text{ on } D_1.$$

As this holds for all such u, we deduce that

$$(H_{D_2}\varphi_2) \circ f \leq H_{D_1}\varphi_1 \text{ on } D_1.$$

By **Theorem 4.12**,

$$H_{D_j}\varphi_j = P_{D_j}\varphi_j = 1 - P_{D_j}1_{B_j},\ j = 1{,}2$$

and hence

$$(P_{D_2}1_{B_2}) \circ f \geq P_{D_1} \circ 1_{B_1} \text{ on } D_1,$$

which is the desired inequality. Finally, if f is in addition a homeomorphism of $D_1 \cup B_1$ onto $D_2 \cup B_2$, then we can apply the same argument to f^{-1} to obtain the equality.

□

Corollary 4.16.1: Domain Monotonicity for Harmonic Measure

Let D_1 and D_2 be domains in $\mathbb{C}^\infty$ with non-polar boundaries, and suppose that $D_1 \subset D_2$. If B is a Borel subset of $\partial D_1 \cap \partial D_2$ then

$$\omega_{D_1}(z, B) \leq \omega_{D_2}(z, B),\ z \in D_1.$$

Proof:

Take $f : D_1 \cup B \to D_2 \cup B$ to be the inclusion map in **Theorem 4.16**.

□

As an application of these ideas, we shall prove a theorem about asymptotic values.

Definition: Asymptotic Value

Let φ be a function defined on an unbounded domain D in $\mathbb{C}$. Then a is an asymptotic value of φ if there exists a path $\Gamma : [0,\infty) \to D$ such that

$$\lim_{t \to \infty} \Gamma(t) = \infty \text{ and } \lim_{t \to \infty} \varphi\left(\Gamma(t)\right) = a.$$

Theorem 4.17: Asymptotic Value for Subharmonic Growth on Sector of Half-Plane

Let u be a subharmonic function on $\mathbb{H} := \{z : \operatorname{Im}(z) > 0\}$ such that $u \leq 0$ on $\mathbb{H}$. If $a \in [-\infty,0)$ is an asymptotic value of u, then $\forall \alpha \in (0,\pi/2]$,

$$\limsup_{z \to \infty, z \in S_\alpha} u(z) \leq \frac{\alpha}{\pi} a,$$

where S_α is the sector $\left\{z \in \mathbb{H} : \alpha \leq \arg z \leq \pi - \alpha\right\}$.

Proof:

Let $\Gamma : [0,\infty) \to \mathbb{H}$ be a path such that

$$\lim_{t\to\infty} \Gamma(t) = \infty \text{ and } \lim_{t\to\infty} \varphi\big(\Gamma(t)\big) = a.$$

Take $\widetilde{a}$ such that $a < \widetilde{a} < 0$ and choose $R > 0$ sufficiently large such that $u < \widetilde{a}$ on $\Gamma \cap D_R$, where

$$D_R := \big\{z \in \mathbb{H} : |z| > R\big\}.$$

We may also suppose that Γ meets the circle $\{\,|z| = R\}$. Fix $z \in D_R \backslash \Gamma$, and let W be the component of $D_R \backslash \Gamma$ containing z. Then since $u \leq \widetilde{a}$ on $\partial W \backslash \partial D_R$, the two-constant theorem **Theorem 4.15** gives

$$u(z) \leq \widetilde{a}\omega_W\big(z, \partial W \backslash \partial D_R\big).$$

We now seek to estimate the right hand side of this inequality. Notice that since $\widetilde{a} < 0$, this means finding a lower bound for the harmonic measure. To this end we use **Corollary 4.16.1** in the second inequality and obtain

$$\begin{aligned}\omega_W\big(z, \partial W \backslash \partial D_R\big) &= 1 - \omega_W\big(z, \partial W \cap \partial D_R\big)\\ &\geq 1 - \omega_{D_R}\big(z, \partial W \cap \partial D_R\big)\\ &= \omega_{D_R}\big(z, \partial D_R \backslash \partial W\big)\,.\end{aligned}$$

Now ∂W cannot meet both $(-\infty, -R]$ and $[R, \infty)$, for then Γ would disconnect the connected set W. To this end we consider two cases.

Case I: $\partial W \cap (-\infty, -R] = \varnothing$.

If so, using **Corollary 4.16.1** in the first inequalitty, **Theorem 4.12** in the equality, and definition of the sector in the last inequality gives

$$\begin{aligned}\omega_{D_R}\big(z, \partial D_R \backslash \partial W\big) &\geq \omega_{D_R}\big(z, (-\infty, -R]\big)\\ &= H_{D_R} 1_{(-\infty,-R]}(z)\\ &\geq \frac{1}{\pi} \arg z - H_{D_R} 1_{C_R}(z)\end{aligned}$$

where $C_R := \big\{\zeta \in \partial D_R : |\zeta| = R\big\}$.

Case II: $\partial W \cap (R, \infty] = \infty$.

If so, a similar argument as in the first case shows that

$$\omega_{D_R}\big(z, \partial D_R \backslash \partial W\big) \geq \frac{1}{\pi}\big(\pi - \arg z\big) - H_{D_R} 1_{C_R}(z).$$

Finally we can estimate the right hand side of $u(z) \leq \widetilde{a}\omega_W\big(z, \partial W \backslash \partial D_R\big)$.

Claim: $\displaystyle\limsup_{z\to\infty, z\in S_\alpha} u(z) \leq \widetilde{a}\,\frac{\alpha}{\pi}$.

Combining the estimates in the two cases we derive the conclusion that

$$u(z) \leq \widetilde{a}\,\frac{1}{\pi} \min\big(\arg z, \pi - \arg z\big) - \widetilde{a} H_{D_R} 1_{C_R}(z).$$

Note that, although this inequality was proved under the assumption that $z \in D_R \backslash \Gamma$, it evidently holds if $z \in D_R \cap \Gamma$ as well. Hence, in particular,

$$u(z) \leq \widetilde{a}\,\frac{\alpha}{\pi} - \widetilde{a} H_{D_R} 1_{C_R}(z),\ z \in D_R \cap S_\alpha.$$

Finally, $1_{C_R} = 0$ on a neighbourhood on ∞, which is a regular boundary point of D_R, so by **Theorem 4.3** $\lim_{z\to\infty} H_{D_R}1_{C_R}(z) = 0$. It follows that

$$\limsup_{z\to\infty, z\in S_\alpha} u(z) \leq \widetilde{a}\,\frac{\alpha}{\pi},$$

and since $\widetilde{a} > a$ is chosen arbitrarily, sending $\widetilde{a} \downarrow a$ gives the desired result.

□

Remark 4.6: Asymptotic Bound in **Theorem 4.17** Is Sharp

The harmonic function $u = -\arg z$, which has $-\pi$ as an asymptotic value, shows that the above bound is sharp. ⋄

Of course, the function in **Remark 4.6** also has many other asymptotic values. By contrast, a bounded holomorphic function on $\mathbb{H}$ has at most one. This is proved in the following result.

Corollary 4.17.1: Lindelöf Theorem

Let f be a bounded holomorphic function on $\mathbb{H} := \{z : \operatorname{Im}(z) > 0\}$. If α is an asymptotic value of f then, for each sector S_α as in **Theorem 4.17**, $f(z) \to a$ uniformly as $z \to \infty$ in S_α. In particular, f can have at most one asymptotic value.

Proof:

Applying **Theorem 4.17** with $u := \log\left(\frac{|f-a|}{M}\right)$, where $M = \sup_{\mathbb{H}} |f - a|$.

□

These results provide a good illustration of how many problems in potential theory and complex analysis can be reduced to question about harmonic measure. It is therefore of great importance to be able to compare, or at least estimate, the harmonic measure for as many domains as possible. Simple cases can be treated using conformal mapping. As an illustration, we now compute the important example of harmonic measure for the half plane $\mathbb{H}$.

Theorem 4.18: Harmonic Measure for Half-Plane

Let $\mathbb{H} := \left\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\right\}$. If B is a Borel subset of $\mathbb{R}$. Then

$$\omega_{\mathbb{H}}(x+iy, B) = \frac{1}{\pi}\int_B \frac{y}{(x-t)^2+y^2}dt,\ x+iy \in \mathbb{H}.$$

Proof:

Set $\Delta := \Delta(0,1)$ and let $f : \mathbb{H} \to \Delta$ be the conformal mapping

$$f(z) := \frac{z-i}{z+i},\ z \in \mathbb{H}.$$

Then one has, using **Theorem 4.17** conformal case in the first equality, that

$$\begin{aligned}\omega_{\mathbb{H}}(z,B) = \omega_\Delta\big(f(z), f(B)\big) &= \frac{1}{2\pi}\int_{f(B)} \frac{1-|f(z)|^2}{|\zeta - f(z)|^2}|d\zeta| \\ &= \frac{1}{2\pi}\int_B \frac{1-|f(z)|^2}{|f(t)-f(z)|^2}|f'(t)|dt \\ &= \frac{1}{\pi}\int_B \frac{\mathrm{Im}(z)}{|z-t|^2}dt\end{aligned}$$

where the second equality holds by Poisson's integral, the third equality holds by conformal mapping, and the last holds by conformal invariance. Substituting $z := x + iy \in \mathbb{H}$ yields the desired result.

□

The problem of estimating harmonic measure for more complicated domains is vast subject, well beyond the scope of this book. We shall content ourselves with one general estimate for simply connected domains, which will be a by-product of the Carleman-Milloux theorem in **Section 4.5**. We conclude this section by relating the harmonic measure to the equilibrium measure.

Theorem 4.19: Equilibrium and Harmonic Measure Agree on Component with ∞

Let K be a compact non-polar subset of $\mathbb{C}$. Then its equilibrium measure ν is given by

$$\nu := \omega_D(\infty, \cdot\,),$$

where D is the component of $\mathbb{C}^\infty \backslash K$ containing ∞.

Proof:

Denote ω for $\omega_D(\infty, \cdot\,)$ so ω is a Borel probability measure on K by definition. If we define

$$u(z) := \begin{cases} p_\omega(z) - p_\nu(z) + I(\nu), & z \in D\backslash\{\infty\} \\ I(\nu), & z = \infty \end{cases}$$

then u is subharmonic on D and

$$\limsup_{z\to\zeta} u(z) \le p_\omega(\zeta)\ \forall \zeta \in \partial D.$$

Denote $\varphi_n := \max(p_\omega, -n)$ on ∂D, it follows that

$$u \le H_D\varphi_n = P_D\varphi_n \text{ on } D,$$

where the equality holds by **Theorem 4.12**. Sending $n \to \infty$ we deduce that

$$u(z) \le \int_{\partial D} p_\omega(\zeta) d\omega_D(z,\zeta),\ z \in D.$$

In particular, setting $z = \infty$ we obtain $I(\nu) \le I(\omega)$. This implies that ω is an equilibrium measure for K, and by uniqueness **Theorem 3.21** it follows that $\nu = \omega$, as desired.

□

4.4 Green Functions

The harmonic measure of a domain is intimately related to another important invariant, the Green function. In essense, a Green function is a family of fundamental solutions of the Laplacian, each of which is zero on the boundary. Here is the precise definition.

Definition: Green Function

Let D be a proper subdomain of $\mathbb{C}^\infty$. A Green function for D is a map

$$g_D : D \times D \to (-\infty, \infty]$$

such that $\forall w \in D$ one has

(a) $g_D(\,\cdot\,, w)$ is harmonic on $D\backslash\{w\}$, and bounded outside each neighbourhood of w.

(b) $g_D(w, w) = \infty$ and as $z \to w$

$$g_D(z, w) = \begin{cases} \log|z| + O(1), & w = \infty \\ -\log|z - w| + O(1), & w \neq \infty \end{cases}$$

(c) $g_D(z, w) \to 0$ as $z \to \zeta$ for n.e. $\zeta \in \partial D$.

Example 4.4: Green Function for Unit Disk

Consider $\Delta := \Delta(0,1)$ then

$$g_\Delta(z, w) := \log\left|\frac{1 - z\overline{w}}{z - w}\right|$$

is a Green function for Δ. ◇

As usual, to justify the definition we now verify the existence and the uniqueness.

Theorem 4.20: Existence and Uniqueness of Green Function

If D is a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar, then there exists a **unique** Green function g_D for D.

Once again, the case when ∂D is polar is less interesting, see **Exercise 1**.

Proof of **Theorem 4.20**:

Step I: Uniqueness

Suppose that g_1 and g_2 are two Green functions for D. Given $w \in D$, define

$$h(z) := g_1(z, w) - g_2(z, w),\ z \in D\backslash\{w\}.$$

Then by the definition of Green function (a), h is harmonic and bounded on $D\backslash\{w\}$, and by (b) $\lim_{z\to\zeta} h(z) = 0$ for n.e. $\zeta \in \partial D$, so by the extended maximum principle **Theorem 3.17** (a),

$$h \equiv 0 \text{ on } D\backslash\{w\}.$$

As w is chosen arbitrarily, it follows that $g_1 = g_2$ on $D \times D$.

Step II: Existence

By the definition of Green function (b), we shall prove the existence of $w = \infty$ and $w \neq \infty$ respectively.

Step II.1: $g_D(z, w)$ exists when $w = \infty \in D$

Set $K := \mathbb{C}^\infty\backslash D$, so that K is a compact non-polar subset of $\mathbb{C}$, and let ν be its equilibrium measure. If we define

$$g_D(z, \infty) := \begin{cases} p_\nu(z) - I(\nu), & z \in D\backslash\{\infty\} \\ \infty, & z = \infty \end{cases}$$

using Frostman's theorem **Theorem 3.7** in each condition, (a) holds by using in addition **Theorem 3.1** (i) and (b) holds by using addition the definition of $g_D(z, w)$.

Step II.2: $g_D(z, w)$ exists when $w \neq \infty$, $w \in D$.
Now, for $w \in D$, $w \neq \infty$, define

$$g_D(z,w) := g_{\widetilde{D}}\Big(\frac{1}{z-w}, \infty\Big), z \in D$$

where $\widetilde{D}$ is the image of D under the map $z \mapsto (z-w)^{-1}$. By **Example 4.4** applying to the domain $\widetilde{D} := \Delta(0,1)$, result follows as z and w are arbitrary.

□

We now start to investigate the properties of Green functions, the most elementary one is its positivity.

Theorem 4.21: Green Function Is Positive

Let D be a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar. Then

$$g_D(z,w) > 0 \ \forall z, w \in D.$$

Proof:

Fix $w \in D$, and define

$$u(z) := -g_D(z,w), z \in D.$$

Then u is subharmonic and bounded above on D and $\lim_{z \to \zeta} u(z) = 0$ for n.e. $\zeta \in \partial D$. Hence by the extended maximum principle **Theorem 3.17** (b) $u \leq 0$ on D. Moreover, if we were at the case $u(z) = 0$ for some $z \in D$, then by the standard maximum principle **Theorem 2.5** (i) it would follow that $u \equiv 0$ on D, which is impossible! For example

$$u(w) := -g_D(w,w,) = -\infty$$

by definition. Hence $u < 0$ on D and the positivity follows from the definition of u.

□

As with the Harnack distance and harmonic measure, Green function admits a subordination principle for meromorphic functions.

Theorem 4.22: Subordination Principle for Green Function

Let D_1 and D_2 be domains in $\mathbb{C}^\infty$ with non-polar boundaries, and let $f : D_1 \to D_2$ be a meromorphic function. Then

$$g_{D_2}\big(f(z), f(w)\big) \geq g_{D_1}(z,w), z, w \in D_1,$$

with equality holds if and only if f is a conformal mapping of D_1 onto D_2.

Proof:

By the definition of Green function, we will consider the case $w = \infty$ and $w \neq \infty$ respectively. Then in the last step we prove the case for conformal mapping.

Step I: $w \neq \infty$

Suppose so and define

$$u(z) := g_{D_1}(z,w) - g_{D_2}\big(f(z), f(w)\big), z \in D_1 \backslash \{w\}.$$

Then u is subharmonic on $D_1 \backslash \{w\}$ and u is bounded above outside each neighbourhood of w. Moreover, as $z \to w$,

$$u(z) = \log\left|\frac{f(z) - f(w)}{z-w}\right| + O(1) = \log|f'(w)| + O(1),$$

so in fact u is bounded above on $D_1\backslash\{w\}$. Finally, by **Theorem 4.21** $g_{D_2} > 0$ hence

$$\limsup_{z\to\zeta} u(z) \le \lim_{z\to\zeta} g_{D_1}(z,w) = 0 \text{ for n.e. } \zeta \in \partial D_1.$$

Hence by the extended maximum principle **Theorem 3.17** (b), $u \le 0$ on $D_1\backslash\{w\}$, which gives the desired inequality.

Step II: $w = \infty$

Suppose so, recall **Step II.2** in the proof of **Theorem 4.20**, we have

$$g_{D_1}(z,\infty) = g_{\widetilde{D}_1}(1/z, 0),$$

where $\widetilde{D}_1$ is the image of D_1 under the map $z \mapsto \frac{1}{z}$. Hence the result follows by applying **Step I** to the function

$$z \mapsto f(1/z) : \widetilde{D}_1 \to D_2.$$

The case when $f(w) = \infty$ is treated similarly.

Step III: Equality when f is conformal

Finally, if f is conformal from D_1 onto D_2, applying the inequality we just proved in the first two steps to f and f^{-1} respectively yields the desired result.

□

This result allows us to compute Green functions for some simple domains by means of conformal mapping. A few examples will be given in **Example 4.5**.

Another consequence of **Theorem 4.22** is that g_D increases with D.

Corollary 4.22.1: Domain Monotonicity for Green Function

Let D_1 and D_2 be domains in $\mathbb{C}^\infty$ with non-polar boundaries. If $D_1 \subset D_2$ then

$$g_{D_1}(z,w) \le g_{D_2}(z,w),\ z,w \in D_1.$$

Proof:

Take $f : D_1 \to D_2$ to be the inclusion map.

□

In fact g_D increases continuously with D, in the following sense.

Theorem 4.23: Green Function Is Continuous in Increase of Domain

Let D be a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar, and let $\{D_n\}_{n\ge 1}$ be subdomains of D such that $D_1 \subset D_2 \subset \cdots$ and $\bigcup_{n\ge} D_n = D$. Then

$$\lim_{n\to\infty} g_{D_n}(z,w) = g_D(z,w),\ z,w \in D.$$

Proof of **Theorem 4.23**:

Fix $w \in D$. Then $w \in D_{n_0}$ for some n_0, and by renumerating the sequence $\{D_n\}_{n\ge 1}$, we may suppose that $n_0 = 1$. For $n \ge 1$ define

$$h_n(z) := g_D(z,w) - g_{D_n}(z,w),\ z \in D_n\backslash\{w\}.$$

Then h_n is harmonic on $D_n\backslash\{w\}$ and bounded near w, so by the removale singularity theorem **Corollary 3.13.1** in conjunction with **Remark 3.1** (i), h_n extends to be harmonic on D_n. Now **Corollary 4.22.1** implies that

$$h_n \ge h_{n+1} \text{ on } D_n \text{ for each } n,$$

thus $u := \lim_{n\to\infty} h_n$ is subharmonic on D by **Theorem 2.12**. Since

$$h \le g_D(\,\cdot\,, w) \text{ on } D_n \text{ for each } n,$$

it follows that

$$u \le g_D(\,\cdot\,, w) \text{ on } D.$$

Hence u is bounded above on D, and also

$$\limsup_{z\to\zeta} u(z) \le 0 \text{ for n.e. } \zeta \in \partial D$$

by the definition of Green function. Therefore using the extended maximum principle **Theorem 3.17** we have $u \le 0$ on D. This tells us that

$$\liminf_{n\to\infty} g_{D_n}(z, w) \ge g_D(z, w),\ z \in D.$$

But from **Corollary 4.22.1** we also have

$$\limsup_{n\to\infty} g_{D_n}(z, w) \le g_D(z, w),\ z \in D.$$

Combining the two displays yields the desired result.

□

Example 4.5: Some Examples of Green Function

Domain D	Green Function $g_D(z, w)$
$\{\,\lvert z\rvert < \rho\}$	$\log\left\lvert \dfrac{\rho^2 - z\overline{w}}{\rho(z-w)} \right\rvert$
$\{\mathrm{Im}(z) > 0\}$	$\log\left\lvert \dfrac{z - \overline{w}}{z - w} \right\rvert$
$\{\mathrm{Re}(z) > 0\}$	$\log\left\lvert \dfrac{z + \overline{w}}{z - w} \right\rvert$
$\left\{\,\lvert \arg z\rvert < \dfrac{\pi}{2\alpha}\right\}$	$\log\left\lvert \dfrac{z^\alpha + \overline{w}^\alpha}{z^\alpha - w^\alpha} \right\rvert$
$\left\{\,\lvert \mathrm{Re}(z)\rvert < \dfrac{\pi}{2\alpha}\right\}$	$\log\left\lvert \dfrac{e^{i\alpha z} + e^{-i\alpha\overline{w}}}{e^{i\alpha z} - e^{i\alpha w}} \right\rvert$

For bounded domains there is an integral formula for Green function in terms of the harmonic measure. In some literature the following result is also known as the fundamental identity for Green function (or sometimes Green kernel).

Theorem 4.24: Fundamental Identity for Logarithmic Potential

Let D be a bounded domain in $\mathbb{C}$. Then

$$g_D(z, w) = \int_{\partial D} \log\lvert \zeta - w\rvert\, d\omega_D(z, \zeta) - \log\lvert z - w\rvert$$

for $z, w \in D$.

Proof:

Given $w \in D$, define $\varphi_w : \partial D \to \mathbb{R}$ by

$$\varphi_w(\zeta) := \log\lvert \zeta - w\rvert,\ \zeta \in \partial D.$$

Then $P_D\varphi_w$ is harmonic and bounded on D and

$$\lim_{z\to\zeta} P_D\varphi_w(z) = \varphi_w(\zeta) \text{ for n.e. } \zeta \in \partial D.$$

Therefore the function

$$(z, w) \mapsto P_D\varphi_w(z) - \log|z - w|$$

satisfies condition (a), (b), and (c) for being Green function, and so by the uniqueness part of **Theorem 4.20** it must be the Green function g_D.

□

The importance of this formula is that it tells us how $g_D(z, w)$ depends on w, which is the key to proving the following symmetry theorem for Green functions. In view of the asymmetry way that $g_D(z, w)$ was defined, this is perhaps a surprising result.

Theorem 4.25: Symmetry Theorem for Green Function

Let D be a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar. Then

$$g_D(z, w) = g_D(w, z)$$

for $z, w \in D$.

Proof:

Applying a conformal invariance, we can suppose that $D \subset \mathbb{C}$. Then D can be exhausted by an increasing sequence of bounded subdomains $\{D_n\}_{n\geq 1}$, and by **Theorem 4.23**, g_D will be symmetric provided that each g_{D_n} is symmetric. It is thus sufficient to prove the result in the case when D is a bounded subdomain of $\mathbb{C}$.

Fix such a domain D, and let $w \in D$. Define u on $D\backslash\{w\}$ by

$$u(z) := g_D(z, w) - g_D(w, z), z \in D\backslash\{w\}.$$

Switching the rôles of z and w in **Theorem 4.24** we have

$$u(z) = g_D(z, w) + \log|z - w| - \int_{\partial D} \log|\zeta - z|\, d\omega_D(w, \zeta)$$

for $z \in D\backslash\{w\}$. By **Theorem 2.14**, this formula shows that u is subharmonic on $D\backslash\{w\}$. It also tells us that u is bounded above there. In addition, from the original definition of u, we have

$$\limsup_{z\to\zeta} u(z) \leq \lim_{z\to\zeta} g_D(z, w) = 0$$

for n.e. $\zeta \in \partial D$. Hence by the extended maximum principle **Theorem 3.17** it follows that $u \leq 0$ on $D\backslash\{w\}$. Thus

$$g_D(z, w) \leq g_D(w, z), z \in D.$$

Finally, since w is arbitrary, result follows.

□

As part of our definition of Green function,

$$\lim_{z\to\zeta} g_D(z, w) = 0 \text{ for n.e. } \zeta \in \partial D,$$

but it is not clear whether the exceptional set depends on w. The next result shows that it doesn't, and identifies it precisely.

Theorem 4.26: Criterion for Solvability of Dirichlet Problem via Green Function

Let D be a domain in $\mathbb{C}^\infty$ such that ∂D is non-polar, let $w \in D$, and let $\zeta \in \partial D$. Then the following statements are equivalent:

(a) $\lim_{z \to \zeta} g_D(z, w) = 0.$

(b) ζ is a regular boundary point of D.

Proof:

Step I: (a) $\Rightarrow$ (b)

If $\lim_{z \to \zeta} g_D(z, w) = 0$ when $-g_D(\,\cdot\,, w)$ is a barrier at ζ by definition and so ζ is regular.

Step II: (b) $\Rightarrow$ (a)

Conversely, suppose that ζ is a regular boundary point of D. Let N be a relatively compact neighbourhood of w in D, and define $\varphi : \partial(D\backslash\overline{N}) \to \mathbb{R}$ by

$$\varphi(\zeta) := \begin{cases} 0, & \zeta \in \partial D \\ g_D(z, w), & \zeta \in \partial N \end{cases}$$

Then $g_D(\,\cdot\,, w)$ solves the generalized Dirichlet problem on $D\backslash\overline{N}$ with boundary function φ (see **Corollary 4.10.1**), so by uniqueness **Theorem 1.5** one has

$$g_D(z, w) = H_{D\backslash\overline{N}}\varphi(z), z \in D\backslash\overline{N}.$$

Hence, as ζ is a regular point for D, and thus also for $D\backslash\overline{N}$. By **Theorem 4.3** it follows that

$$\lim_{z \to \zeta} g_D(z, w) = \varphi(\zeta) = 0.$$

□

This result provides a characterization of regular points which is internal to D. This has some interesting consequences, for example:

Example 4.6: Regularity Is Stable Under Conformal Mapping

If a domain D is regular, then so is every domain $\widetilde{D}$ conformally equivalent to D, regardless of how $\widetilde{D}$ is embedded in $\mathbb{C}^\infty$. ◇

In fact, **Example 4.6** is a consequence of the Kelvin transform, see Port and Sidney **Section 4.3**.

We now use **Example 4.6**, in conjunction with the symmetry property of Green function, to prove a strong converse to the subordination principle we proved for Green function in **Theorem 4.22**.

Theorem 4.27: Characterization of Conformal Mapping via Green Function

Let D_1 and D_2 be domains in $\mathbb{C}^\infty$ with non-polar boundaries, and let $f : D_1 \to D_2$ be a meromorphic function.

(a) If there exist distinct points $z_0, w_0 \in D_1$ such that

$$g_{D_2}\big(f(z_0), f(w_0)\big) = g_{D_1}(z_0, w_0).$$

Then $g_{D_2}\big(f(z), f(w)\big) = g_{D_1}(z, w)$ $\forall z, w \in D_1$ and f injective.

(b) If in addition D_1 is a regular domain, then f is also surjective, and is therefore a conformal mapping of D_1 onto D_2.

Proof:

Step I: (a)

Define, for $z \in D_1\backslash\{w_0\}$, that

$$u(z) := g_{D_1}(z, w_0) - g_{D_2}\big(f(z), f(w_0)\big).$$

Then u is subharmonic on $D_1\backslash\{w_0\}$ and by the subordination principle **Theorem 4.22** $u \leq 0$ there. Since, by assumption in (a), $u(z_0) = 0$, it follows from the maximum principle **Theorem 2.5** that $u \equiv 0$ and hence

$$g_{D_2}\big(f(z), f(w_0)\big) = g_{D_1}(z, w_0),\ z \in D_1.$$

Now by the symmetry **Theorem 4.25** we can switch the rôles of z and w_0, repeating the argument gives

$$g_{D_2}\big(f(z), f(w)\big) = g_{D_1}(z, w),\ z, w \in D_1.$$

This implies that f is injective since

$$\begin{aligned} z \neq w &\Rightarrow g_{D_1}(z, w) < \infty && \text{(Definition of Green Function (a))} \\ &\Rightarrow g_{D_2}\big(f(z), f(w)\big) < \infty && \text{(Above Display)} \\ &\Rightarrow f(z) \neq f(w) && \text{(Definition of Green Function (b))} \end{aligned}$$

Step II: (b)

Suppose not, that is, $f(D_1) \neq D_2$. Then an elementary connectedness argument shows that $\partial f(D_1) \cap D_2 \neq \varnothing$. Let η be a point in this set, and choose $\{z_n\}_{n\geq 1} \subset D_1$ such that $f(z_n) \to \eta$. Replacing $\{z_n\}_{n\geq 1}$ by a sequence, if necessary, we may also suppose that $z_n \to \zeta \in \partial D_1$. Then for any $w \in D_1$, we have

$$\begin{aligned} \lim_{n\to\infty} g_{D_1}(z_n, w) &= \lim_{n\to\infty} g_{D_2}\big(f(z_n), f(w)\big) \\ &= g_{D_2}\big(\eta, f(w)\big) \\ &> 0 \end{aligned}$$

where the first equality holds by (a), the second equality holds by continuity in **Theorem 4.23**, and the last equality holds by positivity in **Theorem 4.21**. Therefore, by **Theorem 4.26** ζ must be an irregular point of ∂D_1, contradicting assumption in (b). Thus if D_1 is a regular domain, then necessarily $f(D_1) = D_2$.

□

As an application of this result, we obtain a simple proof of the Riemann mapping theorem.

Theorem 4.28: Riemann Mapping Theorem

If D is a simply connected proper subdomain of $\mathbb{C}$, then there exists a conformal mapping of D onto the unit disc Δ.

Proof:

By **Theorem 4.6**, D is a regular domain. In particular, ∂D is non-polar, so D has a Green function g_D by **Theorem 4.20**. Fix $w \in D$, we define

$$h(z) := g_D(z, w) + \log|z - w|,\ z \in D\backslash\{w\}.$$

Then h is harmonic on $D\backslash\{w\}$ and bounded near w, so by the removable singularity theorem **Corollary 3.13.1** h extends to be harmonic on D. Applying **Theorem 1.1** (i), we can write $h := \mathrm{Re}(f_1)$ for some holomorphic function f_1 on D. Define

$$f(z) := (z - w)^{e^{-f_1(z)}},\ z \in D.$$

Then f is holomorphic on D and $f(w) = 0$. Moreover,

$$\log|f(z)| = -g_D(z, w),\ z \in D,$$

which shows that f maps D into Δ and that

$$g_\Delta\big(f(z), f(w)\big) = g_D(z, w),\ z \in D.$$

Theorem 4.27 (b) applies and we conclude that f is the desired conformal mapping from D onto Δ.

□

Remark 4.7: Conformal Mapping Does Not Extend to Homeomorphism of Closures

In general, the conformal mapping $f : D \to \Delta$ will not extend to a homeomorphism of the closures. ⋄

For this to be possible, it is clear that every boundary point of D must be accessible, in the following sense.

Definition: Accessible Point

A point $\zeta \in \partial D$ is said to be accessible if, for each sequence $\{z_n\}_{n\geq 1} \subset D$ with $\lim\limits_{n\to\infty} z_n = \zeta$, there exists a path $\Gamma : [0,\infty) \to D$ with $\lim\limits_{t\to\infty} \Gamma(t) = \zeta$ such that $\Gamma(t_n) = z_n$ for some increasing sequence $t_n \to \infty$.

It turns out that this simple necessary condition is also sufficient.

Theorem 4.29: Sufficiency for Extension to Homeomorphism on Closure

Let D be a bounded simply connected domain in $\mathbb{C}$, and let $f : D \to \Delta$ be a conformal mapping of D onto the unit disc Δ.

(a) If $\zeta \in \partial D$ is accessible then f extends continuously to $D \cup \{\zeta\}$ and $|f(\zeta)| = 1$.

(b) If $\zeta, \tilde{\zeta} \in \partial D$ are distinct accessible points then $f(\zeta) \neq f(\tilde{\zeta})$.

(c) If every boundary point of D is accessible then f extends to a homeomorphism of $\overline{D}$ onto $\overline{\Delta}$.

Proof:

Step I: $|f(\zeta)| = 1$ in (a)

One has

$$\lim_{z\to\zeta} |f(z)| = \lim_{z\to\zeta} e^{-g_\Delta(f(z),0)} = \lim_{z\to\zeta} e^{-g_D(z,f(0))} = 1, \qquad (4.5)$$

where the first equality holds by **Theorem 4.22** conformal case, the second equality holds by **Theorem 4.27** (a), and the last by **Theorem 4.26** (a). So any continuous extension of f to ζ must satisfy $|f(\zeta)| = 1$.

Step II: Extension in (a) exists

To show that such an extension exists, we argue by contradiction. Suppose not, then there exists a sequence $\{z_n\}_{n\geq 1} \subset D$ with $\lim\limits_{n\to\infty} z_n = \zeta$ such that

$$f(z_{2n}) \to \alpha \text{ and } f(z_{2n+1}) \to \beta$$

for some $\alpha \neq \beta$. From (4.5) it is clear that

$$|\alpha| = |\beta| = 1.$$

Multiplying a constant to f, we may assume that $\beta = \overline{\alpha}$. Let N be an integer such that $\dfrac{2\pi}{N} < |\alpha - \beta|$, and define

$$u(z) := \log\left|f^{-1}(z) - \zeta\right|,\ z \in \Delta$$

$$v(z) := \sum_{k=1}^{N} \left(u\left(e^{2\pi ik/N} z\right) + u\left(e^{2\pi ik/N} \overline{z}\right) \right), z \in \Delta.$$

Then u and v are subharmonic on Δ by **Corollary 1.1.1** and

$$v(0) = 2Nu(0) = 2N \log \left| f^{-1}(0) - \zeta \right|.$$

We now seek to estimate this quantity. Choose a path Γ as in the definition of accessible point. Given $\varepsilon > 0$, there exists t_0 such that

$$|\Gamma(t) - \zeta| < \varepsilon \ \forall t \geq t_0.$$

Then

$$u \leq \log \varepsilon \text{ on } S := f\left(\{\Gamma(t)\}_{t \geq 0}\right),$$

therefore

$$v \leq (2N - 1) \sup_{\Delta} u + \log \varepsilon \text{ on } T := \bigcup_{k=1}^{N} e^{2\pi ik/N}(S \cup S^*),$$

where S^* is the reflection of S in the x-axis. Moreover, since $f(\Gamma)$ accumulates at both α and β by assumption, the choice of N implies that T separates $\{0\}$ from $\partial\Delta$. Hence by the ordinary maximum principle **Theorem 2.5**,

$$v(0) \leq \sup_{T} v \leq (2N - 1) \sup_{\Delta} u + \log \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ yields $v(0) = -\infty$, thus $f^{-1}(0) = \zeta$, contradicting the fact that $f^{-1}(0) \in D$.

Step III: (b)

We argue by contradiction. Suppose that $f(\zeta) = f(\widetilde{\zeta}) = \alpha$. As both ζ and $\widetilde{\zeta}$ are accessible, by definition, we can find paths

$$\Gamma, \widetilde{\Gamma} : [0,\infty) \to D$$

such that

$$\lim_{t \to \infty} \Gamma(t) = \zeta \text{ and } \lim_{t \to \infty} \widetilde{\Gamma}(t) = \widetilde{\zeta}.$$

Then $f(\Gamma)$ and $f(\widetilde{\Gamma})$ are two paths in Δ, both ending at α, along which f^{-1} has limits ζ and $\widetilde{\zeta}$, respectively. It follows that the function

$$z \mapsto f^{-1}\left(\alpha \frac{z - i}{z + i}\right)$$

which is bounded and holomorphic in the upper half-plane, has distinct asymptotic values ζ and $\widetilde{\zeta}$, which contradicts Lindelöf's theorem **Corollary 4.17.1**.

Step IV: (c)

Using (a) and (b), if every boundary point of D is accessible, then f extends to a continuous injection of $\overline{D}$ onto $\overline{\Delta}$ by definition of accessible points. A standard compactness argument now shows that

$$f(\overline{D}) = \overline{\Delta}$$

and therefore f^{-1} is continuous on $\overline{\Delta}$, proving f is indeed the desired homeomorphism.

□

4.5 The Poisson-Jensen's Formula

If u is a subharmonic function on a domain containing a closed disc $\overline{\Delta}$, then we saw in **Theorem 2.9** (b) that $u \le P_\Delta u$ on Δ. The difference $P_\Delta u - u$ measures how far u is from being harmonic on Δ, and one would expect this to depend on the size of the generalized Laplacian Δu. The following theorem not only makes this precise, but extends it to a wider class of other domains. It is the culmination of a whole sequence of earlier results.

Theorem 4.30: Poisson-Jensen's Formula for Subharmonic Functions

Let D be a bounded regular domain in $\mathbb{C}$, and let u be a function subharmonic on a neighbourhood of $\overline{D}$ with $u \not\equiv -\infty$ on D. Then

$$u(z) = \int_{\partial D} u(\zeta) d\omega_D(z, D) - \frac{1}{2\pi} \int_D g_D(z, w) \Delta u(w)$$

where $z \in D$.

Proof:

We begin with the claim that, if $z \in D$, then

$$\int_{\partial D} \log|\zeta - w| d\omega_D(z, \zeta) = \begin{cases} \log|z-w| + g_D(z, w), & w \in D \\ \log|z-w|, & w \in \mathbb{C}\backslash D \end{cases} \quad (4.6)$$

In proving **(4.6)** we consider $w \in D$, $w \in \mathbb{C}\backslash\overline{D}$, and $w \in \partial D$, respectively.

Step I: **(4.6)** holds when $w \in D$.

Suppose $w \in D$, then **(4.6)** follows from the fundamental identity for logarithmic potential **Theorem 4.24**.

Step II: **(4.6)** holds when $w \in \mathbb{C}\backslash\overline{D}$.

Suppose $w \in \mathbb{C}\backslash\overline{D}$, then the function

$$\tilde{z} \mapsto \log|\tilde{z} - w|$$

is harmonic on a neighbourhood of $\overline{D}$ by **Corollary 1.1.1**, and so in this case **(4.6)** follows from the definition of harmonic measure (b).

Step III: **(4.6)** holds when $w \in \partial D$.

Finally, suppose that $w \in \partial D$ then as D is connected one has

$$\begin{aligned} \log_{\partial D} \log|\zeta - w| d\omega_D(z, \zeta) &= \limsup_{\widetilde{w} \to w, \widetilde{w} \in D} \left(\int_{\partial D} \log|\zeta - \widetilde{w}| d\omega_D(z, \zeta) \right) \\ &= \limsup_{\widetilde{w} \to w, \widetilde{w} \in D} \left(\log|z - \widetilde{w}| + g_D(z, \widetilde{w}) \right) \\ &= \log|z - w| + \lim_{\widetilde{w} \to w, \widetilde{w} \in D} g_D(\widetilde{w}, z) \\ &= \log|z - w|, \end{aligned}$$

where the first equality holds by the definition of non-thin, for which holds by **Theorem 3.26** as D is connected, the second equality holds by the fundamental identity for logarithmic potential **Theorem 4.24**, the third equality holds by the continuity of $\log|z - w|$, and the last equality holds by **Theorem 4.26** (a) and the assumption that D is a regular domain. Thus **(4.6)** is proved.

Step IV: Desired equality

Now choose a bounded domain D_1 containing $\overline{D}$ such that u is subharmonic on a neighbourhood of $\overline{D_1}$. By the Riesz decomposition **Theorem 3.23** we can write

$$u = p_\mu + h \text{ on } D_1,$$

where $\mu = (2\pi)^{-1}\Delta u\Big|_{D_1}$ and h is harmonic on D_1. Then for $z \in D_1$ we have

$$\begin{aligned}
&\int_{\partial D} u(\zeta)d\omega_D(z,\zeta) \\
&= \int_{\partial D}\left(\int_{D_1}\log|\zeta - w|d\mu(w)\right)d\omega_D(z,\zeta) + \int_{\partial D} h(\zeta)d\omega_D(z,\zeta) \\
&= \int_{D_1}\left(\int_{\partial D}\log|\zeta - w|d\omega_D(z,\zeta)\right)d\mu(w) + h(z) \\
&= \int_D g_D(z,w)d\mu(w) + \int_{D_1}\log|z-w|d\mu(w) + h(z) \\
&= \frac{1}{2\pi}\int_D g_D(z,w)\Delta u(w) + u(z)
\end{aligned}$$

where the first equality holds since $u = p_\mu + h$ on D_1 and h is harmonic on D_1, the red term in the second equality holds by Fubini's theorem and the last term in the second equality holds by **Theorem 4.13** (b) as ∂D is non-polar for the existence of a harmonic measrue (see **Theorem 4.11**), the third equality holds by equation **(4.6)**, and the last equality holds by the definition of μ. Rearranging the terms yields the desired result.

□

As a special case, we recapture the classical Poisson-Jensen's formula for holomorphic functions on a disc, use, for example, in value-distribution theory.

Corollary 4.30.1: Poisson-Jensen's Formula for Holomorphic Functions on Disc

Let f be a function holomorphic on a neighbourhood of $\overline{\Delta}(0,1)$ with $f \not\equiv 0$. Then

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\log|f(e^{i\theta})|d\theta - \sum_{j=1}^{n}\log\left|\frac{1-z\overline{w}_j}{z-w_j}\right|,$$

where $|z| < 1$, $w_1, \cdots, w_n$ are the zeros of f in $\Delta(0,1)$, counted according to multiplicity.

Proof:

Set $\Delta := \Delta(0,1)$ and recall that

$$d\omega_\Delta(z,e^{i\theta}) = \frac{1}{2\pi}1 - |z|^2|e^{i\theta} - z|^2 d\theta$$

by **Example 4.2** and

$$g_\Delta(z,w) = \log\left|\frac{1-z\overline{w}}{z-w}\right|$$

by **Example 4.4**. Moreover, by **Theorem 3.22**, $\Delta\big(\log|f|\big)$ consists of 2π -masses at the zeros of f. The result follows by feeding these facts into Poisson-

Jensen's formula for subharmonic functions **Theorem 4.30**.

□

Remark 4.8: u Being Harmonic on N.B.D. of $\overline{D}$ Is Necessary in **Theorem 4.30**

If we merely suppose that u is subharmonic on D, rather than on a neighbourhood of $\overline{D}$, then Δu may be an infinite measure on D, and it is no longer clear whether the integral

$$\int_D g_D(z,w)\Delta u(w)$$

converges. ⋄

In fact, the convergence turns out to be dependent on whether u has a harmonic majorant, a concept which we shall now define.

Definition: Harmonic Majorant

Let u be a subharmonic function on a domain D. A harmonic majorant of u is a harmonic function h on D such that $h \geq u$ there.

Definition: Least Harmonic Majorant

Let u be a subharmonic function on a domain D and let h be its harmonic majorant. Then h is called the least harmonic majorant if for every other harmonic majorant k of u, $h \leq k$.

In some literature, the harmonic majorant (respectively, harmonic minorant) is also called the harmonic correction. The following result tells us that the least harmonic majorant exists and without the harmonic majorant problem prescribed in **Remark 4.8** may occur.

Theorem 4.31: Existence of Harmonic Majorant Prevents Δu Being Infinite Measure

Let D be a subdomain of $\mathbb{C}$ such that ∂D is non-polar, and let u be a subharmonic function on D with $u \not\equiv -\infty$.

(a) If u has a harmonic majorant on D, then it has at least one least harmonic majorant and

$$u(z) = h(z) - \frac{1}{2\pi}\int_D g_D(z,w)\Delta u(w),\ z \in D.$$

(b) If u has no harmonic majorant on D then

$$\frac{1}{2\pi}\int_D g_D(z,w)\Delta u(w) = \infty,\ z \in D.$$

Proof:

Step I: Construct harmonic functions via **Theorem 1.14**

Let $\{D_n\}_{n\geq 1} \subset D$ be a sequence of relatively compact subdomains of D such that

$$D_1 \subset D_2 \subset D_3 \subset \cdots \text{ and } \bigcup_{n\geq 1} D_n = D.$$

Without loss of generality, we may assume that each component of $\mathbb{C}^\infty \backslash D_n$ contains at least two points, so that by **Theorem 4.7**, each D_n is a regular domain. For $n \geq 1$, define

$$h_n(z) := \int_{\partial D_n} u(\zeta) d\omega_{D_n}(z, \zeta), z \in D_n,$$

then by the definition of harmonic measure (b), h_n is harmonic on D_n. Now by Poisson-Jensen's formula for subharmonic functions **Theorem 4.30** one has

$$u(z) = h_n(z) - \frac{1}{2\pi} \int_{D_n} g_{D_n}(z, w) \Delta u(w)$$

for each $z \in D_n$. By **Corollary 4.22.1** $h_n \uparrow h$ on D, where h satisfies

$$u(z) = h(z) - \frac{1}{2\pi} \int_D g_D(z, w) \Delta u(w), z \in D \tag{4.7}$$

by **Theorem 1.14**, also, either h is harmonic on D or $h \equiv \infty$ there. We now consider these two cases, for which should agree with (a) and (b) respectively.

Step I: (a)

Suppose first that u has a harmonic majorant k on D. Then for each n, it follows from the definition of h_n that

$$h_n(z) \leq \int_{\partial D_n} k(\zeta) d\omega_{D_n}(z, \zeta) = k(z), z \in D_n,$$

where the inequality holds by the definition of harmonic majorant, and hence $h \leq k$ on D. In particular $h \not\equiv \infty$, so h must be harmonic on D. Now equation (4.7) then shows that h is a harmonic majorant of u, and it is the least as k is arbitrary. Thus (a) is proved.

Step II: (b)

Now suppose that u has no harmonic majorant on D. Then h cannot be harmonic, for otherwise it would be such a marjorant. Consequently $h \equiv \infty$ on D, and we conclude from (4.7) that

$$\frac{1}{2\pi} \int_D g_D(z, w) \Delta u(w) = \infty, z \in D.$$

This completes the proof for (b).

□

This has an interesting consequence for holomorphic functions. For the sakeness of simplicity we shall use h.m. to denote the harmonic majorant whenever necessary.

Corollary 4.31.1: Criterion for Finite Growth of Holomorphic Zeros via H.M.

Let D be a domain in $\mathbb{C}$ such that ∂D is non-polar, let f be a holomorphic function on D, and let z_0 be a point in D such that $f(z_0) \neq 0$. Then the followings are equivalent:

(a) $\log|f|$ has a harmonic majorant on D.

(b) $\sum_{j \geq 1} g_D(z_0, w_j) < \infty$, where $w_1, w_2, \cdots$ are the zeros of f. In particular, this series must converge if f is bounded.

Proof:

If we write $u := \log|f|$, then Δu consists of 2π-masses at the zeros of f, and so

$$\int_D g_D(z,w)\Delta u(w) = \sum_{j\geq 1} g_D(z,w_j).$$

Result follows from **Theorem 4.31**.

□

The Carleman-Milloux theorem, which arises naturally out of the problem of estimating the harmonic measure, is to find the best upper bound for a subharmonic function u on $\Delta(0,1)$ satisfying

$$\sup_{|z|=r} u(z) \leq 0 \text{ and } \inf_{|z|=r} u(z) \leq -1, 0 \leq r < 1. \tag{4.8}$$

As an application of the "Green machinery", we now present the beautiful solution found by Beurling and Nevanlinna.

Theorem 4.32: Beurling-Nevanlinna Theorem

Let u be a subharmonic function on $\Delta(0,1)$ satisfying **(4.8)**. Then

$$u(z) \leq -\frac{2}{\pi}\sin^{-1}\left(\frac{1-|z|}{1+|z|}\right), \ |z| < 1,$$

and this bound is sharp.

The proof of **Theorem 4.32** relies on two lemmas, the first of them is an elementary inequality for Green functions on the disc.

Lemma 4.33: Rotational Bounds for Green Function Over Unit Disk

If $\Delta = \Delta(0,1)$ then

$$g_\Delta(-|z|,|w|) \leq g_\Delta(z,w) \leq g_\Delta(|z|,|w|)$$

for $z, w \in \Delta$.

Proof:

Let $z, w \in \Delta$ and write $z := |z|e^{i\alpha}$ and $w = |w|e^{i\beta}$. Then

$$\left|\frac{1-z\overline{w}}{z-w}\right|^2 = 1 + \frac{(1-|z|^2)(1-|w|^2)}{|z|^2+|w|^2-2|z||w|\cos(\alpha-\beta)},$$

which is maximized when $\cos(\alpha-\beta)=1$, and minimized when $\cos(\alpha-\beta)=-1$. Thus we obtain the natural bounds

$$\left|\frac{1+|z||w|}{|z|+|w|}\right|^2 \leq \left|\frac{1-z\overline{w}}{z-w}\right|^2 \leq \left|\frac{1--|z||w|}{|z|-|w|}\right|^2,$$

finally, using the formula in **Example 4.4** we obtain

$$g_\Delta(z,w) = \left|\frac{1-z\overline{w}}{z-w}\right|.$$

Plugging $g_\Delta(z,w)$ into the above display yields the desired result.

□

Lemma 4.34: Subharmonic Function Formula in Unit Disk via Harmonic Majorant

Let $\Delta := \Delta(0,1)$ and $I := [0,1)$, and define v on Δ by

$$v(z) := \begin{cases} -\omega_{\Delta\backslash I}(z,I), & z \in \Delta\backslash I \\ -1, & z \in I \end{cases}$$

Then v is subharmonic on Δ and harmonic on $\Delta\backslash I$. Moreover,

$$v(z) = \frac{1}{2\pi}\int_I g_\Delta(z,w)\Delta v(w), z \in \Delta$$

and

$$v(-x) = -\frac{2}{\pi}\sin^{-1}\left(\frac{1-x}{1+x}\right), x \in I.$$

Proof:

By **Theorem 4.13** (a) v is harmonic on $\Delta \backslash I$ and thus the Gluing **Theorem 2.11** applies, it follows that v is subharmonic on Δ. It left us to prove the moreover statement.

Step I: First identity in the moreover statement

Since v is defined to be non-positive, 0 is clearly a harmonic majorant of v. In fact, it is also the least one, for if k is another least harmonic majorant of v then

$$\limsup_{z \to \zeta} -k(z) \le \lim_{z \to \zeta} -v(z) = 0, \zeta \in \partial\Delta \backslash \{1\},$$

where the inequality holds since $k \ge v$ thus $-k \le -v$ and the equality holds by **Corollary 4.10.1**, so by the extended maximum principle **Theorem 3.17** (b) $k \ge 0$ on Δ. Applying **Theorem 4.31** (a), we deduce that

$$v(z) = 0 - \frac{1}{2\pi}\int_I g_\Delta(z, w)\Delta v(w), z \in \Delta,$$

where 0 is the least harmonic majorant of v and the integral is taken over I since v is harmonic on $\Delta \backslash I$ and thus $\Delta v = 0$ there.

Step II: Second identity in the moreover statement

We calculate the harmonic measure under conformal mapping as given in **Example 4.3** and obtain

$$\omega_{\Delta \backslash I}(z, I) = 1 - \frac{2}{\pi}\arg\left(\frac{1+\sqrt{z}}{z-\sqrt{z}}\right)$$

for $z \in \Delta \backslash I$.

□

Proof of **Theorem 4.32**:

We first verify the bound and then show that it is sharp. First, we set up our assumptions for the above two lemmas to apply. Let $\Delta := \Delta(0,1)$ and

$$U := \{z \in \Delta : u(z) < -1\},$$

where -1 is taken in order to match the form in **Lemma 4.34**.

We may, without loss of generality, assume that

$$\inf_{|z|=r} u(z) < -1 \text{ for all } r$$

as otherwise we can work with $u - \varepsilon$ and sending $\varepsilon \downarrow 0$. Thus if we define

$$T : \Delta \to I$$

by $T(z) = |z|$ and $T(U) = I$, where I is the same as in **Lemma 4.34**. Now both lemmas are ready to be applied. We proceed to verify the bound.

Step I: Verifying the bound in the assertion

Let v be defined in **Lemma 4.34**. Given $\rho < 1$, we can find a compact subset K of U such that $T(K) = [0,\rho]$. Then by "Existence of Pushforward Measure under Surjection”[5] there exists a finite Borel measure μ on K such that $\mu T^{-1} = \Delta v\Big|_{[0,\rho]}$. Define a function h on Δ by

$$h(z) = -\frac{1}{2\pi}\int_K g_\Delta(z,w)d\mu(w),\ z \in \Delta.$$

Now by **Lemma 4.34** h is harmonic on $\Delta\backslash K$ and $\lim_{z\to\zeta} h(z) = 0\ \forall\zeta \in \partial\Delta$.

Moreover, if $z \in \Delta$ we get

$$\begin{aligned} h(z) &\geq -\frac{1}{2\pi}\int_K g_\Delta(|z|,|w|)d\mu(w) \\ &= -\frac{1}{2\pi}\int_{[0,\rho]} g_\Delta(|z|,w)\Delta v(w) \\ &\geq -\frac{1}{2\pi}\int_I g_\Delta(|z|,w)\Delta v(w) \\ &= v(|z|) = -1 \end{aligned}$$

where the first inequality holds by the upper bound in **Lemma 4.33** and the second inequality holds since $[0,\rho] \subseteq I$ and taking negation, the first equality holds by “Existence of Pushforward Measure under Surjection”, the second equality holds by holds by the first identity in **Lemma 4.34**, and the last equality holds by the definition of v on I.

Hence, if $\zeta \in \partial(\Delta\backslash K)$ then

$$\limsup_{z\to\zeta, z\in\Delta\backslash K} \big(u-h\big)(z) \leq \begin{cases} 0, & \zeta \in \partial\Delta \\ u(\zeta) - (-1), & \zeta \in \partial K \end{cases} \leq 0,$$

and so by the maximum principle **Theorem 2.5** (b) $u \leq h$ on $\Delta\backslash K$. Since also $u \leq -1 \leq h$ on K, we in fact have $u \leq h$ on the whole of Δ. Applying now the lower bound in **Lemma 4.33**, we deduce that, for each $z \in \Delta$,

$$u(z) \leq -\frac{1}{2\pi}\int_K g_\Delta(-|z|,|w|)d\mu(w) = -\frac{1}{2\pi}\int_{[0,\rho]} g_\Delta(-|z|,w)\Delta v(w),$$

where the inequality holds by the lower bound in **Lemma 4.33** and the equality holds by the second identity in **Lemma 4.34**. As this holds for each $\rho < 1$, we can send $\rho \uparrow 1$ and obtain

$$u(z) \leq -\frac{1}{2\pi}\int_I g_\Delta(-|z|,w)\Delta v(w) = v(-|z|) = -\frac{2}{\pi}\sin^{-1}\Big(\frac{1-|z|}{1+|z|}\Big),$$

---

[5] **Theorem**: (Existence of Pushforward Measure under Surjection): Let X and Y be compact metric spaces, and let $T : X \to Y$ be a continuous surjection. Then, given $\nu \in \mathscr{P}(Y)$, the collection of all Borel probability measures, there exists $\mu \in \mathscr{P}(X)$ such that $\mu T^{-1} = \nu$ so that

$$\int_X \varphi \circ T(x)d\mu(x) = \int_Y \varphi(y)d\nu(y)$$

for $\varphi \in C(Y)$, the space of all continuous functions $\varphi : Y \to \mathbb{R}$.

where in the first inequality the limit can be passed since the integral is bounded by **Theorem 4.31** (a), the second equality holds by definition of v, and the last equality holds by the second identity in **Lemma 4.34**. This proves the desired bound for u.

Step II: The desired bound is sharp

To show that the bound is sharp, we note that for each θ, the function

$$u_\theta(z) := v(e^{i\theta}z)$$

satisfies the hypotheses of the theorem, and so any general upper bound for $u(z)$ must be at least as large as

$$\sup_\theta u_\theta(z) = \sup_\theta v(e^{i\theta}z) = v(-|z|) = -\frac{2}{\pi}\sin^{-1}\left(\frac{1-|z|}{1+|z|}\right)$$

where the first equality holds by definition and the last equality holds by the second identity in **Lemma 4.34**. This observation concludes the proof.

□

As a consequence of this result, we can derive some general estimates for harmonic measure on a simply connected domain.

Corollary 4.32.1: Bounds for Harmonic Measure of Connected Domain without Zero

Let D be a simply connected subdomain of $\mathbb{C}$ such that $0 \notin D$, and let $\rho > 0$.

(a) If $z \in D$ and $|z| < \rho$ then

$$\omega_D\big(z, \partial D \cap \Delta(0,\rho)\big) \geq \frac{2}{\pi}\sin^{-1}\left(\frac{\rho-|z|}{\rho+|z|}\right).$$

(b) If $z \in D$ and $|z| > \rho$ then

$$\omega_D\big(z, \partial D \cap \Delta(0,\rho)\big) \leq \frac{2}{\pi}\cos^{-1}\left(\frac{|z|-\rho}{|z|+\rho}\right).$$

Proof:

Step I: Assertion (a)

Define u on $\Delta(0,\rho)$ by

$$u(z) := \begin{cases} -\omega_D\big(z, \partial D \cap \Delta(0,\rho)\big), & z \in \Delta(0,\rho) \cap D \\ -1, & z \in \Delta(0,\rho) \backslash D \end{cases}$$

As D is simply connected, **Theorem 4.6** guarantees that it is a regular domain, and hence the gluing **Theorem 2.11** applies to show that u is subharmonic on $\Delta(0,\rho)$. Evidently, $u \leq 0$. Also no circle $|z| = r$ can be entirely contained in D for then it would separate 0 and ∞, both of which lie outside D, contradicting the fact that D is simply connected. Hence

$$\inf_{|z|=r} u(z) = -1, 0 \leq r < \rho.$$

Applying **Theorem 4.32** to the function $\tilde{z} \mapsto u(\rho\tilde{z})$ on $\Delta(0,1)$, we deduce that

$$u(z) \leq -\frac{2}{\pi}\sin^{-1}\left(\frac{1-|z/\rho|}{1+|z/\rho|}\right), z \in \Delta(0,\rho),$$

which proves (a).

Step II: (b)

Let D^* be the image of D under the inversion $z \mapsto \dfrac{1}{z}$. Then if $z \in D$ one has

$$\omega_D\big(z, \partial D \cap \Delta(0,\rho)\big) = \omega_{D^*}\Big(\frac{1}{z}, \partial D^* \backslash \overline{\Delta}(0,1/\rho)\Big)$$

$$\leq 1 - \omega_{D^*}\Big(\frac{1}{z}, \partial D^* \cap \Delta(0,1/\rho)\Big)$$

Applying part (a) to D^*, it follows that if also $|z| > \rho$ then

$$\omega_D\big(z, \partial D \cap \Delta(0,\rho)\big) \leq 1 - \frac{2}{\pi}\sin^{-1}\Big(\frac{1/\rho - 1/|z|}{1/\rho + 1/|z|}\Big)$$

$$= \frac{2}{\pi}\cos^{-1}\Big(\frac{|z| - \rho}{|z| + \rho}\Big)$$

which completes the proof.

□

Summary of Chapter 4

Since the ordinary Dirichlet problem not necessarily has a solution for general domains, it is desired to derive a natural reformulation of the Dirichlet problem that always has a solution. We generalize the domain D to be any proper subdomain of $\mathbb{C}^\infty$ and we generalize the continuous boundary condition to bounded boundary condition. Then we define the Perron function, for which is defined in a way that if the generalized Dirichlet problem has a solution then it is the Perron function. Therefore we proved "Perron Function Is Always Bounded Harmonic", for the proof we used the lemma "Poisson Modification". But so far there are still cases for the generalized Dirichlet probelm not having solutions, and the reason is that the isolated boundary point lack sufficient influence to the subharmonic function and thus the Perron function has wrong boundary limit there. To this end we defined the Barrier, from which we defined the regularity and irregularity of boundary points, and finally regularity for the domain. Then our construction allows us to prove "Sufficiency for Perron Function Solving Dirichlet Problem", for the proof relies on a property that "Perron Function Is Antisymmetric" and a globalization for barriers "Bouligand's Lemma". As a consequence for Perron function solving Dirichlet problem, we can prove "Existence and Unique Solution to the Dirichlet Problem". Moreover the regularity is necessary and sufficient for the existence of the solution. This answers part of our motivation and the other part will be told after Kellogg's theorem.

In the second section, we aim to find the criterion for regularity. We first proved that "Simply Connected Domain Smaller than $\mathbb{C}^\infty$ Is Regular", and then we localize the result to obtain a sufficient condition for regularity of a single point "Boundary Point in Non-Trivial Component Is Regular". As the other extreme, we are also able to tell the irregularity "Boundary Point with Polar Neighbourhood Is Irregular". Summarizing we derive the desired "Criterion for Regularity". As a consequence we can show that the set of irregular points is always small, this result is also known as "Kellogg's Theorem", a consequence of this is to finish the construction that the Generalized Dirichlet problem always has a solution, namely, "Solution of the Generalized Dirichlet Problem".

In the third setion, we need to verify that explicit solution to the Dirichlet problem over disk holds also for the generalized one. To this end we need to extend the Poisson integral to general domains. This motivates us to define the harmonic measure, for which is the common value between the Perron function and the generalized Poisson integral. This definition is confirmed by "Existence and Uniqueness for Harmonic Measure", as in our definition of harmonic measure the boundary condition is assumed to be continuous, it is natural to extend the definition of harmonic measure to bounded boundary condition, namely "$H_D\varphi = P_D\varphi$ for All Bounded Borel Function φ On Non-Polar ∂D". Since the Perron function solves the generalized Dirichlet problem, it should be harmonic, this is confirmed by the "Characterization of Harmonic Measure". Moreover, a measure property for harmonic measure is proved, namely, "Mutual Absolute Continuity for Harmonic Functions". As desired, the harmonic measure does not charge polar sets, this is proved by "Borel Polar Subset Has Harmonic Measure Zero". Unfortunately the converse is not true. Since the harmonic measure is itself subharmonic, it has the desired properties we proved, and even better. "Two Constant Theorem for Harmonic Measure" gives a generalized extended Maximum principle, "Subordination Principle for Harmonic Measure", and "Domain Monotonicity for Harmonic Measure". We can then tell the growth rate of subharmonic functions by introducing the concept of asymptotic value and "Asymptotic Value for Subharmonic Growth on Sector of Half-Plane", for which the bound is sharp. Furthermore, we proved "Lindelöf Theorem" which tells us that the bounded holomorphic function over half-line can have at most one asymptotic value. These two results allow us to find the "Harmonic Measure for Half-Plane". Finally, we compare the equilibrium measure and the harmonic measure by showing that "Equilibrium and Harmonic Measure Agree on Component with ∞".

In the fourth section we introduced the Green function, for which the existence and uniqueness is verified by "Existence and Uniqueness of Green Function". Some properties are derived: "Green Function Is Positive", "Subordination Principle for Green Function", "Domain Monotonicity for Green Function", and "Green Function Is Continuous in Increase of Domain". With the help of Green function we derived the "Fundamental Identity for Logarithmic Potential", which in turn tells us that the Green function is symmetric in the space variables, namely, "Symmetry Theorem for Green Function". Moreover, the relation between solvability of Generalized Dirichlet Problem and Green function is found in "Criterion for Solvability of Dirichlet Problem via Green Function". A consequence of the symmetry enables us to prove the strong converse of subordination principle, which is "Characterization of Conformal Mapping via Green Function", this yields a simple proof of the "Riemann Mapping Theorem". However, the conformal mapping will not extend to a homeomorphism of the closures, it is then natural to ask when it is possible. It is clear that every boundary point must be accessible, this is also sufficient: "Sufficiency for Extension to Homeomorphism on Closure".

Finally, in the last section, we proved "Poisson-Jensen’s Formula for Subharmonic Functions" and compared it with the "Poisson-Jensen’s Formula for Holomorphic Functions on Disc", for the latter is a consequence of the former. For the first result to hold, the function has to be harmonic on the neighbourhood of the closure for the domain, as otherwise the Laplacian may be an infinite Radon measure. For us to control this term we defined the harmonic marjorant and the least harmonic majorant, and indeed they did their job as "Existence of Harmonic Majorant Prevents Δu Being Infinite Measure". As a corollary, we derived "Criterion for Finite Growth of Holomorphic Zeros via Harmonic Majorant". This motivates us to give bound estimates for harmonic measure, namely the Carleman-Milloux theorem, and we derived the green function version "Beurling-Nevanlinna Theorem". The proof relies on two technical lemma, the first is "Rotational Bounds for Green Function Over Unit Disk" and the second is "Subharmonic Function Formula in Unit Disk via Harmonic Majorant". Finally, as a consequence, we are able to find the "Bounds for Harmonic Measure of Connected Domain without Zero".

5. Capacity

5.1 Capacity as a Set Function

Even though polar sets have played a prominent rôle in the theory developed so far, we still lack an effective means of determining whether or not a given set is polar. Thus it was only by a very indirect method that we were able to demonstrate the existence of uncountable polar sets in **Section 3.5**, and nothing we have yet proved will tell us whether, for example, the Cantor set is polar.

More generally, it is desirable to be able to gauge, in some way, how close a set is to being polar. In the case of a compact set, the energy $I(\nu)$ of its equilibrium measure ν, a quantity that has already cropped up several times, provides just such an indicator. Taking exponentials in order to make it positive, we are led to the following definition.

Definition: Logarithmic Capacity

The logarithmic capacity of a subset $E \subseteq \mathbb{C}$ is given by

$$c(E) := \sup_{\mu} e^{I(\mu)},$$

where the supremum is taken over all Borel probability measures μ on $\mathbb{C}$ whose support is a compact subset of E. In particular, if K is a compact set with equilibrium measure ν then $c(K) = e^{I(\nu)}$.

Here it is understood that $e^{-\infty} = 0$, so that $c(E) = 0$ precisely when E is polar. There are several other capacities with this property, but the logarithmic capacity enjoys the advantage of particularly close links with complex analysis. Since it is the only one we shall study, it will henceforth be referred to simply as ‘the capacity’.

We begin with proving some of its elementary properties.

Theorem 5.1: Some Elementary Properties of Logarithmic Capacity

(a) If $E_1 \subset E_2$ then $c(E_1) \leq c(E_2)$. (Monotone)

(b) If $E \subset \mathbb{C}$ then $c(E) = \sup \left\{ c(K) : K \subset E \text{ are compact subsets} \right\}$.

(c) If $E \subset \mathbb{C}$ then $c(\alpha E + \beta) = |\alpha| c(E)\ \forall \alpha, \beta \in \mathbb{C}$.
(Positive Homogeneous in Scaling, Invariant in Constant Translation)

(d) If K is a compact subset of $\mathbb{C}$ then $c(K) = c(\partial_e K)$.

Proof:

Both (a) and (b) are immediate consequences of the definition for logarithmic capacities. An application of **Theorem 3.21** gives (d). It left us to prove (c). Let $T : \mathbb{C} \to \mathbb{C}$ be the map $T(z) := \alpha z + \beta$. Then $\operatorname{supp}(\mu) \subset E$ if and only if $\operatorname{supp}(\mu T^{-1}) \subset \alpha E + \beta$, and

$$I(\mu T^{-1}) = I(\mu) + \log|\alpha|$$

by the definition of energy. This proves (c).

□

Since capacity is a monotone set function, it is natural to ask if it is continuous with respect to increasing or decreasing sequences. The following result gives the answer.

Theorem 5.2: Capacity Is Continuous in Monotone Sequences

(a) If $K_1 \supset K_2 \supset \cdots$ are compact subsets of $\mathbb{C}$ and $K := \bigcap_{n\geq 1} K_n$ then

$$c(K) = \lim_{n\to\infty} c(K_n).$$

(b) If $B_1 \subset B_2 \subset \cdots$ are Borel subsets of $\mathbb{C}$ and $B := \bigcup_{n\geq 1} B_n$ then

$$c(B) = \lim_{n\to\infty} c(B_n).$$

Proof:

Step I: (a)

By **Theorem 5.1** (a) we certainly have

$$c(K_1) \geq c(K_2) \geq c(K_2) \geq \cdots. \tag{5.1}$$

In the other direction, for each $n \geq 1$ let ν_n be an equilibrium measure for K_n. Then $\nu_n \in \mathscr{P}(K_1)$ for all $n \geq 1$. By a diagonal argument, there is a subsequence $\{\nu_{n_k}\}_{k\geq 1}$ which is weak*-convergent to some $\nu \in \mathscr{P}(K_1)$. Using **Lemma 3.6**, we deduce that

$$\limsup_{n\to\infty} I(\nu_{n_k}) \leq I(\nu).$$

Moreover, since $\operatorname{supp}(\nu_n) \subset K_n$ for all n, it follows that $\operatorname{supp}(\nu) \subset K$, and also $e^{I(\nu)} \leq c(K)$. Thus we obtain

$$\limsup_{k\to\infty} c(K_{n_k}) \leq c(K),$$

and combining this with **(5.1)** yields the desired conclusion.

Step II: (b)

Again using **Theorem 5.1** (a), we have

$$c(B_1) \leq c(B_2) \leq c(B_3) \leq \cdots. \tag{5.2}$$

In the other direction, let K be a compact subset of B, and let ν be an equilibrium measure for K. Since

$$\nu(B_n \cap K) \to \nu(K) \text{ as } n \to \infty.$$

The regularity of finite Borel measure gives compact sets $K_n \subset B_n \cap K$ such that

$$K_1 \subset K_2 \subset \ K_3 \subset \cdots \text{ and } \nu(K_n) \to 1.$$

For n sufficiently large we have $\nu(K_n) > 0$, and for these n we define

$$\mu_n := \frac{\nu|_{K_n}}{\nu(K_n)}.$$

Thus μ_n is a Borel probability measure on K_n and

$$I(\mu_n) = \frac{1}{\nu(K_n)^2}\int_K\int_K \log|z-w|1_{K_n}1_{K_n}(w)d\nu(z)d\nu(w).$$

As $n\to\infty$, we have $\nu(K_n)\to 1$ and $1_{K_n}\uparrow 1_K$ ν-almot everywhere, so

$$\lim_{n\to\infty} I(\mu_n) = \int_K\int_K \log|z-w|d\nu(z)d\nu(w) =: I(\nu).$$

Since each μ_n is supported on a compact subset of B_n, we have

$$c(B_n) \geq e^{I(\mu_n)}$$

and it follows that

$$\liminf_{n\to\infty} c(B_n) \geq c(K).$$

Finally, since K is an arbitrary compact subset of B, **Theorem 5.1** (b) implies that

$$\liminf_{n\to\infty} c(B_n) \geq c(B),$$

together with **(5.2)** yields the desired result.

□

Theorem 5.2 (a) is false for general Borel sets, indeed even for bounded opens ets.

Example 5.1: **Theorem 5.2** (a) Fails for Bounded Open Sets

Consider the sequence

$$U_n := \left\{z\in\mathbb{C} : -1 < \mathrm{Re}(z) < 1, 0 < \mathrm{Im}(z) < \frac{1}{n}\right\}$$

for $n\geq 1$. Then clearly $U_1 \supset U_2 \supset U_3 \supset \cdots$ and $\bigcap_{n\geq 1} U_n = \varnothing$. But also each set U_n contains a translation of the non-polar set $[0,1]$, and so

$$c(U_n) \geq c\big([0,1]\big) > 0 \ \forall n\geq 1.$$

On the other hand, since $\varnothing$ is polar, $c(U) = 0$, therefore

$$c(U) = 0 < \lim_{n\to\infty} c(U_n),$$

thus the continuity fails. ◇

However, it can be shown that, given a bounded Borel set B, we have

$$c(B) = \inf\left\{c(U) : \text{open } U\supset B\right\}. \tag{5.3}$$

This result, due to Choquet, looks like dual to **Theorem 5.1** (b), but actually it lies much deeper, and we shall not prove it here (see Port and Sydney Theorem 6.78).

Capacity is not an additive set function, like a measure.

Example 5.2: Capacity Is NOT an Additive Set Function

Consider the unit disk $\overline{\Delta}(0,1)$, which has finite capacity, contains infinitely many disjoint translations of the unit interval $[0,1]$, which has strictly positive capacity since it is non-polar. ◇

There is however a relation between capacity and unions.

Theorem 5.3: Bound Estimates for Capacity of Borel Union

Let $\{B_n\}_{n\geq 1}$ be a (finite or infinite) sequence of Borel subsets of $\mathbb{C}$, let $B := \bigcup_{n\geq} B_n$, and let $d > 0$.

(a) If $\operatorname{diam}(B) \leq d$ then $c(B) \leq d$ and

$$\frac{1}{\log\big(d/c(B)\big)} \leq \sum_{n\geq 1} \frac{1}{\log\big(d/c(B_n)\big)}. \tag{5.4}$$

(b) If $\operatorname{dist}(B_j, B_k) \geq d \ \forall j \neq k$ then

$$\frac{1}{\log^+\big(d/c(B)\big)} \geq \sum_{n\geq 1} \frac{1}{\log^+\big(d/c(B_n)\big)}. \tag{5.5}$$

Hence, we interpret $\frac{1}{0}$ as ∞ and $\frac{1}{\infty}$ as 0. Thus, for example, part (a) re-proves the result that a countable union of Borel polar sets is polar $\big($at least provided the union is bounded, but the unbounded case can then be deduced from **Theorem 5.2** (b)$\big)$, for which we proved in **Corollary 3.4.2**.

Proof of **Theorem 5.3**:

Step I: $c(B) \leq d$ in (a)

We begin noting that if $\operatorname{diam}(B) \leq d$ then, for any probability measure μ that is compactly supported on B, we have

$$I(\mu) = \int_B \int_B \log|z-w|\, d\mu(z)d\mu(w) \leq \int_B \int_B (\log d) d\mu(z)d\mu(w),$$

where the equality holds by the definition of energy and the inequality holds by assumption $\operatorname{diam}(B) \leq d$. Therefore $c(B) \leq d$ by definition of capacity.

Step II: **(5.4)** in (a)

As for **(5.4)**, it suffices to prove it in the case where there are just two sets B_1 and B_2. The case for n set then follows by induction, and for infinitely many sets the result can be deduced from **Theorem 5.2** (b). By scaling, we can also suppose that $d = 1$.

Let K be a compact subset of B and let $\varepsilon > 0$. Our claim is to show that

$$\frac{1-\varepsilon}{\log\big(1/c(K)\big)} \leq \frac{1}{\log\big(1/c(B_1)\big)} + \frac{1}{\log\big(1/c(K_2)\big)}. \tag{5.6}$$

This inequality is clear if $c(K) = 0$, so we may as well assume that $c(K) > 0$. In that case $I(\nu) > -\infty$, where ν is the equilibrium measure for K. Since

$$\nu(B_1 \cap K) + \nu(B_2 \cap K) \geq \nu(K) = 1,$$

where the inequality holds by subadditivity of ν and the equality holds since $d = 1$. Now by the regularity of finite Borel measure we can find compact sets

$$K_j \subset B_j \cap K, j = 1{,}2$$

such that

$$\nu(K_1) + \nu(K_2) > 1 - \varepsilon.$$

For $j = 1{,}2$, let ν_j be the equilibrium measure for K_j. Then we have

$$I(\nu) \le \int_{K_j} p_\nu d\nu_j = \int_K p_{\nu_j} d\nu \le \int_{K_j} p_{\nu_j} d\nu = I(\nu_j)\nu(K_j),$$

where the first relation holds as $p_\nu \ge I(\nu)$ on $\mathbb{C}$ by **Theorem 3.7** (a), the second holds by Fubini's theorem, the third holds as $p_{\nu_j} \le 0$ on K (recall that $\operatorname{diam}(K) \le d = 1$), and the fourth holds since $p_{\nu_j} = I(\nu_j)$ n.e. and hence ν-a.e. on K_j by **Theorem 3.7** (b).

Now $I(\nu) = \log c(K) \le 0$ since $\operatorname{diam}(K) \le d = 1$, and likewise for $I(\nu_j)$, so we obtain

$$\frac{\nu(K_j)}{\log\left(1/c(K)\right)} \le \frac{1}{\log\left(1/c(K_j)\right)} \le \frac{1}{\log\left(1/c(B_j)\right)}, j = 1,2.$$

Summing over j gives **(5.6)**. Finally, letting $\varepsilon \downarrow 0$ in **(5.6)**, and taking the supremum over all compact subset $K \subset B$ yields **(5.4)**.

Step III: (b)

As in (a), we can suppose that there are just two sets B_1, B_2, and that $d = 1$. Let K_1, K_2 be compact subsets of B_1, B_2, respectively. This time, our aim is to show that

$$\frac{1}{\log^+\left(1/c(B)\right)} \ge \frac{1}{\log^+\left(1/c(K_1)\right)} + \frac{1}{\log^+\left(1/c(K_2)\right)}. \tag{5.7}$$

We can assume that $0 < c(K_j) \le c(B) < 1$, since otherwise **(5.7)** is clear anyway. For $j = 1,2$, let ν_j be the equilibrium measure of K_j and set

$$\mu := (1 - t)\nu_1 + \nu_2,$$

where $t := \dfrac{I(\nu_1)}{I(\nu_1) + I(\nu_2)}$. Since $-\infty < I(\nu_j) < 0$, it follows that $0 < t < 1$, and hence μ is a probability measure with

$$I(\mu) \ge (1 - t)^2 I(\nu_1) + t^2 I(\nu_2) = \frac{I(\nu_1)I(\nu_2)}{I(\nu_1) + I(\nu_2)}.$$

Now μ is supported on $K_1 \cup K_2 \subset B$ so $I(\mu) \le \log c(B)$ and hence

$$\log c(B) \ge \frac{\log c(K_1) \log c(K_2)}{\log c(K_1) + \log c(K_2)}.$$

Since $\log c(B)$ and $\log c(K_j)$ are all negative, when the inequality is inverted it becomes **(5.7)**. Finally, taking supremum in **(5.7)** over all compact subsets K_1 of B_1 and K_2 of B_2 gives **(5.5)**.

□

We conclude by mentioning that capacity can behave badly with respect to complements.

Example 5.3: Capacity Behave Bad in Set Complements

Given $E \subset \mathbb{C}$, one can show that there exists an F_σ set $F \subset E$ such that $c(F) = c(E)$. Let S be a subset of $[0,1]$ which is not F_σ. Then every F_σ subset F of $[0,1] \times S$ satisfies

$$c\big(([0,1]\times S)\backslash F\big) \ge c([0,1]) > 0.$$

Based on this one can construct a set E of positive capacity such that every F_σ subset F of E satisfies $c(E\backslash F) = c(E)$. ⋄

5.2 Computation of Capacity

Though our definition for capacity is fine for the purpose of deriving theoretical properties of capacity, it is not well studied to computing the capacity of specific sets. Even for the simplest one, that of a disc, requires some work, and most other sets are virtually impossible.

Fortunately, for compact sets at least, there are easier alternatives. They are based on the following relation between capacity and Green functions.

Theorem 5.4: Capacity of Compact Non-Polar Set via Green Function

Let K be a compact non-polar set and let D be the component of $\mathbb{C}^\infty\backslash K$ which contains ∞. Then, as $z \to \infty$,

$$g_D(z,\infty) = \log|z| - \log c(K) + o(1). \tag{5.8}$$

Proof:

Let ν be the equilibrium measure for K. From the way that g_D was constructed in **Theorem 4.20** we have

$$g_D(z,\infty) = p_\nu(z) - I(\nu) = p_\nu(z) - \log c(K),\ z \in D\backslash\{\infty\},$$

where the first equality holds by **Theorem 4.20** Step II.1 and the second holds by the definition of capacity. Now using **Theorem 3.1** (ii) we also know that

$$p_\nu(z) = \log|z| + o(1) \text{ as } z \to \infty.$$

Combining these two facts yields the desired result.

□

As a consequence, we can read off the capacity of a disc.

Corollary 5.4.1: Capacity of Closed Disc

If $w \in \mathbb{C}$ and $r > 0$, then $c\big(\overline{\Delta}(w,r)\big) = r$.

Proof:

Setting $D := \mathbb{C}^\infty\backslash\overline{\Delta}(w,r)$, we have

$$g_D(z,\infty) = \log\left|\frac{z-w}{r}\right| = \log|z| - \log r + o(1),$$

where the first equality holds by the definition of D and the second holds by **Theorem 5.4**. Combining this with (5.8) we deduce that $c\big(\overline{\Delta}(w,r)\big) = r$.

□

The subordination principle for Green functions gives rise to a useful inequality for capacity, as the inequality goes the other direction, we refer to it the inversed subordination principle.

Theorem 5.5: Inversed Subordination Principle for Capacity

Let K_1, K_2 be compact subsets of $\mathbb{C}$, and let D_1, D_2 be the components containing ∞ of $\mathbb{C}^\infty\backslash K_1$ and $\mathbb{C}^\infty\backslash K_2$ respectively. If there is a meromorphic function $f : D_1 \to D_2$ such that

$$f(z) = z + O(1) \text{ as } z \to \infty. \tag{5.9}$$

Then

$$c(K_2) \le c(K_1),$$

with equality if f is a conformal mapping from D_1 onto D_2.

Proof:

If K_2 is polar then $c(K_2) = 0$ and the inequality is clear. Thus, without loss of generality, we may assume that K_2 is non-polar. We consider two cases for the inequality, namely, K_1 is non-polar and K_1 is general.

Case I: K_1 is non-polar

Suppose that K_1 is also non-polar. Then the Green function g_{D_1} and g_{D_2} both exists by **Theorem 4.20**, and by subordination principle **Theorem 4.22**,

$$g_{D_2}\big(f(z), \infty\big) \ge g_{D_1}(z, \infty), z \in D_1.$$

Now from **Theorem 5.4**, as $z \to \infty$,

$$g_{D_1}(z, \infty) = \log|z| - \log c(K_1) + o(1),$$

and from **(5.9)**,

$$\begin{aligned} g_{D_2}\big(f(z), \infty\big) &= \log|f(z)| - \log c(K_2) + o(1) && \big(\text{by } (5.9)\big) \\ &= \log|z| - \log c(K_2) + o(1) && (\text{meromorphic}) \end{aligned}$$

Combining these facts, we deduce that $c(K_2) \le c(K_1)$ in this case.

Case II: K_1 not necessarily non-polar

For a general K_1, take $\varepsilon > 0$ and set

$$K_1^\varepsilon := \big\{z : \text{dist}(z, K_1) \le \varepsilon\big\}.$$

This set is non-polar by **Corollary 3.4.1**, so by Case I, we have

$$c(K_2) \le c(K_1^\varepsilon).$$

Sending $\varepsilon \downarrow 0$ and using **Theorem 5.2** (a), we again obtain $c(K_2) \le c(K_1)$, and so, in fact, K_1 was non-polar anyway.

Finally, assume that f is a conformal mapping from D_1 onto D_2. Then we can apply the same argument to f^{-1} to deduce that

$$c(K_2) \ge c(K_1),$$

together with the meromorphic case we conclude the proof.

□

Using this, we can find the capacity of an interval.

Corollary 5.5.1: Capacity for Interval

If $a \le b$ then $c\big([a,b]\big) = \dfrac{b-a}{4}$.

Proof:

The function $f(z) := z + \dfrac{1}{z}$ maps $\mathbb{C}^\infty \backslash \overline{\Delta}(0,1)$ conformally onto $\mathbb{C}^\infty \backslash [-2,2]$ and satisfies **(5.9)**, so using **Theorem 5.5** in the first equality,

$$c\big([-2,2]\big) = c\big(\overline{\Delta}(0,1)\big) = 1,$$

where the second equality holds by **Corollary 5.4.1**. For a general a, b, the result follows by translating and scaling.

□

In principle, the same technique works for any compact connected set K with more than one point.

Example 5.4: Capacity for Non-Trivial Connected Compact Set

Let K be a non-trvial connected compact set. Using Riemann mapping theorem **Theorem 4.28**, $\mathbb{C}^\infty \backslash K$ can be mapped conformally onto the unit disc, and, by composiing with a suitable Möbius transformation, we can find $r > 0$, and a conformal mapping

$$f : \mathbb{C}^\infty \backslash K \to \mathbb{C}^\infty \overline{\Delta}(0,r)$$

which satisfies **(5.9)**. The capacity of K is then given by

$$c(K) = c\left(\overline{\Delta}(0,r)\right) = r,$$

where the first equality holds by the conformal case in **Theorem 5.5** and the second holds by **Corollary 5.4.1**. ⋄

In practice, however, it is only possible to compute the conformal map f explicitly for relatively simple sets K, such as those bounded by a finite number of straight lines and circular arcs. A table of some calculations is available in the appendix.

Capacities also behaves well under taking inverse images by polynomials.

Theorem 5.6: Capacity under Inverse Image of Polynomials

Let K be a compact set, and let $q(z) := \sum_{j=0}^{d} a_j z^j$, where $a_d \neq 0$. Then

$$c\left(q^{-1}(K)\right) = \left(\frac{c(K)}{|a_d|}\right)^{1/d}.$$

Proof:

Let D and $\widetilde{D}$ be components containing ∞ of $\mathbb{C}^\infty \backslash K$ and $\mathbb{C}^\infty \backslash q^{-1}(K)$ respectively. Then, as is easily checked,

$$q(\widetilde{D}) = D \text{ and } q(\partial \widetilde{D}) = \partial D.$$

Case I: D is a regular domain

Assume so, then by **Theorem 4.26** (a),

$$\lim_{z \to \zeta, z \in \widetilde{D}} g_D\left(q(z), \infty\right) = 0, \zeta \in \partial \widetilde{D}.$$

Moreover, by the definition of Green function (a), $g_D\left(f(z), \infty\right)$ is harmonic on $\widetilde{D} \backslash \{\infty\}$, and as $z \to \infty$,

$$g_D\left(q(z), \infty\right) = \log|q(z)| + O(1) = d \log|z| + O(1),$$

where the first equality holds by the definition of Green function (b) and the term d appearing in the second is due to the degree of the polynomial. Now by the uniquness part of **Theorem 4.20**, it follows that

$$g_D\left(q(z), \infty\right) = d \cdot g_{\widetilde{D}}(z, \infty), z \in \widetilde{D}.$$

From **Theorem 5.4** we also know that as $z \to \infty$,

$$\begin{aligned} g_D\left(q(z), \infty\right) &= \log|q(z)| - \log c(K) + o(1) \\ &= d \log|z| + \log|a_d| - \log c(K) + o(1), \end{aligned}$$

where the first equality holds by **Theorem 5.4** and the second holds by the blue display above, the constant $\log|a_d|$ is the $O(1)$ term by assumption. Also,

$$g_{\widetilde{D}}(z, \infty) = \log|z| - \log c\left(q^{-1}(K)\right) + o(1)$$

by **Theorem 5.4** once more. Putting these facts together, we obtain

$$d \log c\big(q^{-1}(K)\big) = \log c(K) - \log|a_d|,$$

which gives the desired result.

Case II: General D

For a general K and hence a general D, take $\varepsilon > 0$ and set

$$K^\varepsilon := \{z : \operatorname{dist}(z, K) \ge \varepsilon\}.$$

Since no component of K^ε is a singleton, it follows that the corresponding domain D_ε is regular by **Theorem 4.6**. Therefore, by Case I,

$$c\big(q^{-1}(K^\varepsilon)\big) = \left(\frac{c(K^\varepsilon)}{|a_d|}\right)^{1/d}.$$

The desired result now follows by letting $\varepsilon \downarrow 0$ and using **Theorem 5.2** (a).

□

This result can be used to compute the capacity of a few disconnected set which possesses symmetry. As an illustration, we do this for a union of two intervals of equal length.

Corollar 5.6.1: Capacity for Simple Symmetric Disconnected Set

If $0 \le a \le b$ then $c\big([-b, -a] \cup [a, b]\big) = \dfrac{\sqrt{b^2 - a^2}}{2}$.

Proof:

Taking $q(z) := z^2$, we have

$$c\big([-b, -a] \cup [a, b]\big) = c\big(q^{-1}[a^2, b^2]\big) = c\big([a^2, b^2]\big)^{1/2} = \left(\frac{b^2 - a^2}{4}\right)^{1/2},$$

where the first equality holds by **Theorem 5.6**, the second holds by the definition of q, and the last holds by **Corollary 5.5.1**.

□

5.3 Estimation of Capacity

Even for relatively simple sets, such as a square, calculation of the capacity requires some effort. For more complicated sets it is usually impossible, and we have to be content with estimates.

In this section we shall derive various upper and lower bounds for capacity in terms of other, more easily computed geometric quantities. As in the previous section, we shall restrict attention to compact sets, relying on the results such as **Theorem 5.1** (b) to cater for more general sets.

Many of the estimates rely on the following elementary result.

Theorem 5.7: Upper Bound Estimate for Capacity under Bounded Mapping

Let K be a compact subset of $\mathbb{C}$ and let $T : K \to \mathbb{C}$ be a mapping satisfying

$$|T(z) - T(w)| \le A \cdot |z - w|^\alpha, \; z, w \in K \tag{5.10}$$

where A and α are positive constants. Then

$$c\big(T(K)\big) \le A c(K)^\alpha.$$

Proof:

Let ν be an equilibrium measure for the compact set $T(K)$. By the "Existence of Pushforward Measure under Surjection" (see page 106), there exists a Borel probability measure μ on K such that $\mu T^{-1} = \nu$. Then

$$\begin{aligned} I(\nu) &= \int_K \int_K \log |T(z) - T(w)| d\mu(z)d\mu(w) \\ &\le \int_K \int_K \log(A|z-w|^\alpha) d\mu(z)d\mu(w) \\ &= \log A + \alpha I(\mu) \end{aligned}$$

where the equalities hold y the definition of energy and the inequality holds by the assumption **(5.10)**. Hence, from the definition of capacity, we have

$$c\big(T(K)\big) := e^{I(\nu)} \le Ae^{\alpha I(\mu)} \le Ac(K)^\alpha,$$

where the first and the last relations hold by the definition of capacity, and the second by the above display.

□

Using this theorem in conjunction with **Corollary 5.5.1**, we deduce a number of "1/4 estimates" for capacity.

Theorem 5.8: Quarter Estimates for Capacity of Certain Compact Sets

Let K be a compact subset of $\mathbb{C}$.

(a) If K is connected and has diameter d, then $c(K) \ge \dfrac{d}{4}$.

(b) If K is a rectifiable curve of length ℓ, then $c(K) \le \dfrac{\ell}{4}$.

(c) If K is a subset of the real axis of Lebesgue measure m, then $c(K) \ge \dfrac{m}{4}$.

(d) If K is a subset of the unit circle of arc-length measure a, then $c(K) \ge \sin\left(\dfrac{a}{4}\right)$.

The example of a line segment $\big($or that of a circular arc in case (d)$\big)$ shows that all these inequalities are sharp.

Proof of **Theorem 5.8**:

Step I: (a)

Rotating and translating, we can suppose that $0,d \in K$. Let $T : \mathbb{C} \to \mathbb{R}$ denote the orthogonal projection onto the real-axis. Then $T(K)$ is a connected set containing $0,d$, so it contains $[0,d]$, and hence

$$c\big(T(K)\big) \ge c\big([0,d]\big) = \frac{d}{4},$$

where the first relation holds by **Theorem 5.1** and the last by **Corollary 5.5.1**. On the other hand, T satisfies **(5.10)** with $A = \alpha = 1$, so by **Theorem 5.7**

$$c\big(T(K)\big) \le c(K) \le \frac{d}{4},$$

result follows.

Step II: (b)

Let $T : [0,\ell] \to K$ be the arc-length parametrization of K. Then T satisfies **(5.10)** with $A = \alpha = 1$, so by **Theorem 5.7** using in the first realtion and **Corollary 5.5.1** using in the second, one has

$$c(K) \leq c\big([0,\ell]\big) = \frac{\ell}{4},$$

as desired.

Step III: (c)

Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) := \text{Leb}\big(K \cap (-\infty, \infty]\big)$, where Leb denotes the Lebesgue measure. Then $T(K) = [0,m]$, so

$$c\big(T(K)\big) = c\big([0,m]\big) = \frac{m}{4}.$$

Again T satisfies **(5.10)** with $A = \alpha = 1$, so the result follows as in (a).

Step IV: (d)

If K is contained within a semicircle, then one can employ an argument similar to that in (c), using the **circular version**[6] of **Corollary 5.5.1**. For a general set K, however, it is necessary to proceed somewhat differently.

Define $f_1 : \mathbb{C}^\infty \backslash K \to \mathbb{C}$ by

$$f_1(z) := \frac{1}{4}\int_K \frac{z+\zeta}{z-\zeta}|d\zeta|,$$

so that f_1 is holomorphic on $\mathbb{C}^\infty \backslash K$, with $f_1(\infty) = \dfrac{a}{4}$ and $f_1(0) = \dfrac{-a}{4}$. Also,

$$\text{Re}\big(f_1(z)\big) = \frac{1}{4}\int_K \frac{|z|^2-1}{|z-\zeta|^2}|d\zeta| = \frac{-\pi}{2}\int_K P(z,\zeta)|d\zeta|,$$

where the second equality holds by Poisson integral, from which it follows that

$$-\frac{\pi}{2} \leq \text{Re}\big(f_1(z)\big) \leq \frac{\pi}{2}, z \in \mathbb{C}^\infty \backslash K. \tag{5.11}$$

Now define $f_2 : \mathbb{C}^\infty \backslash K \to \mathbb{C}$ by

$$f_2(z) := \frac{e^{if_1(z)} - e^{-ia/4}}{e^{if_1(z)} + e^{ia/4}},$$

so that f_2 is holomorphic on $\mathbb{C}^\infty \backslash K$, with $f_2(\infty) = ie^{ia/4}\sin(a/4)$ and $f_2(0) = 0$. Also **(5.11)** implies that $|f_2(z)| \leq 1$ $\forall z \in \mathbb{C}^\infty \backslash K$, and so, using Schwartz's lemma, it follows that

$$\left|\frac{f_2(z)}{z}\right| < 1, z \in \mathbb{C}^\infty \backslash K.$$

Finally, define $f_3 : \mathbb{C}^\infty \backslash K \to \mathbb{C}^\infty$ by

$$f_3(z) = f_2(\infty)\frac{z}{f_2(z)},$$

so that f_3 is meromorphic on $\mathbb{C}^\infty \backslash K$, with $f_3(z) = z + O(1)$ as $z \to \infty$. Then

$$|f_3(z)| > |f_2(\infty)| = \sin(a/4), z \in \mathbb{C}^\infty \backslash K.$$

---

[6] **Theorem**: (Capacity of Simple Symmetric Disconnected Set - Circular Version) Let K be the circular arc $\{e^{i\theta} : |\theta| \leq \alpha/2\}$, where $0 < \alpha < 2\pi$, and let

$$f(z) := \frac{1}{2}\Big(z - 1 + \sqrt{(z-e^{i\alpha/2})(z-e^{-i\alpha/2})}\Big),$$

where the square root is taken so that $f(z) = z + O(1)$ as $z \to \infty$. Then f maps $\mathbb{C}^\infty \backslash K$ conformally onto $\mathbb{C}^\infty \backslash \overline{\Delta}\big(0, \sin(\alpha/4)\big)$, and $c(K) = \sin(\alpha/4)$.

Now by **Theorem 5.5** we deduce that

$$c(K) \geq \sin\left(\frac{a}{4}\right),$$

as claimed.

□

As an application of this result, we prove the celebrated Koebe one-quarter theorem for univalent functions.

Theorem 5.9: Koebe's One-Quarter Theorem

If f is an injective holomorphic function on $\Delta(0,1)$ with $f(0) = 0$ and $f'(0) = 1$. Then

$$f\big(\Delta(0,1)\big) \supset \Delta\left(0,\frac{1}{4}\right).$$

That $f\big(\Delta(0,1)\big)$ contains some disc about the origin is a consequence of the open mapping theorem. The point of Koebe's theorem is that this disc always has radius at least $\frac{1}{4}$. The constant $\frac{1}{4}$ is sharp, as can be seen by considering the function $f(z) = \dfrac{z}{(1-z)^2}$.

Proof of **Theorem 5.9**:

Let K be the compact set given by

$$K := \left\{z \in \mathbb{C} : \frac{1}{z} \notin f\big(\Delta(0,1)\big)\right\}$$

and define $f_1 : \mathbb{C}^\infty \backslash \overline{\Delta}(0,1) \to \mathbb{C}^\infty \backslash K$ by

$$f_1(z) = \frac{z}{f(1/z)}.$$

Then f_1 is a conformal homeomorphism, and $f_1(z) = z + O(1)$ as $z \to \infty$. Thus by **Theorem 5.5** conformal case using in the first and **Corollary 5.4.1** using in the second, we have

$$c(K) = c\big(\overline{\Delta}(0,1)\big) = 1.$$

Moreover, $\mathbb{C}^\infty \backslash K$ is homeomorphic to $\mathbb{C}^\infty \backslash \overline{\Delta}(0,1)$, which is simply connected, and hence K is connected. Therefore by **Theorem 5.8** (a),

$$\operatorname{diam}(K) \leq 4c(K) = 4.$$

As $0 \in K$, we deduce that $K \subset \overline{\Delta}(0,4)$, from which the result follows.

□

As we saw in **Theorem 5.3**, it is an easy consequence of the definition of capacity that

$$c(K) \leq \operatorname{diam}(K)$$

for every compact set K. But in fact this can be improved.

Theorem 5.10: Capacity Upper Bound for Compact Set with Finite Diameter

If K is a compact subset of $\mathbb{C}$ with diameter d. Then

$$c(K) \leq \frac{d}{2}.$$

The example of a disc shows that this inequality is sharp.

Proof of **Theorem 5.10**:

Replacing K by its convex hull, which increases the capacity but leaves the diameter unchanged, we can assume that K is convex. We may also suppose that K contains more than one point, so by the Riemann mapping **Theorem 4.28** there is a conformal map $f : \mathbb{C}^\infty \backslash K \to \mathbb{C}^\infty \backslash \overline{\Delta}(0,1)$ with $f(\infty) = \infty$. Define

$$u : \mathbb{C}\backslash K \to [-\infty, \infty)$$

by

$$u(z) := \log\left|\frac{z - f^{-1}\big(-f(z)\big)}{d}\right| - g_{\mathbb{C}^\infty \backslash K}(z, \infty),$$

so that u is subharmonic on $\mathbb{C}\backslash K$. Then, using **Theorem 5.4**, we have

$$u(z) = \log\left|\frac{2z}{d}\right| - \log|z| + \log c(K) + o(1) \text{ as } z \to \infty,$$

and so we can remove the singularity at ∞ by **Theorem 3.13** via setting

$$u(\infty) = \log\left(\frac{2}{d}\right) + \log c(K).$$

Now

$$\operatorname{dist}\Big(f^{-1}\big(-f(z)\big), \partial K\Big) \to 0 \text{ as } \operatorname{dist}(z, \partial K) \to 0.$$

Therefore,

$$\limsup_{z \to \zeta} u(z) \leq \log\left|\frac{d}{d}\right| - 0 = 0, \zeta \in \partial K.$$

Hence by the maximal principle **Theorem 2.5** (b), $u \leq 0$ on $\mathbb{C}^\infty \backslash K$, and in particular $u(\infty) \leq 0$, result follows from **Theorem 5.4**.

□

Since there are sets, such as line segments, which have positive capacity but zero area, we would not expect to find an upper bound for capacity in terms of area. But there is a lower bound, which can be viewed as a kind of isoperimetric inequality for capacity.

Theorem 5.11: Lower Bound for Capacity of Compact Sets with Finite Area

If K is a compact susbet of $\mathbb{C}$ with area A, then $c(K) \geq \sqrt{A/\pi}$.

The example of a disc shows that this inequality is sharp, though if K is connected then it can be generalized to take account of the "dispersion" of K (see Exercise 5). The proof of **Theorem 5.11** proceeds with a lemma, which is of interest in its own right.

Lemma 5.12: Ahlfors-Beurling Inequality

If K is a compact subset of $\mathbb{C}$ with area A, then

$$\left|\int_K \frac{1}{w - z} dA(w)\right| \leq \sqrt{\pi A}, z \in \mathbb{C}.$$

Proof:

We begin by making some reductions. First of all, if K has zero area then the inequality is obvious, so we may as well assume that $A > 0$. Also, it is enough to prove the inequality for the special case $z = 0$; as the general case then follows by applying this to the translate $K - z$. Finally, we can suppose that

$$\int_K w^{-1} dA(w) \geq 0,$$

otherwise just rotate K about the origin until it became true.

Let Δ be the disc $\left\{ w \in \mathbb{C} : \mathrm{Re}\left(\frac{1}{2}\right) > \frac{1}{2a} \right\}$, where the radius a is chosen so that Δ and K have the same area, in other words, $\pi a^2 = A$. Then

$$\begin{aligned}
\int_K \frac{1}{w} dA(w) &= \int_K \mathrm{Re}\left(\frac{1}{w}\right) dA(w) \\
&\leq \int_{K\cap\Delta} \mathrm{Re}\left(\frac{1}{w} dA(w) + \int_{K\backslash\Delta} \frac{1}{2a} dA(w)\right. \\
&= \int_{K\cap\Delta} \mathrm{Re}\left(\frac{1}{w}\right) dA(w) + \int_{\Delta\backslash K} \frac{1}{2a} dA(w) \\
&\leq \int_{\Delta} \mathrm{Re}\left(\frac{1}{w}\right) dA(w) \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} \frac{\cos\theta}{r} r dr d\theta \\
&= \pi a = \sqrt{\pi A},
\end{aligned}$$

where the first relation holds as a is chosen so that Δ and K have the same area, the second holds by the subadditive and assumption $\mathrm{Re}(1/w) > 1/2a$, the third holds as Δ and K have the same area, the fourth by the assumption $\mathrm{Re}(1/w) > 1/2a$, and the fifth by change of coordinates. This display gives the desired inequality.

□

This result tells us that the size of the kernel over a compact set is bounded above by a constant multiple of the measure of the set (in our case the area of the set).

Proof of **Theorem 5.11**:

Let D be the component of $\mathbb{C}^\infty \backslash K$ containing ∞, and define $f : D \to \mathbb{C}^\infty$ by

$$f(z) := \left(\frac{1}{A}\int_K \frac{1}{z-w} dA(w)\right)^{-1}.$$

Then f is meromorphic, $f(z) = z + O(1)$ as $z \to \infty$, and by **Lemma 5.12** f maps D onto $\mathbb{C}^\infty \backslash \overline{\Delta}\left(0, \sqrt{A/\pi}\right)$. Hence by **Theorem 5.5** using in the first and **Corollary 5.4.1** using in the second, one has

$$c(K) \geq c\left(\overline{\Delta}\left(0, \sqrt{A/\pi}\right)\right) = \sqrt{A/\pi},$$

as claimed.

□

Finally, in this section, we return to the problem mentioned at the beginning of this chapter, to determine whether or not the Cantor set is polar. In fact we shall study the generalized Cantor set, constructed as follows.

Definition: Generalized Cantor Set

Let $s := \{s_n\}_{n\geq 1}$ be a sequence of numbers such that $0 < s_n < 1\ \forall n \geq 1$.
Define $C(s_1)$ to be the set obtained from [0,1] by removing an open interval of length s_1 from the center.
At the n-th stage, let $C(s_1, \cdots, s_n)$ be the set obtained by removing from the middle of each interval in $C(s_1, \cdots, s_{n-1})$ an open sub-interval whose length is a proportion s_n of the whole interval. We thereby obtain a decreasing sequence of compact sets $\left\{C(s_1, \cdots, s_n)\right\}_{n\geq 1}$, and the corresponding generalized Cantor set is defined to be

$$C(s) := \bigcap_{n\geq 1} C(s_1, \cdots, s_n).$$

It is readily checked that $C(s)$ is a compact, perfect, totally disconnected set of Lebesgue measure $\Pi_{n\geq 1}(1 - s_n)$.

We now investigate the capacity of the generalized Cantor set.

Theorem 5.13: Capacity Bounds for Generalized Cantor Set

Let $p := \Pi_{n\geq 1}(1 - s_n)^{1/2^n}$ and $q := \Pi_{n\geq 1} s_n^{1/2^n}$. Then

$$\frac{pq}{2} \leq c\big(C(S)\big) \leq \frac{p}{2}.$$

Thus, for example, the standard one-third Cantor set, which is obtained by taking $s_n = \dfrac{1}{3}$ for all n, has capacity at least $\dfrac{1}{9}$, and in particular it is non-polar.

Example 5.5: Uncountable Polar Set

If we set $s_n := 1 - (1/2)^{2^n}$, then $C(S)$ is polar, thereby providing the long-promised example of uncountable polar set. ⋄

Proof of **Theorem 5.13**:

Step I: Upper Bound

We begin with proving the upper bound. Put $K := C(s_1, \cdots, s_n)$, and let K_1, K_2 denote the left hand side and right hand side of K, respectively.
As $\operatorname{diam}(K) = 1$, we can apply **Theorem 5.3** (a) with $d = 1$ to obtain

$$\frac{1}{\log\big(1/c(K)\big)} \leq \sum_{j=1}^{2} \frac{1}{\log\big(1/C(K_j)\big)}.$$

By the symmetry $c(K_1) = c(K_2)$, so the above inequality simplifies to

$$\log c(K) \leq \frac{1}{2} \log c(K_1).$$

Now $K = C(s_1, \cdots, s_n)$, and K_1 is just the set $C(s_2, \cdots, s_n)$ scaled down by a factor $\dfrac{1 - s_1}{2}$, so the inequality becomes

$$\log\Big(c\big(C(s_1, \cdots, s_n)\big)\Big) \leq \sum_{j=1}^{n} \frac{1}{2^j} \log\left(\frac{1 - s_j}{2}\right) + \frac{1}{2^n} \log c\big([0,1]\big).$$

Sending $n \to \infty$ yields

$$\log c\big(C(s)\big) \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \log\left(\frac{1 - s_j}{2}\right),$$

which gives the desired upper bound.

Step II: Lower Bound

The lower bound is proved in a similar fashion. With K, K_1, K_2 as before, we have $\operatorname{dist}(K_1, K_2) = s_1$, so applying **Theorem 5.3** (b) with $d = s_1$ gives

$$\frac{1}{\log^+\big(s_1/c(K)\big)} \geq \sum_{j=1}^{2} \frac{1}{\log^+\big(s_1/c(K_j)\big)}.$$

If $c(K) < s_1$, then this simplifies to

$$\log c(K) \geq \frac{1}{2}\log s_1 + \frac{1}{2}\log c(K_1).$$

If $c(K) \geq s_1$, then this inequality is clear anyway, since $c(K) \geq c(K_1)$. Repeating the argument used in the first step leads to

$$\log c\big(C(s)\big) \geq \sum_{j=1}^{\infty} \frac{1}{2^j}\log s_j + \sum_{j=1}^{\infty} \frac{1}{2^j}\log\Big(\frac{1-s_j}{2}\Big),$$

which yields the lower bound.

□

5.4 Criterion for Thinness

As we saw in **Theorem 4.9** (b), the question of whether a given point is regular for the Dirichlet problem on a domain D is equivalent to whether $\mathbb{C}_\infty \backslash D$ is non-thin at the point. Unfortunately, at that time we had no general criterion for thinness, but with the theory of capacity at our disposal, we are now in a position to put that right.

Theorem 5.14: Wiener's Criterion for Thinness

Let F be an F_σ subset of $\mathbb{C}$ and let $\zeta_0 \in \mathbb{C}$. Let γ be a constant with $0 < \gamma < 1$, and for $n \geq 1$ define

$$F_n := \{z \in F : \gamma^n < |z - \zeta_0| \leq \gamma^{n-1}\}.$$

Then F is thin at ζ_0 if and only if

$$\sum_{k\geq 1} \frac{n}{\log\big(2/c(F_n)\big)} < \infty. \tag{5.12}$$

Proof:

Since thinness and capacity both remain invariant under translation, we may as well suppose from the outset that $\zeta_0 = 0$. We can also suppose that $0 \notin F$, and that $F_n \neq \emptyset \ \forall n \geq 1$ (otherwise just remove 0, and add an appropriate countable set).

Step I: $\Leftarrow$

Assume first that **(5.12)** holds, we shall show that F is thin at 0. As each F_n is an F_σ set, we may write it as $F_n := \bigcup_{n\geq 1} K_{nm}$, where $\{K_{nm}\}_{m\geq 1}$ is an increasing sequence of compact sets. For each pair n, m, let ν_{nm} be an equilibrium measure for K_{nm} (the existence is guaranteed by **Theorem 3.21**). Then

$$p_{\nu_{nm}} = I(\nu_{nm}) = \log c(K_{nm}) \leq \log c(F_n) \text{ n.e. on } K_{nm},$$

where the first relation holds by Frostman's **Theorem 3.7**, the second holds by definition of capacity, and the last holds by **Theorem 5.1** (a). Moreover, as $K_{nm} \subset \overline{\Delta}(0,1)$, which has diameter 2, we have

$$p_{\nu_{nm}} \leq \log 2 \text{ on } \overline{\Delta}(0,1).$$

Lastly, since $K_{nm} \cap \overline{\Delta}(0,\gamma^n) = \emptyset$, it follows that

$$p_{\nu_{nm}}(0) = \int_{K_{nm}} \log |w| \, d\nu_{nm}(w) \geq n \log \gamma,$$

where the first relation holds by the definition of logarithmic potential and the second holds by our assumption on F_n. Now set

$$\alpha_n := \frac{1}{\log\big(2/c(F_n)\big)}.$$

By our assumption, $\Sigma_{n\geq 1} n\alpha_n < \infty$, so we can find a sequence of positive numbers $\{\beta_n\}_{n\geq 1}$ such that $\beta_n \to \infty$ and still $\Sigma_{n\geq 1} n\alpha_n\beta_n < \infty$. For each $m \geq 1$, define u_m on $\Delta(0,1)$ by

$$u_m := \sum_{n\geq 1} \alpha_n\beta_n\big(p_{\nu_{nm}} - \log 2\big).$$

Then by **Theorem 2.12**, u_m is subharmonic on $\Delta(0,1)$, and

$$u_m \leq -\beta_n \text{ n.e. on } K_{nm}$$
$$u_m \leq 0 \text{ on } \Delta(0,1)$$
$$u_m(0) \geq \sum_{n\geq 1} \alpha_n\beta_n\big(n \log \gamma - \log 2\big)$$

Next, define u on $\Delta(0,1)$ by

$$u := \big(\limsup_{m\to\infty} u_m\big)^*,$$

where $*$ denotes the upper semicontinuous regularization, thus **Theorem 3.9** (a) tells us that u is subharmonic on $\Delta(0,1)$, and by **Theorem 3.9** (b)

$$u \leq -\beta_n \text{ n.e. on } F_n$$
$$u \leq 0 \text{ on } \Delta(0,1)$$
$$u(0) \geq \sum_{n\geq} \alpha_n\beta_n\big(n \log \gamma - \log 2\big).$$

In particular, if we set

$$E := \bigcup_{n\geq 1} \{z \in F_n : u(z) \geq -\beta_n\},$$

then

$$\limsup_{z\to 0, z\in F\backslash E} u(z) \leq \lim_{n\to\infty} -\beta_n = -\infty < u(0),$$

where the first relation holds since $u \leq -\beta_n$ n.e. on F_n, the second holds by assumption of β_n, and the last by the bound of $u(0)$ we showed above. Therefore, $F\backslash E$ is thin at 0 by definition. But E is an F_σ polar set (since $u \leq -\beta_n$ n.e. on F_n, thus $u \geq -\beta_n$ on polar sets, **Corollary 3.4.2** then tells us that E is polar), therefore by **Theorem 3.25**, E is thin at 0 too. It follows that F is thin at 0, as desired.

Step II: $\Rightarrow$

Now we suppose that F is thin at 0. By the definition of thinness, there exists a subharmonic function u on a neighbourhood of 0 such that

$$\limsup_{z\to 0, z\in F} u(z) < u(0).$$

By the Riesz decomposition **Theorem 3.23**, we may take u to be of the form $u = p_\mu$, where μ is a finite Borel measure on $\overline{\Delta}(0,1)$. Then in particular

$$p_\mu(0) > -\infty,$$

and hence, writing

$$A_k := \left\{w : \gamma^k < |w| \le \gamma^{k-1}\right\},$$

we have

$$p_\mu(z) - p_\mu(0) = \sum_{k\ge 1} \int_{A_k} \log\left|1 - \frac{z}{w}\right| d\mu(w),\ z \in \mathbb{C}.$$

Now we decompose the summation into three parts and consider the value on the right hand side respectively:

$$1 \le k \le n-2,\ n+2 \le k < \infty, \text{ and } n-1 \le k \le n+1.$$

Case I: $1 \le k \le n-2$

Now, if $z \in A_n$ and $w \in A_k$, where $k \le n-2$, then $|z/w| \le \gamma^{n-k-1}$, and so

$$\inf_{z\in A_n} \sum_{k=1}^{n-2} \int_{A_k} \log\left|1 - \frac{z}{w}\right| d\mu(w) \ge \sum_{k=1}^{n-2} \log\left(1 - \gamma^{n-k-1}\right)\mu(A_k) \xrightarrow[n\to\infty]{} 0.$$

Case II: $n+2 \le k < \infty$

Also, if $z \in A_n$ and $w \in A_k$, where $k \ge n+2$, then $|z/w| \ge \gamma^{-1}$, so

$$\inf_{z\in A_n} \sum_{k=n+2}^{\infty} \int_{K} \log\left|1 - \frac{z}{w}\right| d\mu(w) \ge \sum_{k=n+2}^{\infty} \log\left(\gamma^{-1} - 2\right)\mu(A_k) \xrightarrow[n\to\infty]{} 0.$$

Case III: $n-1 \le k \le n+1$

Lastly, since μ is supported on $\overline{\Delta}(0,1)$, we have

$$\sum_{k=n-1}^{n+1} \int_{A_k} \log|w|\, d\mu(w) \le 0.$$

It follows that, given any $\varepsilon > 0$, there exists n_0 such that $\forall n \ge n_0$,

$$\sum_{k=n-1}^{n+1} \int_{A_k} \log|z-w|\, d\mu(w) \le p_\mu(z) - p_\mu(0) + \varepsilon,\ z \in A_n.$$

Thus, combining the three cases we have proved, we can choose an ε sufficie-ntly small so that

$$\limsup_{z\to 0, z\in F} p_\mu(z) < p_\mu(0) - 2\varepsilon.$$

Then, increasing n_0 if necessary, we have that, for all $n \ge n_0$,

$$\sum_{k=n-1}^{n+1} \int_{A_k} \log|z-w|\, d\mu(w) \le -\varepsilon,\ z \in F_n.$$

For each $n \ge n_0$, write $F_n := \bigcup_{m\ge 1} K_{nm}$, where $\{K_{nm}\}_{m\ge 1}$ is an increasing seque-nce of compact sets, and let ν_{nm} be the equilibrium measure for K_{nm} (again, the existence is guaranteed by **Theorem 3.21**). Then replacing the last inequality by $p_{\nu_{nm}}$ yields

$$\sum_{k=n-1}^{n+1} \int_{A_k} p_{\nu_{nm}}(w)d\mu(w) \leq -\varepsilon.$$

Now by Frostman's **Theorem 3.7** (a),

$$p_{\nu_{nm}} \geq I(\nu_{nm}) = \log c(K_{nm}) \text{ on } \mathbb{C}.$$

Hence,

$$\sum_{k=n-1}^{n+1} \log c(K_{nm})\mu(A_k) \leq -\varepsilon \text{ for } n \geq n_0, m \geq 1.$$

Sending $m \to \infty$ and rearranging the terms yields

$$\frac{1}{\log\left(1/c(F_n)\right)} \leq \frac{1}{\varepsilon} \sum_{k=n-1}^{n+1} \mu(A_k), \, n \geq n_0.$$

Thus to show that **(5.12)** holds, it suffices to show that

Claim: $\sum_{n\geq 1} (n-1)\mu(A_n) < \infty$.

This is done by observing that if $w \in A_n$ then

$$\log|w| \leq -(n-1)\log(1/\gamma).$$

Therefore,

$$\sum_{n\geq 1} (n-1)\mu(A_n) \leq -\sum_{n\geq 1} \int_{A_n} \frac{\log|w|}{\log(1/\gamma)} d\mu(w) = -\frac{p_\mu(0)}{\log(1/\gamma)} < \infty.$$

This completes the proof.

□

As we have seen in the proof, we interchange the use of energy, capacity, and potential whenever one makes us more advantageous. This interchange can be done via the bridge given by Frostman's **Theorem 3.7**. Moreover, we can add subharmonic functions into display by Riesz's decomposition **Theorem 3.23**.

Even though the criterion **(5.12)** is rather complicated, it can be combined with the results of the previous section to provide simpler conditions which are necessary for thinness, or, equivalently, ones which are sufficient for non-thinness.

Theorem 5.15: Set Thin at Zero Has Finite Logarithmic Measure

Let F be an F_σ subset of $\mathbb{C}$. If F is thin at 0, then

$$E := \left\{r \in (0,1] : re^{i\theta} \in F \text{ for some } \theta\right\}$$

is a set of finite logarithmic measure, that is,

$$\int_E \frac{1}{x} dx < \infty.$$

Proof:

Let $0 < \gamma < 1$, and for $n \geq 1$ define

$$F_n := \left\{z : \gamma^n < |z| < \gamma^{n-1}\right\}.$$

Let $T : \mathbb{C} \to \mathbb{R}$ denote the circular projection $T(z) := |z|$. Then, by applying **Theorem 5.8** (c) in the first relation and **Theorem 5.7** in the second to a sequence of compact sets increasing to F_n, we obtain

$$\int_{T(F_n)} dx \le 4c\big(T(F_n)\big) \le 4c(F_n).$$

Since $E := \bigcup_{n\ge1} T(F_n)$, it follows that

$$\int_E \frac{1}{x}dx \le \sum_{n\ge1}\int_{T(F_n)} \frac{1}{\gamma^n}dx \le \sum_{n\ge1}\frac{4c(F_n)}{\gamma^n},$$

where the first inequality holds since $T(z) := |z|$ and $E = \cup_{n\ge1} T(F_n)$, and the definition of F_n; the second inequality holds by the above display. Note that

$$t \le \frac{1}{\log(1/t)} \text{ for } t \in (0,1),$$

thus by **Theorem 5.3** (a),

$$\frac{c(F_n)}{\gamma^n} \le \frac{2}{\log\big(2\gamma^n/c(F_n)\big)}.$$

Hence

$$\int_E \frac{1}{x}dx \le 8\sum_{n\ge1}\frac{1}{\log\big(2\gamma^n/c(F_n)\big)},$$

and since F is thin at 0, this result is finite by **Theorem 5.14**.

□

Using radial projection instead of circular projection leads to a different type of result. In particular, **Theorem 5.15** and **Theorem 5.16** would allow us to construct polar set from F_σ set.

Theorem 5.16: Polar Set Derived from Thin Set via Radial Projection

Let F be an F_σ subset of $\mathbb{C}$. If F is thin at 0, then

$$E := \left\{e^{i\theta} : r_n e^{i\theta} \in F \text{ for some sequence } r_n \to 0\right\}$$

is a polar set.

Proof:

Again, let $0 < \gamma < 1$ and define

$$F_n := \left\{z : \gamma^n < |z| < \gamma^{n-1}\right\}.$$

This time, let $T : \mathbb{C}\backslash\{0\} \to \partial\Delta(0,1)$ be the radial projection $T(z) := \dfrac{z}{|z|}$.

Then, by applying **Theorem 5.7** to a sequence of compact sets increasing to F_n we have

$$c\big(T(F_n)\big) \le \frac{c(F_n)}{\gamma^n}.$$

Now,

$$E := \bigcap_{m\ge1}\bigcup_{n\ge m} T(F_n),$$

it follows that for every $m \ge 1$, using **Theorem 5.1** (a) in the first relation, **Theorem 5.3** (a) in the second relation, and the display above for capacity bound, one obtains that

$$\frac{1}{\log\left(2/c(E)\right)} \leq \frac{1}{\log\left(2/c\left(\cup_{n\geq m} T(F_n)\right)\right)}$$
$$\leq \sum_{n\geq m} \frac{1}{\log\left(2/c\left(T(F_n)\right)\right)}$$
$$\leq \sum_{n\geq m} \frac{1}{\log\left(2\gamma^n/c(F_n)\right)}$$

Again, if F is thin at 0, then by **Theorem 5.14**, the last series converges, and, hence, sending $m \to \infty$ gives $c(E) = 0$ and it follows that E is polar.

□

This has the following pleasant consequence.

Corollary 5.16.1: Radial Convergence for Subharmonic Functions Near Origin

If u is a function subharmonic on a neighbourhood of 0 then
$$\lim_{r\to 0} u(re^{i\theta}) = u(0) \text{ for n.e. } e^{i\theta}.$$

Proof:

For each $k \geq 1$, define
$$U_k := \left\{ z \in \mathbb{C} : u(z) < u(0) - \frac{1}{k} \right\}.$$

Then U_k is an open set which is thin at 0, so by **Theorem 5.16**,
$$\liminf_{r\to 0} u(re^{i\theta}) \geq u(0) - \frac{1}{k} \text{ for n.e. } e^{i\theta}.$$

As a countable union of Borel polar set is polar by **Corollary 3.4.2**, we have
$$\liminf_{r\to 0} u(re^{i\theta}) \geq u(0) \text{ for n.e. } e^{i\theta}.$$

On the other hand, by upper semicontinuity, we ccertainly have
$$\limsup_{r\to 0} u(re^{i\theta}) \leq u(0) \text{ for all } e^{i\theta}.$$

Combining these inequalities yields the desired result.

□

5.5 Transfinite Diameter

There is another approach to capacity which is actually more direct than our definition in the first section. As well as giving further useful estimates for capacity, it has close links with the theory of uniform approximation. It is based on the following definition.

Definition: n-th Diameter

Let K be a compact subset of $\mathbb{C}$ and let $n \geq 2$. The n-th diameter of K is given by
$$\delta_n(K) := \sup\left\{ \prod_{j,k:j<k} |w_j - w_k|^{2/n(n-1)} : w_1, \cdots, w_n \in K \right\}.$$

Definition: Fekete n-Tuple

An n-tuple $w_1, \cdots, w_n \in K$ for which the supremum is attained is called the

Fekete n-tuple for K.

As K is compact, a Fekete n-tuple always exists, though it needs not to be unique. The maximum principle **Theorem 2.5** shows that in fact it must lie in $\partial_e K$.

Evidently, $\delta_2(K)$ is just the usual diameter of K, and $\delta_n(K) \leq \delta_2(K)$ for all $n \geq 2$. Indeed, as we shall shortly see, the sequence $\{\delta_n(K)\}_{n\geq 2}$ is decreasing, so it has a limit, often called the transfinite diameter of K. Actually, as the following theorem suggests, this is nothing than the capacity.

Theorem 5.17: Fekete-Szegö Theorem

Let K be a compact subset of $\mathbb{C}$. Then the sequence $\{\delta_n(K)\}_{n\geq 1}$ is decreasing and

$$\lim_{n\to\infty} \delta_n(K) = c(K).$$

Proof:

In order to simplify the notation, throughout the proof we shall denote $\delta_n(K)$ as δ_n.

Claim I: $\{\delta_n\}_{n\geq 2}$ is decreasing

Let $n \geq 2$ and choose $w_2, \cdots, w_{n+1} \in K$ such that

$$\delta_{n+1}^{n(n+1)/2} = \prod_{i\leq j<k\leq n+1} |w_j - w_k|.$$

Then, since $w_2, \cdots, w_{n+1}$ is an Fekete n-tuple in K, by definition one has

$$\delta_n^{n(n-1)/2} \geq \prod_{2\leq j<k\leq n+1} |w_j - w_k|.$$

There are $n+1$ such inequalities in all, the m-th tuple $(1 \leq m \leq n+1)$ one obtained by omitting the terms involving w_m. Multiplying them all together gives

$$\left(\delta_n^{n(n-1)/2}\right)^{n+1} \geq \prod_{1\leq j<k\leq n+1} |w_j - w_k|^{n-1} = \left(\delta_n^{n(n+1)/2}\right)^{n-1}.$$

Hence $\delta_n \geq \delta_{n+1}$ as desired.

Claim II: $\delta_n \geq c(K)$ $\forall n \geq 2$.

Next, we show that $c(K) \leq \delta_n$ for all $n \geq 2$. If $z_1, \cdots, z_n \in K$. Then taking log on both sides and using the definition of energy give

$$\frac{2}{n(n-1)} \sum_{1\leq j<k\leq n} \log|z_j - z_k| \leq \log \delta_n.$$

Integrating this inequality with respect to $d\nu(z_1), \cdots, d\nu(z_n)$, where ν is an equilibrium measure for K (existence and uniqueness by **Theorem 3.21**), we have

$$\frac{2}{n(n-1)} \sum_{1\leq j<k\leq n} \int_K \int_K \log|z_j - z_k|\, d\nu(z_j) d\nu(z_j) \leq \log \delta_n.$$

Hence $I(\nu) \leq \log \delta_n$, thus by the definition of capacity $c(K) \leq \delta_n$ $\forall n \geq 2$.

Claim III: $c(K) \geq \limsup_{n\to\infty} \delta_n$ $\forall n \geq 2$.

Choose $\varepsilon > 0$ and set

$$\delta_n^{n(n-1)/2} = \prod_{j<k} |w_j - w_k|.$$

For each j, let μ_j be normalized Lebesgue measure on the circle $\partial\Delta(w_j, \varepsilon)$ and put $\mu := n^{-1}\sum_{j=1}^{n}\mu_j$. Then $I(\mu)$ is given by

$$\frac{1}{n^2}\sum_{j\geq 1}\iint \log|z-w|\,d\mu_j(z)d\mu_j(w) + \frac{2}{n^2}\sum_{j<k}\iint \log|z-w|\,d\mu_j(z)d\mu_k(w).$$

Now for each j,

$$\iint \log|z-w|\,d\mu_j(z)d\mu_j(w) =: I(\mu_j) = \log\varepsilon,$$

where the last equality holds by **Corollary 3.21.1** and **Corollary 5.4.1**.
Moreover, for each pair $j<k$,

$$\iint \log|z-w|\,d\mu_j(z)d\mu_k(w) = \int p_{\mu_j}(w)d\mu_k(w) \geq p_{\mu_j}(w_k),$$

where the first relation holds by the definition of potential and the second by the upper semicontinuous since p_{μ_j} is subharmonic by **Theorem 3.1** (a).
Furthermore, using the same argument, we have

$$p_{\mu_j}(w_k) = \int \log|z-w_k|\,d\mu_j(z) \geq \log|w_j - w_k|$$

since $\log|w_j-w_k|$ is subharmonic by **Theorem 2.19**.
Summing these together yields

$$\begin{aligned} I(\mu) &\geq \frac{1}{n^2}\sum_{j=1}^{n+1}\log\varepsilon + \frac{2}{n^2}\sum_{1\leq j<k\leq n+1}\log|w_j-w_k| \\ &= \frac{1}{n}\log\varepsilon + \frac{n-1}{n}\log\delta_n \end{aligned}$$

where the second term in the last equality comes from the definition for n-th diameter. Since μ is supported on K^ε, it follows that

$$c(K^\varepsilon) \geq \varepsilon^{1/n}\delta_n(K)^{(n-1)/n}.$$

Hence $\limsup_{n\to\infty}\delta_n \leq c(K^\varepsilon)$. Finally, since $\varepsilon > 0$ is arbitrary, sending $\varepsilon \downarrow 0$ and using **Theorem 5.2** (a) yield the desired result.

□

Much of the importance of this theorem derives from its connection with polynomial approximation. For several reasons, it is of interest to find monic polynomials $q(z)$ for which the sup-norm on K,

$$\|q\|_K := \sup\{|q(z)| : z \in K\},$$

is relatively small. We now consider one such class.

Definition: Fekete Polynomial

Let K be a compact subset of $\mathbb{C}$ and let $n \geq 2$. A Fekete polynomial for K of degree n is a polynomial of the form

$$q(z) := \prod_{j=1}^{n}(z - w_j),$$

where $w_1, \cdots, w_n$ is a Fekete n-tuple for K.

Theorem 5.18: Capacity Bounds via Fekete Polynomial

Let K be a compact subset of $\mathbb{C}$.

(a) If q is a monic polynomial of degree $n \geq 1$, then $\|q\|_K^{1/n} \geq c(K)$.

(b) If q is a Fekete polynomial of degree $n \geq 2$, then $\|q\|_K^{1/n} \leq \delta_n(K)$.

Proof:

Step I: Assertion (a)

Since

$$\begin{aligned} q^{-1}\left(\overline{\Delta}(0,\|q\|_K)\right) &= \left\{z \in \mathbb{C} : q(z) \in \overline{\Delta}(0,\|q\|_K)\right\} \\ &= \left\{z \in \mathbb{C} : |q(z)| \leq \|q\|_K\right\} \end{aligned}$$

we have $\forall z \in K, z \in \left\{z \in \mathbb{C} : |q(z)| \leq \|q\|_K\right\}$. Thus $K \subset q^{-1}\left(\overline{\Delta}(0,\|q\|_K)\right)$.

It follows that

$$c(K) \leq c\left(q^{-1}\left(\overline{\Delta}(0,\|q\|_K)\right)\right) = \left(\frac{c\left(\overline{\Delta}(0,\|q\|_K)\right)}{1}\right)^{1/n} = \|q\|_K^{1/n}$$

where the first realation holds by monotonicity of capacity in **Theorem 5.1** (a), the second holds by **Theorem 5.6** where $|a_d| = 1$ since monic, and the last by **Corollary 5.4.1**.

Step II: Assertion (b)

Suppose that $q(z) = \prod_{i=1}^{n}(z - w_i)$, where $w_1, \cdots, w_n$ is an Fekete n-tuple for K.

If $z \in K$, then $z, w_1, \cdots, w_n$ is an $(n+1)$-tuple for K, so

$$\prod_{i=1}^{n}|z - w_i|\prod_{j<k}|w_j - w_k| \leq \delta_{n+1}(K)^{n(n+1)/2},$$

and hence

$$|q(z)| \leq \frac{\delta_{n+1}(K)^{n(n+1)/2}}{\delta_n(K)^{n(n-1)/2}} \leq \frac{\delta_n(K)^{n(n+1)/2}}{\delta_n(K)^{n(n-1)/2}} = \delta_n(K)^n,$$

where the first relation holds by the above display, the second holds since δ_n is decreasing as we have shown in the first claim for the proof of **Theorem 5.17**, and the last holds by the definition of the Fekete n-th diamter.

Finally, since z is chosen arbitrarily, the desired inequality follows.

□

In particular, the second inequality in **Theorem 5.18** together with **Theorem 5.17** tells us that, sending $n \to \infty$, the left hand side is the sup norm and the right hand side is nothing but the capacity for K. Thus we have a lower bound for capacity via Fekete n-th diameter. This aligned with our intuition.

Combining the last two theorems leads immediately to another characterization of capacity.

Corollary 5.18.1: Characterization of Capacity via Monic Polynomial

Let K be a compact subset of $\mathbb{C}$, and for each $n \geq 1$ let

$$m_n := \inf\left\{\|q\|_K : q \text{ is a monic polynomial of degree } n\right\}.$$

Then

$$\lim_{n\to\infty} m_n(K)^{1/n} = \inf_{n\geq 1} m_n(K)^{1/n} = c(K).$$

Definition: Chebyshev Polynomial

A monic polynomial q of degree n for which $\|q\|_K = m_n(K)$ is said to be a Chebyshev polynomial.

Remark 5.1: Comparison of Chebyshev Polynomial and Fekete Polynomial

It can be shown that the Chebyshev polynomial exists and, provided that K has at least n points, is unique. However, the Fekete polynomials have the advantage that, unlike the Chebyshev polynomials, their zeros always belong to K. ⋄

As an illustration of this, we now use them to prove a strong form of **Lemma 3.12**. It states that for every compact polar set, there exists a Borel probability measure whose potential is minus infinity, hence zero energy.

Theorem 5.19: Evan's Theorem

Let E be a compact polar set. Then there exists a Borel probability measure μ on E such that

$$p_\mu(z) = -\infty \ \forall z \in E.$$

Proof:

Given $n \geq 2$, let $w_1, \cdots, w_n$ be a Fekete n-tuple for K, and let q_n be the corresponding Fekete polynomial. If μ_n denotes the probability measure on K consisting of $\dfrac{1}{n}$ - masses at $w_1, \cdots, w_n$, then

$$p_{\mu_n}(z) = \sum_{j=1}^{n} \log|z - w_j| = \frac{1}{n}\log|q_n(z)| \leq \log \delta_n(E),\ z \in E$$

where the first relation holds by the definition of potential and Fekete n-tuple, the second holds by the definition of Fekete polynomial, and the last holds by **Theorem 5.18** (b). Now by Fekete-Szegö **Theorem 5.17**,

$$\lim_{nt\to\infty} \delta_n(E) = c(E) = 0.$$

So, replacing $\{\mu_n\}_{n\geq 1}$ by a sequence, we may suppose that

$$p_{\mu_n} \leq -2^n \text{ on } E \text{ for all } n \geq 1.$$

If we set $\mu = \displaystyle\sum_{n=1}^{\infty} 2^{-n}\mu_n$, then μ is a Borel probability measure on K, and

$$p_\mu(z) = \sum_{n=1}^{\infty} 2^{-n} p_{\mu_n}(z) \leq \sum_{n=1}^{\infty} 2^{-n}(2^{-n}) = -\infty,\ z \in E.$$

Thus μ has the desired property.

□

Knowledge of $\|q\|_K$ also gives us information about how q behaves off K. If D is a bounded component of $\mathbb{C}^\infty \backslash K$ then

$$|q(z)| \leq \|q\|_K \ \forall z \in D,$$

by the maximum principle **Theorem 2.5**. The next result tells us what happens when D is the unbounded component. The first part contains the basic inequality, and the second part gives some indication of its sharpness.

Theorem 5.20: Berstein's Lemma

Let K be a non-polar compact subset of $\mathbb{C}$ and let D be the component of $\mathbb{C}^\infty \backslash K$ containing ∞.

(a) If q is a polynomial of degree $n \geq 1$ then

$$\left(\frac{|q(z)|}{\|q\|_K}\right)^{1/n} \leq e^{g_D(z,\infty)}, z \in D\backslash\{\infty\},$$

where g_D is the Green function of D.

(b) If q is a Fekete polynomial for K of degree $n \geq 2$ then

$$\left(\frac{|q(z)|}{\|q\|_K}\right)^{1/n} \geq e^{g_D(z,\infty)}\left(\frac{c(K)}{\delta_n(K)}\right)^{\tau_D(z,\infty)}, z \in D\backslash\{\infty\},$$

where τ_D denotes the Harnack's distance for D.

Proof:

Step I: Assertion (a)

Multiplying q by a constant, we can suppose that it is monic. If we define

$$u(z) := \frac{1}{n}\log|q(z)| - \frac{1}{n}\log\|q\|_K - g_D(z,\infty), z \in D\backslash\{\infty\}.$$

(To see this, take logarithm on both sides of the desired inequality). Then $u(z)$ is subharmonic on $D\backslash\{\infty\}$. Moreover, as $z \to \infty$, by **Theorem 5.4**,

$$u(z) = \log|z| - \frac{1}{n}\log\|q\|_K - \log|z| + \log c(K) + o(1).$$

Therefore, setting

$$u(\infty) := \log c(K) - \frac{1}{n}\log\|q\|_K$$

makes u subharmonic on D. Now since $\partial D \subset K$, we have

$$\limsup_{z\to\zeta} u(z) \leq \frac{1}{n}\log|q(\zeta)| - \frac{1}{n}\log\|q\|_K \leq 0, \zeta \in \partial D,$$

where the first inequality holds by the upper semicontinuity of u and the second holds by the definition of $\|q\|_K$, which is the supremum of $|q(z)|$. Thus, by the maximum principle **Theorem 2.5**, $u \leq 0$ on D, as desired.

Step II: Assertion (b)

If q is a Fekete polynomial, then in particular by **Remark 5.1** all its zeros lie in K, and therefore u is actually harmonic on D. Also, from part (a), $u \leq 0$ on D, so we may apply Harnack's inequality **Corollary 1.10.2** to $-u$ to obtain

$$u(z) \geq \tau_D(z,\infty)u(\infty), z \in D.$$

Now by **Theorem 5.18** (b) using in the second relation, one has

$$u(\infty) = \log c(K) - \frac{1}{n}\log\|q\|_K \geq \log c(K) - \log\delta_n(K).$$

Combining the above displays yields the desired result.

□

We offer a heuristic interpretation for Bernstein's lemma. Since Chebyshev polynomial offers us an approximation result for the capacity of K via components in $\mathbb{C}^{\infty}\backslash K$, it is naturally to ask what happens for unbounded component in $\mathbb{C}^{\infty}\backslash K$. Bernstein's lemma tells us that, the growth rate of the approximation is bounded by the exponential of Green function on that component, thus the approximation near infinity does not go wild. Moreover, if the polynomial is further Fekete, then all zeros of this polynomial lay in K, thus a convexity argument yields the sharpness.

We end this section with an application to polynomial convexity.

Definition: Polynomially Convex

A compact subset K of $\mathbb{C}$ is polynomially convex if for each $z \in \mathbb{C}\backslash K$, there exists a polynomial q such that

$$|q(z)| > \|q\|_K.$$

Remark 5.2: Necessary and Sufficient Condition for Polynomially Convex

The definition will not make sense if z belongs to a bounded component of $\mathbb{C}\backslash K$, so for K to be polynomially convex it is necessary that $\mathbb{C}\backslash K$ is connected. This condition also turns out to be sufficient. ⋄

Example 5.6: Connectedness of $\mathbb{C}\backslash K$ Is Sufficient for Polynomially Convex

A simple compactedness argument shows that, given an open neighbourhood U of K, there is a finite set of polynomials $q_1, \cdots, q_n$ such that

$$\max_{1 \le j \le n} \frac{|q_j(z)|}{\|q_j\|_K} > 1, \forall z \in \mathbb{C}\backslash U. \qquad \diamond$$

What is less obvious in **Example 5.6** is that in fact one polynomial will do the job.

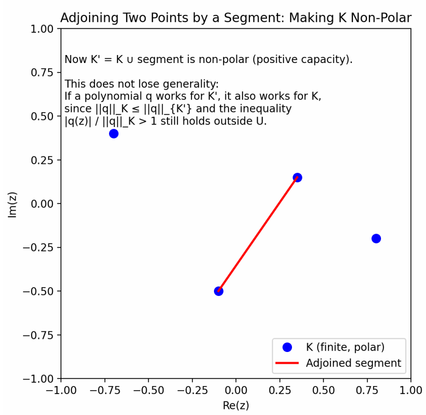
Theorem 5.21: Hilbert-Lemniscate Theorem

Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C}\backslash K$ is connected, and let U be an neighbourhood of K. Then there exists a polynomial q such that

$$\frac{|q(z)|}{\|q\|_K} > 1, \forall z \in \mathbb{C}\backslash U.$$

Proof:

We can suppose that K is non-polar, otherwise just adjoin a small line segment in U.



(Figure 5.1: Adjoining line segment in compact polar set make it non-polar)

Let $D := \mathbb{C}^{\infty} \backslash K$ and put

$$L := \inf_{z \in \mathbb{C}^{\infty} \backslash U} g_D(z, \infty) \text{ and } M := \sup_{z \in \mathbb{C}^{\infty} \backslash U} \tau_D(z, \infty),$$

so that $L > 0$ and $M < \infty$. Then by **Theorem 5.20** (b), if q is a Fekete polynomial for K of degree n, then

$$\left(\frac{|q(z)|}{\|q\|_K} \right)^{1/n} \geq e^L \left(\frac{c(K)}{\delta_n(K)} \right)^M, z \in \mathbb{C} \backslash U.$$

Since $\delta_n(K) \to c(K)$ as $n \to \infty$ by **Theorem 5.17**, the right hand side will exceed 1 for all sufficiently large n.

□

Summary of Chapter 5

Lacking the criterion for polarity, we introduce the concept of capacity to provide a characterization. In the first section, we constructed the capacity. Moreover, as it is a set function, some of its properties can be compared with the ones of measures: We first proved some of “Elementary Properties of Logarithmic Capacity”, then “Capacity Is Continuous in Monotone Sequences”. So far the connection with measures looks perfect, but capacity is not additive, so for the union operations under capacity we proved “Bound Estimates for Capacity of Borel Union” instead.

In the second section we studied the computation of capacity for certain sets, which are based on the connection of capacity and Green function, namely, “Capacity of Compact Non-Polar Set via Green Function”. Then immediately “Capacity of Closed Disc” is derived. We proved an inversed version “Inversed Subordination Principle for Capacity”, as a result, “Capacity for Interval” is computed explicitly. Then we proved “Capacity under Inverse Image of Polynomials”, which has a corollary providing a formula for certain disconnected sets, that is, “Capacity for Simple Symmetric Disconnected Set”.

For some other sets that do not have good shapes or properties, the computation for capacity can be very hard. Thus in the third section we proved some estimation results. The first among these is “Upper Bound Estimate for Capacity under Bounded Mapping”, then we derived a collection of quarter estimates for certain compact sets - “Quarter Estimates for Capacity of Certain Compact Sets”. As an application, a result in complex analysis is proved via capacity, that is, “Koebe’s One-Quarter Theorem”. For compact sets with finite diameter, we proved “Capacity Upper Bound for Compact Set with Finite Diameter”. The area and the capacity is also related by “Lower Bound for Capacity of Compact Sets with Finite Area”, for the proof of this result we introduced a lemma “Ahlfors-Beurling Inequality” stating that the size of the kernel over a compact set is bounded above by a constant multiple of the area of the set. Finally, we constructed the generalized Cantor set and proved “Capacity Bound for Generalized Cantor Set”. All of these bounds are **SHARP**! However, for the generalized Cantor set, the polarity really depends on how it is constructed.

In the fourth section, we studied the Criterion for thinness, which enables us to tell if a set is polar or not. This is the famous “Wiener’s Criterion for Thinness”. As a result, we proved “Set Thin at Zero Has Finite Logarithmic Measure” and “Polar Set Derived from Thin Set via Radial Projection”. These two results offered us a way to construct certain polar sets. Finally we proved a corollary - "Radial Convergence for Subharmonic Functions Near Origin".

In the last section we introduced the concept of transfinite diameter. We started with the definition of Fekete tuple and proved "Fekete-Szegö Theorem", which tells us that the Fekete tuple for a compact set converges to the capacity of the same set. This enables us to introduce the Fekete polynomial and "Capacity Bounds via Fekete Polynomial". In particular, if we introduce the Chebyshev polynomial, then we can characterize the capacity via "Characterization of Capacity via Monic Polynomial". Then we proved "Evan's Theorem", which tells us that compact polar set has Borel probability measure with minus infinity potential. It is natural to ask how the approximation behaves on unbounded domains, the growth rate is bounded and the bound is sharp, which is proved in "Bernstein's Lemma". Finally, as the sharpness requires convexity, we defined the polynomial convexity and proved that connectedness of its component is necessary for a set to be polynomially convex, and, also, sufficient, by "Hilbert-Lemniscate Theorem".

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