Lecture Notes Week 7

Part I:

Uniform circular motion is motion in a circle at constant speed. Although this is the simplest case of rotational motion, it is very useful for many situations, and we use it here to introduce rotational variables.

The coordinate system is fixed and serves as a frame of reference to define the particle's position. Its position vector from the origin of the circle to the particle sweeps out the angle θ , which increases in the counterclockwise direction as the particle moves along its circular path. The angle θ is called the **angular position** of the particle. As the particle moves in its circular path, it also traces an arc length s.

The angle is related to the radius of the circle and the arc length by

$$\theta = \frac{s}{r}. \qquad (1.1)$$

The angle θ, the angular position of the particle along its path, has units of radians (rad). There are 2π radians in 360°. Note that the radian measure is a ratio of length measurements, and therefore is a dimensionless quantity. As the particle moves along its circular path, its angular position changes and it undergoes angular displacements $\Delta\theta$.

We can assign vectors to the quantities (1.1) and then by elementart arithmetic one can then derive the equation

$$s = \theta \times r. \qquad (1.2)$$

The magnitude of the angular velocity, denoted by ω, is the time rate of change of the angle θ as the particle moves in its circular path. The instantaneous angular velocity is defined as the limit in which $\Delta t \to 0$ in the average angular velocity $\overline{\omega} = \dfrac{\Delta\theta}{\Delta t}$. It follows that

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}, \qquad (1.3)$$

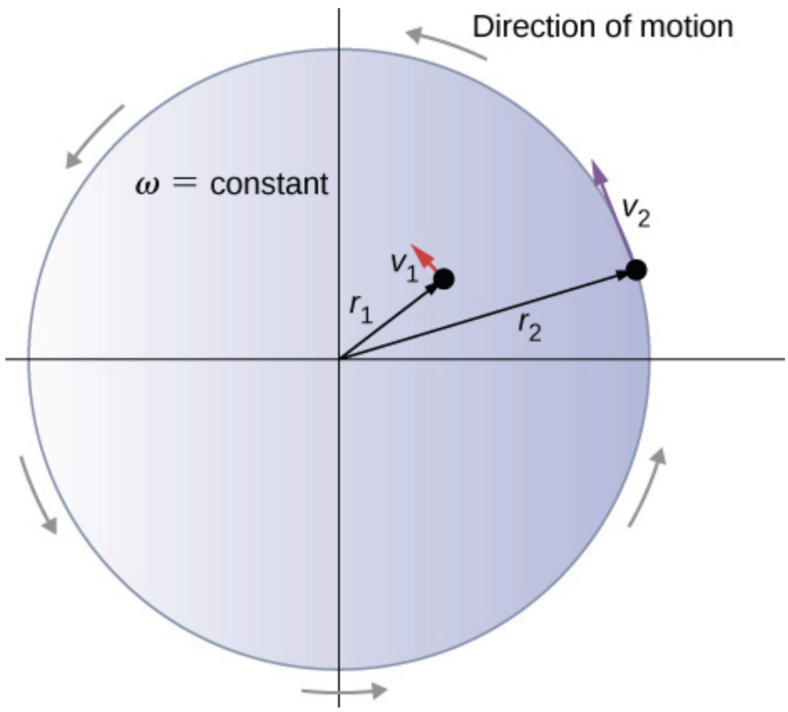
where θ is the angle of rotation. The units of angular velocity are radians per second (rad/s). Angular velocity can also be referred to as the rotation rate in radians per second. In many situations, we are given the rotation rate in revolutions/s or cycles/s. To find the angular velocity, we must multiply revolutions/s by 2π, since there are 2π radians in one complete revolution. Since the direction of a positive angle in a circle is counterclockwise, we take counterclockwise rotations as being positive and clockwise rotations as negative.

Since $s = r\theta$, we then taking derivative on both side with respect to t and noting that the radius r is a constant, one has

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = \theta\frac{dr}{dt} + r\frac{d\theta}{dt} = r\frac{d\theta}{dt},$$

where $\theta\dfrac{dr}{dt} = 0$. Here $\dfrac{ds}{dt}$ is just the tangential speed v_t of the particle, thus, by (1.3) one has

$v_t = r\omega.$

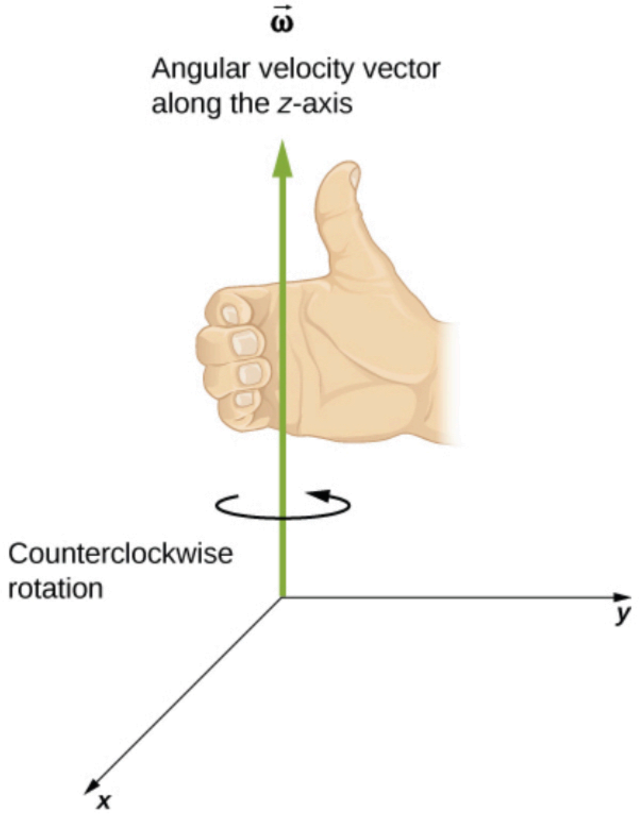


That is, the tangential speed of the particle is its angular velocity times the radius of the circle. That is, the tangential speed of the particle is its angular velocity times the radius of the circle.

(Figure 1.1)

Two particles are placed at different radii on a rotating disk with a constant angular velocity. As the disk rotates, the tangential speed increases linearly with the radius from the axis of rotation. We see that in the figure $v_1 = r_1\omega_1$ and $v_2 = r_2\omega_2$. But the disk has a constant angular velocity, so $\omega_1 = \omega_2$. This means that $\frac{v_1}{r_1} = \frac{v_2}{r_2}$ or $v_2 = \frac{r_2}{r_1}v_1$. Thus, since $r_2 > r_1$, we know $v_2 > v_1$.

Up until now, we have discussed the magnitude of the angular velocity $\omega = \frac{d\theta}{dt}$, which is a scalar quantity — he change in angular position with respect to time. The vector ω s the vector associated with the angular velocity and points along the axis of rotation. This is useful because when a rigid body is rotating, we want to know both the axis of rotation and the direction that the body is rotating about the axis, clockwise or counterclockwise. The angular velocity ω ives us this information. The angular velocity ω has a direction determined by what is called the right-hand rule. The right-hand rule is such that if the fingers of your right hand wrap counterclockwise from the x-axis (the direction in which θ increases) toward the y -axis , your thumb points in the direction of the positive z-axis. An angular velocity ω hat points along the positive z-axis therefore corresponds to a counterclockwise rotation, whereas an angular velocity ω hat points along the negative z-axis corresponds to a clockwise rotation.



(Figure 1.2)

We have just discussed angular velocity for uniform circular motion, but not all motion is uniform. Envision an ice skater spinning with his arms outstretched—when he pulls his arms inward, his angular velocity increases. Or think about a computer's hard disk slowing to a halt as the angular velocity decreases. We will explore these situations later, but we can already see a need to define an **angular acceleration** for describing situations where ω changes. The faster the change in ω, the greater the angular acceleration. We define the **instantaneous angular acceleration** α as the derivative of angular velocity with respect to time:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2},$$

where we have taken the limit of the average angular acceleration, $\overline{\alpha} = \frac{\Delta\omega}{\Delta t}$ as $\Delta t \to 0$. The units of angular acceleration are (rad/s)/s, or rad/s^2.

In the same way as we defined the vector associated with angular velocity ω, we can define α, the vector associated with angular acceleration. If the angular velocity is along the positive z-axis and $\frac{d\omega}{dt}$ is positive, then the angular acceleration α is positive and points along the $+z$-axis. Similarly, if the angular velocity ω is along the positive z-axis and $\frac{d\omega}{dt}$ is negative, then the angular acceleration is negative and points along the $-z$-axis.

We can express the tangential acceleration vector as a cross product of the angular acceleration and the position vector. This expression can be found by taking the time derivative of $v = \omega \times r$.

We can relate the tangential acceleration of a point on a rotating body at a distance from the axis of rotation in the same way that we related the tangential speed to the angular velocity. If we differentiate with respect to time, noting that the radius r is constant, we obtain $a_t = r\alpha$.

Part II:

Using our intuition, we can begin to see how the rotational quantities $\theta, \omega, \alpha,$ and t are related to one another. For example, we saw in the preceding section that if a flywheel has an angular acceleration in the same direction as its angular velocity vector, its angular velocity increases with time and its angular displacement also increases. On the contrary, if the angular acceleration is opposite to the angular velocity vector, its angular velocity decreases with time. We can describe these physical situations and many others with a consistent set of rotational kinematic equations under a constant angular acceleration. The method to investigate rotational motion in this way is called **kinematics of rotational motion**.

To begin, we note that if the system is rotating under a constant acceleration, then the average angular velocity follows a simple relation because the angular velocity is increasing linearly with time. The average angular velocity is just half the sum of the initial and final values:

$$\overline{\omega} = \frac{\omega_0 + \omega_f}{2}.$$

From the definition of the average angular velocity, we can find an equation that relates the angular position, average angular velocity, and time:

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}.$$

Solving for θ, we have

$$\theta_f = \theta_0 + \overline{\omega} t,$$

where we have set $t_0 = 0$. his equation can be very useful if we know the average angular velocity of the system. Then we could find the angular displacement over a given time period. Next, we find an equation relating $\omega, \alpha,$ and t. To determine this equation, we start with the definition of angular acceleration:

$$\alpha = \frac{d\omega}{dt}.$$

We rearrange this to get $\alpha dt = d\omega$ nd then we integrate both sides of this equation from initial values to final values, that is, from t_0 to t and ω_0 to ω_f. In uniform rotational motion, the angular acceleration is constant so it can be pulled out of the integral, yielding two definite integrals

$$\alpha \int_{t_0}^{t} dt = \int_{\omega_0}^{\omega_f} d\omega.$$

Setting $t_0 = 0$, one has

$$\alpha t = \omega_f - \omega_0.$$

Rearranging yields

$$\omega_f = \omega_0 + \alpha t,$$

where ω_0 is the initial angular velocity. Note that this is the rotational counterpart to the linear kinematics equation $v_f = v_0 + at$.

Let's now do a similar treatment starting with the equation $\frac{d\omega}{dt}$. We rearrange it to obtain $\omega dt = d\theta$ and integrate both sides from initial to final values again, noting that the angular acceleration is constant and does not have a time dependence. However, this time, the angular velocity is not constant (in general), so we substitute in what we derived above:

$$\int_{t_0}^{t_f} (\omega_0 + \alpha t)dt = \int_{\theta_0}^{\theta_f} d\theta.$$

Then

$$\int_{t_0}^{t} \omega_0 dt + \int_{t_0}^{t} \alpha t dt = \int_{\theta_0}^{\theta_f} d\theta = \left[\omega_0 t + \alpha\left(\frac{t^2}{2}\right)\right]_{t_0}^{t} = \omega_0 t + \alpha\left(\frac{t^2}{2}\right) = \theta_f - \theta_0,$$

where we have set $t_0 = 0$. Rearranging yields

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2.$$

This is the rotational counterpart to the linear kinematics equation found in the motion in linear equation $x_f = x_0 + v_0 t + \frac{1}{2}at^2$.

We can find an equation that is independent of time by solving for t and then we have

$$\begin{aligned}\theta_f &= \theta_0 + \omega_0\left(\frac{\omega_f - \omega_0}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega_f - \omega_0}{\alpha}\right)^2 \\ &= \theta_0 + \frac{\omega_0\omega_f}{\alpha} - \frac{\omega_0^2}{\alpha} + \frac{1}{2}\frac{\omega_f^2}{\alpha} - \frac{\omega_0\omega_f}{\alpha} + \frac{1}{2}\frac{\omega_0^2}{\alpha} \\ &= \theta_0 + \frac{1}{2}\frac{\omega_f^2}{\alpha} - \frac{1}{2}\frac{\omega_0^2}{\alpha}.\end{aligned}$$

It follows that $\theta_f - \theta_0 = \frac{\omega_f^2 - \omega_0^2}{2\alpha}$. Hence $\omega_f^2 = \omega_0^2 + 2\alpha\Delta\theta$.

Part III:

In Rotational Variables, we introduced angular variables. If we compare the rotational definitions with the definitions of linear kinematic variables from Motion Along a Straight Line and Motion in Two and Three Dimensions, we find that there is a mapping of the linear variables to the rotational ones. Linear position, velocity, and acceleration have their rotational counterparts, as we can see when we write them side by side:

	Linear	Rotational
Position	x	θ
Velocity	$v = \dfrac{dx}{dt}$	$\omega = \dfrac{d\theta}{dt}$
Acceleration	$a = \dfrac{dv}{dt}$	$\alpha = \dfrac{d\omega}{dt}$

Let's compare the linear and rotational variables individually. The linear variable of position has physical units of meters, whereas the angular position variable has dimensionless units of radians, as can be seen from the definition of $\theta = \frac{s}{r}$, which is the ratio of two lengths. The linear velocity has units of m/s, and its counterpart, the angular velocity, has units of rad/s. In Rotational Variables, we saw in the case of circular motion that the linear tangential speed of a particle at a radius r from the axis of rotation is related to the angular velocity by the relation $v_t = r\omega$. This could also apply to points on a rigid body rotating about a fixed axis. Here, we consider only circular motion. In circular motion, both uniform and nonuniform, there exists a centripetal acceleration (Motion in Two and Three Dimensions). The centripetal acceleration vector points inward from the particle executing circular motion toward the axis of rotation. The derivation of the magnitude of the centripetal acceleration is given in Motion in Two and Three Dimensions. From that derivation, the magnitude of the centripetal acceleration was found to be

$$a_c = \frac{v_t^2}{r}, \tag{1.4}$$

where *r* is the radius of the circle.

Thus, in uniform circular motion when the angular velocity is constant and the angular acceleration is zero, we have a linear acceleration—that is, centripetal acceleration—since the tangential speed in (1.4) is a constant. If nonuniform circular motion is present, the rotating system has an angular acceleration, and we have both a linear centripetal acceleration that is changing (since v_t is changing) as well as a linear tangential acceleration.

The centripetal acceleration is due to the change in the direction of tangential velocity, whereas the tangential acceleration is due to any change in the magnitude of the tangential velocity. The tangential and centripetal acceleration vectors a_t and a_c are always perpendicular to each other. To complete this description, we can assign a **total linear acceleration** vector to a point on a rotating rigid body or a particle executing circular motion at a radius r from a fixed axis. The total linear acceleration vector *a* is the vector sum of the centripetal and tangential accelerations,

$$a = a_t + a_c.$$

Since $a_t \perp a_c$ one then has

$$\|a\| = \sqrt{a_t^2 + a_c^2}.$$

Note that if the angular acceleration is zero, the total linear acceleration is equal to the centripetal acceleration.

Generally speaking, the linear kinematic equations have their rotational counterparts.

Rotational	Translational
$\theta_f = \theta_0 + \overline{\omega} t$	$x = x_0 + \overline{v} t$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x_f = x_0 + v_0 t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_0^2 + 2\alpha\Delta\theta$	$v_f = v_0^2 + 2a\Delta x$

The second correspondence has to do with relating linear and rotational variables in the special case of circular motion.

Rotational	Translational	Relationship
θ	s	$\theta = \frac{s}{r}$
ω	v_t	$\omega = \frac{v_t}{r}$
α	a_t	$\alpha = \frac{a_t}{r}$
	a_c	$a_c = \frac{v_t^2}{r}$

Part IV:

Any moving object has kinetic energy. We know how to calculate this for a body undergoing translational motion, but how about for a rigid body undergoing rotation? This might seem complicated because each point on the rigid body has a different velocity. However, we can make use of angular velocity—which is the same for the entire rigid body—to express the kinetic energy for a rotating object. This system has considerable energy, some of it in the form of heat, light, sound, and vibration. However, most of this energy is in the form of **rotational kinetic energy**.

Energy in rotational motion is not a new form of energy; rather, it is the energy associated with rotational motion, the same as kinetic energy in translational motion. However, because kinetic energy is given by $K = \frac{1}{2}mv^2$, translating to the rotational variable, we have

$$K = \frac{1}{2}mv_t^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}(mr^2)\omega^2.$$

Applying the center of mass $M = \sum_j m_j$, one then has

$$K = \sum_j \frac{1}{2}m_j v_j^2 = \sum_j \frac{1}{2}m_j(r_j\omega_j)^2,$$

Since $\omega_j = \omega$ for all masses, it follows that

$$K = \frac{1}{2}\Big(\sum_j m_j r_j^2\Big)\omega^2. \quad (1.5)$$

If we compare (1.5) to the way we wrote kinetic energy in Work and Kinetic Energy, that is, $\frac{1}{2}mv^2$, this suggests we have a new rotational variable to add to our list of our relations between rotational and translational variables. The quantity $\sum_j m_j r_j^2$ is the counterpart for mass in the equation for rotational kinetic energy. This is an important new term for rotational motion. This quantity is called the **moment of inertia *I*,** with units of $kg \cdot m^2$, given by

$$I = \sum_j m_j r_j^2.$$

For now, we leave the expression in summation form, representing the moment of inertia of a system of point particles rotating about a fixed axis. We note that the moment of inertia of a single point particle about a fixed axis is simply mr^2, with r being the distance from the point particle to the axis of rotation. In the next section, we explore the integral form of this equation, which can be used to calculate the moment of inertia of some regular-shaped rigid bodies. By substituting, the expression for the kinetic energy of a rotating rigid body becomes

$$K = \frac{1}{2}I\omega^2.$$

We see from this equation that the kinetic energy of a rotating rigid body is directly proportional to the moment of inertia and the square of the angular velocity. We summarize the comparison into the following table

Rotational	Translational
$I = \sum_j m_j r_j^2$	m
$K = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mv^2$

Part V:

We defined the moment of inertia I of an object to be $I = \sum_i m_i r_i^2$ for all the point masses that make up the object. Because r is the distance to the axis of rotation from each piece of mass that makes up the object, the moment of inertia for any object depends on the chosen axis. The need to use an infinitesimally small piece of mass dm suggests that we can write the moment of inertia by evaluating an integral over infinitesimal masses rather than doing a discrete sum over finite masses:

$$I = \sum_i m_i r_i^2 \text{ becomes } I = \int r^2 dm.$$

This, in fact, is the form we need to generalize the equation for complex shapes. It is best to work out specific examples in detail to get a feel for how to calculate the moment of inertia for specific shapes. This is the focus of most of the rest of this section.

The similarity between the process of finding the moment of inertia of a rod about an axis through its middle and about an axis through its end is striking, and suggests that there might be a simpler method for determining the moment of inertia for a rod about any axis parallel to the axis through the center of mass. Such an axis is called a **parallel axis**. There is a theorem for this, called the **parallel-axis theorem**, which we state here but do not derive in this text.

Theorem 1.1: Parallel-Axis Theorem

Let m be the mass of an object and let d be the distance from an axis through the object's center of mass to a new axis. Then we have

$$I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2.$$

we can reason that a compound object's moment of inertia can be found from the sum of each part of the object:

$$I_{\text{total}} = \sum_i I_i.$$

Part VI:

So far we have defined many variables that are rotational equivalents to their translational counterparts. Let's consider what the counterpart to force must be. Since forces change the translational motion of objects, the rotational counterpart must be related to changing the rotational motion of an object about an axis. We call this rotational counterpart **torque**.

Definition: Torque

When a force F is applied to a point P whose position is r relative to its center O, the torque τ around O is then $\tau = r \times F$.

By the definition of cross product, one has $\|\tau\| = \|r \times F\| = rF\sin\theta$. The SI unit of torque is newtons times meters, usually written as $N \cdot m$. The quantity $r_\perp = r\sin\theta$ is the perpendicular distance from O to the line determined by the vector F and is called the level arm. Note that the greater the level arm, the greater the magnitude of the torque. In terms of the level arm, the magnitude of the torque is

$$\|\tau\| = r_\perp F.$$

Any number of torques can be calculated about a given axis. The individual torques add to produce a net torque about the axis. When the appropriate sign (positive or negative) is assigned to the magnitudes of individual torques about a specified axis, the net torque about the axis is the sum of the individual torques:

$$\tau_{\text{net}} = \sum_i \|\tau_i\|.$$

Part VII:

Recall that the magnitude of the tangential acceleration is proportional to the magnitude of the angular acceleration by $a = r\alpha$. Substituting this expression into Newton's second law, we obtain $F = mr\alpha$. Multiplying both sides by r yields

$$rF = mr^2\alpha.$$

Since $I = mr^2$, we have

$$\tau = I\alpha.$$

The torque on the particle is equal to the moment of inertia about the rotation axis times the angular acceleration. We can generalize this equation to a rigid body rotating about a fixed axis.

Theorem 1.2: Newton's Second Law for Rotation

If more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration:

$$\sum_i \tau_i = I\alpha.$$

The term $I\alpha$ is a scalar quantity and can be positive or negative (counterclockwise or clockwise) depending upon the sign of the net torque. Remember the convention that counterclockwise angular acceleration is positive. Thus, if a rigid body is rotating clockwise and experiences a positive torque (counterclockwise), the angular acceleration is positive.

Part VIII:

The rotational work done by a rigid body is $W = \int \sum \tau d\theta$. The total work done on a rigid body is the sum of the torques integrated over the angle through which the body rotates. The incremental work is $dW = \left(\sum_i \tau_i\right) d\theta$. Similarly, we found the kinetic energy of a rigid body rotating around a fixed axis by summing the kinetic energy of each particle that makes up the rigid body.

Theorem 1.3: Work-Energy Theorem for Rotation

The work-energy theorem for a rigid body rotating around a fixed axis is $W_{AB} = K_B - K_A$, where $K = \frac{1}{2}I\omega^2$, and the rotational work done by a net force rotating a body from point A to point B is $W_{AB} = \int_{\theta_A}^{\theta_B} \left(\sum_i \tau_i\right) d\theta.$

The relationship to the linear motion is summarized into the following table

Rotational	Translational
$I = \sum_i m_i r_i^2$	m
$K = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mv^2$
$\sum_i \tau_i = I\alpha$	$\sum_i F_i = ma$
$W_{AB} = \int_{\theta_A}^{\theta_B} \left(\sum_i \tau_i\right) d\theta$	$W = \int F ds$
$P = \tau\omega$	$P = Fv$