Physics4A Week 6

There are different types of collisions: explosions, inelastic, and eleastic.

The first possibility is that a sigle object may break apart into two or more pieces. These can be difficult to analyze if the number of fragments after the collision is more than about three or four; but nevertheless, the total momentum of the system before and after the explosion is identical. When such collision happens, the kinetic energy of the system increases.

The second possibility called inelastic is the reverse: that two or more objects collid with each ot-her and stick together, thus after the collision forming one single composite object. The total mass of this composite object is the sum of the masses of the original objects, and the new single object moves with a velocity dictated by the conservation of momentum. However, it turns out again that, although the total momentum of the system of object remains constant, the kinetic energy does decrease.

An collision where the objects stick together will result in the maximum loss of kinetic energy (i.e., K_f will be a minimum). Such a collision is called perfectly inelastic. In the extreme case, multiple objects collide, stick together, and remains motionless after the collision. Since the objects are all motionless after the collision, the final kinetic energy is also zero; therefore, the loss of kinetic energy is a maximum.

$$\begin{cases} 0 < K_f < K_i & \text{, inelastic} \\ K_f \text{ lowest, or the enerygy lost is most} & \text{, perfectly inelastic} \\ K_f = K_i & \text{, elastic} \end{cases}$$

The extreme case on the other end is that if two or more objects approach each other, collide, and bounce off each other, moving away from each other at the same relative speed at which they approached each other. In this case, the total kinetic energy of the system is conserved. Such an interaction is called elastic.

In an interaction of closed system of objects, the total momentum of the system is conserved, i.e., $p_f = p_i$ but the kinetic energy may be not.

$$\begin{cases} 0 < K_f < K_i & \text{, inelastic} \\ K_f = 0 & \text{, perfectly inelastic} \\ K_f = K_i & \text{, elastic} \\ K_f > K_i & \text{, explosion} \end{cases}.$$

We have $F = \dfrac{dp}{dt}$, expressing both the force and the momentum in component form yields $F_x = \dfrac{dp_x}{dt}$, $F_y = \dfrac{dp_y}{dt}$, and $F_z = \dfrac{dp_z}{dt}$. Expressing the momentum into decomposed form again yields

$$p_{f,x} = p_{1,i,x} + p_{2,i,x} \text{ and } p_{f,y} = p_{1,i,y} + p_{2,i,y}.$$

Then one has

$$v_{f,x} = \frac{m_1 v_{1,i,x} + m_2 v_{2,i,x}}{m} \text{ and } v_{f,y} = \frac{m_1 v_{1,i,y} + m_2 v_{2,i,y}}{m}.$$

It follows that combining these components using the Pythagorean theorem yields

$$v_f = |v_f| = \sqrt{v_{f,x}^2 + v_{f,y}^2}.$$

Suppose that we have an extended object of mass M, made of N interacting particles. We label their mass as m_j, for $j = 1, \cdots, N$. Then

$$M = \sum_{j=1}^{N} m_j.$$

If we apply some net external force F_{ext} on the object, every particle experriences some "share" or some fraction of that external force. Let

f_j^{ext} := the fraction of the external force that the jth particle experiences.

Notice that these fractions of the total force are not necessarily equal; indeed, they virtually never are. (They can be, but they usually are not.) In general, therefore,

$$f_1^{ext} \neq f_2^{ext} \neq \cdots \neq f_N^{ext}.$$

Next, we assume that each of the particles making up our object can interact every other particle of the object, we refer to them as the internal force:

f_j^{int} := the net internal force that the jth particle experiences from all other particles that make up the object.

Now, the net force is

$$f_j = f_j^{ext} + f_j^{int}, \text{ for } j = 1, \cdots, N.$$

As a result of the fractional force, the momentum of each particle gets changed:

$$f_j = \frac{dp_j}{dt} \text{ and } f_j^{int} + f_j^{ext} = \frac{dp_j}{dt}.$$

The net force F on the object is the vector sum of these forces.

$$F_{net} = \sum_{j=1}^{N} \left(f_j^{ext} + f_j^{int}\right) = \sum_{j=1}^{N} f_j^{int} + \sum_{j=1}^{N} f_j^{ext}.$$

This net force changes the momentum of the object as a whole, and the net change of momentum of the object must be the vector sum of all the individual changes of momentum of all of the particles.

$$F_{net} = \sum_{j=1}^{N} \frac{dp_j}{dt} = \sum_{j=1}^{N} f_j^{int} + \sum_{j=1}^{N} f_j^{ext}.$$

The conservation tells us that

$$\sum_{j=1}^{N} f_j^{int} = 0 \text{ and } \sum_{j=1}^{N} f_j^{ext} = F_{ext}.$$

As a result, one has

$$F_{ext} = \sum_{j=1}^{N} \frac{dp_j}{dt}.$$

Remember that our actual goal is to determine the equation of motion for the entire object (the entire system of particles). To that end, we define

p_{CM} := the total momentum of the system of N particles.

Then we have

$$p_{CM} = \sum_{j=1}^{N} p_j.$$

It follows that

$$F = \frac{dp_{CM}}{dt}.$$

We now want $F = Ma$ being involved, one has

$$Ma = \frac{dp_{CM}}{dt}$$

and thus one has

$$Ma = \sum_{j=1}^{N} \frac{dp_j}{dt} = \frac{d}{dt} \sum_{j=1}^{N} p_j,$$

which follows since the derivative operation is linear (provided they are differentiable). Now, p_j is the momentum of the jth particle. Defining the positions of the constituent particles as $r_j = (x_j, y_j, z_j)$ one then has

$$p_j = m_j v_j = m_j \frac{dr_j}{dt}.$$

Summarizing, we have

$$Ma = \frac{d}{dt} \sum_{j=1}^{N} M_j \frac{dr_j}{dt} = \frac{d^2}{dt^2} \sum_{j=1}^{N} m_j r_j.$$

Dividing both sides by non-zero mass M yields

$$a = \frac{d^2}{dt^2} \Big(\frac{1}{M} \sum_{j=1}^{N} m_j r_j \Big).$$

Then we can define the center of mass of an object to be

$$r_{CM} := \frac{1}{M} \sum_{j=1}^{N} m_j r_j.$$

Indeed, we have an alternative expression

$$r_{CM} = \frac{1}{M} \int r dm,$$

provided r is integrable. Therefore we can regard the first expression as the discrete case and the second expression as the continuous case.

Suppose we have N objects with masses $m_1, \cdots, m_N$ and initial velocities $v_1, \cdots, v_N$. The center of mass of the objects is

$$r_{CM} = \frac{1}{M} \sum_{j=1}^{N} m_j r_j.$$

Its velocity is

$$v_{CM} = \frac{dr_{CM}}{dt} = \frac{1}{M}\sum_{j=1}^{N} m_j \frac{dr_j}{dt}.$$

Thus the initial momentum of the center of mass is

$$\left[M\frac{dr_{CM}}{dt}\right]_i = \sum_{j=1}^{N} m_j \frac{dr_{j,i}}{dt}, \text{ where } Mv_{CM,i} = \sum_{j=1}^{N} m_j v_{j,i}.$$

After these masses move and interact with each other, the momentum of the center of mass is then

$$Mv_{CM,f} = \sum_{j=1}^{N} m_j v_{j,f}.$$

But conservation of momentum tells us that the RHS of both equations must be equal, then we have

$$Mv_{CM,f} = Mv_{CM,i}.$$

And it follows that

$$v_{CM,f} = v_{CM,i}.$$

Lastly we consider the rocket propulsion. We define the rocket's instantaneous velocity to be $v = v\hat{i}$; this velocity is measured relative to an inertial inference system. Thus, the initial momentum of the system is

$$p_i = mv\hat{i}.$$

The rocket's engines are burning fuel at a constant rate and ejecting the exhaust gases in the x-direction. During an infinitesimal time interval dt, the engines eject a (positive) infinitesimal mass of gas dm_g at velocity $u = -u\hat{i}$, note that although the rocket velocity $v\hat{i}$ is measured with respect to Earth, the exhaust gas velocity is measured with respect to the (moving) rocket. Measured with respect to the Earth, therefore, the exhaust gas has velocity $(v-u)\hat{i}$.

Then we have

$$p_f = p_{roc} + p_{gas} = (m - dm_g)(v + dv)\hat{i} + dm_g(v-u)\hat{i}.$$

Since all vectors are in x-direction, we then have

$$p_i = p_f,$$
$$mv = (m - dm_g)(v + dv) + dm_g(v - u),$$
$$mv = mv + mdv - dm_g v - dm_g dv + dm_g v - dm_g u,$$
$$\text{And } mdv = dm_g dv + dm_g u.$$

Now dm_g and d_v are each very small, thus their product is very small hence we can neglect this term.

$$mdv = dm_g u, \text{ where } dm_g = -dm.$$

Then one has

$$mdv = -dmu, \text{ or, } dv = -u\frac{dm}{m}.$$

Integrating both sides yields

$$\int_{v_i}^{v} dv = -u\int_{m_0}^{m} \frac{1}{m} dm \Rightarrow v - v_i = u \ln\left(\frac{m_0}{m}\right).$$

It follows that

$$\Delta v = u \ln\left(\frac{m_0}{m}\right).$$

This result is called the rocket euqation, originally derived by the Soviet physicist Konstantin Tsiolkovsky in 1897.

If we consider the rocket in a gravitational field, it means that we need to consider one more dimension — the y-direction.

We have $F = -mg\hat{j}$ and this force applies an impulse $dJ = Fdt = -mgdt\hat{j}$, which is equal to the change of momentum. This gives us

$$dp = dJ,$$

$$p_f - p_i = -mgdt\hat{j},$$

$$\left[(m - dm_g)(v + dv) + dm_g(v - u) - mv\right]\hat{j} = -mgdt\hat{j},$$

$$\text{and } mdv - dm_g u = -mgdt.$$

Next we replace dm_g with $-dm$, then

$$mdv + dmu = -mgdt \ \Rightarrow mdv = -dmu - mgdt.$$

Dividing both sides by m yields

$$dv = -u\frac{dm}{m} - gdt,$$

Integrating both sides (provided integrable) one has

$$\Delta v = u \ln\left(\frac{m_0}{m}\right) - g\Delta t.$$