Week 5 Lecture Notes Physics4A

Section 7

We first define the increment of work dW done by a force F acting through an infinitesimal displacement dr as the dot product of these two vectors:

$$dW = F \cdot dr = |F| \cdot |dr| \cdot \cos\theta.$$

Then we can add up the contributions for infinitesimal displacements, along a path between two positions, to get the total work.

Definition: Work done by a Force

The work done by a force is the integral of the force with respect to the displacement along the path of the displacement

$$W_{AB} = \int_{\text{path}AB} F \cdot dr.$$

We also have the work done by spring, given by

$$W_{\text{spring},AB} = \int_A^B F_x dx = -k\int_A^B x dx = -k\frac{x^2}{2}\Big|_A^B = -\frac{1}{2}k(x_B^2 - x_A^2).$$

When it comes to the kinetic energy, we have the following result

Definition: Kinetic Energy

The kinetic energy of a particle is one-half the product of the particle's mass and the square of its speed v, i.e., $K = \frac{1}{2}mv^2$.

Theorem 1: Work-Energy Theorem

The net work done on a particle equals the change in the particle's kinetic energy $W_{\text{net}} = K_B - K_A$.

We express the relation between work done and the time interval involved in doing it, by introducing the concept of power. Since work can vary as a function of time, we first define the average power as

Definition: Average Power

The average power is the work done during a time interval divided by the length of the same interval, given by $P_{\text{ave}} = \frac{\Delta W}{\Delta t}$.

Then, we can define the instantaneous power, or simply the power.

Definition: Power

Power is defined as the rate of doing work, or the limit of the average power for time intervals approaching zero, i.e., $P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$, provided differentiable.

Section 8

We can define the difference of potential energy from point A to point B as the negative of the work done:

$$\Delta U_{AB} = U_B - U_A = -W_{AB}.$$

This formula explicitly states a potential energy difference, not just an absolute potential energy. Therefore, we need to define potential energy at a given position in such a way as to state standard values of potential energy on their own, rather than potential energy differences. We do this by rewriting the potential energy function in terms of an arbitrary constant: $\Delta U = U(r) - U(r_0)$.

As long as there is no friction or air resistence, the change in kinetic energy of the particle equals the negative of the change in gravitational potential energy of the particle. This can be generalized to any potential energy

$$\Delta K_{AB} = -\Delta U_{AB}.$$

Indeed, we have

$$\Delta U_{grav} = -W_{grav,AB} = mg(y_A - y_B),$$

that is,

$$U(y) = mgy + C,$$

for some constant C.

We also have the elastic potential energy provided by

$$\Delta U = -W_{AB} = \frac{1}{2}k(x_B^2 - x_A^2),$$

where the object travels from point A to point B. The potential energy function corresponding to this difference is given by

$$U(x) = \frac{1}{2}kx^2 + C,$$

for some constant C.

In potential energy and conservation of energy, any transition between kinetic and potential energy conserved the total energy of the system. This was path independent, meaning that we can start and stop at any two points in the problem, and the total energy of the system — kinetic plus potential — at these points are equal to each other. This is characterized as conservative force. Alternatively, the non-conservative forces are dissipative forces such as friction or air resistence.

Theorem 2: Conservative Force

The work done by a conservative force is independent of the path; in other words, the work done by a conservative force is the same for any path connecting two points,

$$W_{AB,\text{path1}} = \int_{AB,\text{path1}} F_{cons} dr = W_{AB,\text{path2}} = \int_{AB,\text{path2}} F_{cons} dr.$$

The work done by a non-conservative force depends on the path taken.

Equivalently, a force is conservative if the work it does around any closed path is zero:

$$W_{\text{closed path}} = \oint F_{cons} dr = 0.$$

In one-dimensional, the condition for Fdr to be differential is

$$dW_{net} = mv \cdot dv = d\frac{1}{2}mv^2.$$

In two dimensions, the condition for $Fdr = F_X dx + F_y dy$ to be an exact differential is

$$\frac{dF_x}{dy} = \frac{dF_y}{dx}.$$

When we consider the infinitesimal increment of potential energy as the dot product of the force and the infinitesimal displacement, we have $du = -F \cdot dl = -F_l dl$. It follows that $F_l = -\frac{dU}{dl}$. Here we chose to represent the displacement in an arbitrary direction by dl, so as not to be restricted to any particular coordinate direction. We also expressed the dot product in terms of the magnitude of the infinitesimal displacement and the component of the force in its direction. In two dimensions we then have

$$F = F_X\hat{i} + F_Y\hat{j} = -\left(\frac{\partial U}{\partial x}\right)\hat{i} - \left(\frac{\partial U}{\partial y}\right)\hat{j}.$$

The third theorem regards the conservation of energy.

Theorem 3: Conservation of Energy

> The mechanical energy E of a particle stays constant unless forces outside the system or non-conservative forces do work on it, in which case, the change in the mechanical energy is equal to the work done by the non-conservative forces given by

$$W_{nc,AB} = \Delta(K + U)_{AB} = \Delta E_{AB}.$$

This statement expresses the concept of energy conservation for a classical particle as long as there is no non-conservative work. It is sometimes convenient to separate the case where the work done by non-conservative forces is zero, either because no such forces are assumed present, or, like the normal force, they do zero work when the motion is parallel to the surface. Then

$$0 = W_{nc,AB} = \Delta(K + U)_{AB} = \Delta E_{AB}.$$

Section 9

Now we introduce the concept of momentum. Our study of kinetic energy showed that a complete understanding of an object's motion must include both its mass and its velocity, i.e. $K = \frac{1}{2}mv^2$. However, as powerful as this concept is, it does not include any information about the direction of the moving object's velocity vector. We shall now define a physical quantity that includes direction.

Definition: Momentum

The momentum p of an object is the product of its mass and its velocity given by $p = mv$.

Mathematically, if a quantity is proportional to two (or more) things, then it is proportional to the product of those things. The product of a force and a time interval (over which that force acts) is called impulse, and is given by the symbol J.

Definition: Impulse

Let $F(t)$ be the force applied to an object over some differential time interval dt. The resulting impulse on the object is defined as $dJ = F(t)dt$.

The total impulse over the interval $t_f - t_i$ is given by

$$J = \int_{t_i}^{t_f} dJ \text{ or } J = \int_{t_i}^{t_f} F(t)dt.$$

To calculate the impulse we need to know the force function $F(t)$, which we often do not have. However, a result from calculus is useful here. Recall that the average value of a function over some interval is calculated by

$$f(x)_{ave} = \frac{1}{\Delta x}\int_{x_i}^{x_f} f(x)dx,$$

where $\Delta x = x_f - x_i$. Applying this to the time-dependent force function one has

$$F_{ave} = \frac{1}{\Delta t}\int_{t_i}^{t_f} F(t)dt.$$

Therefore, one has

$$J = F_{ave}\Delta t.$$

To calculat the impulse, another result is done by the equation $f(t) = ma(t)$:

$$J = \int_{t_i}^{t_f} F(t)dt = m\int_{t_i}^{t_f} a(t)dt = m\big(v(t_f) - v(t_i)\big).$$

For a constant force $F_{ave} = F = ma$, this simplifies to

$$J = ma\Delta t = mv(t_f) - mv(t_i) = m\big(v(t_f) - v(t_i)\big).$$

That is,

$$J = m\Delta v.$$

Since an impulse is a force acting for some amount of time, it causes an object's motion to change. Recall that $J = m\Delta v$. Because mv is the momentum of a system, $m\Delta v$ is the change of the momentum p. This gives us the following result.

Theorem 4: Impulse-Momentum Theorem

An impulse applied to a system changes the system's momentum, and the change of momentum is exactly equal to the impulse that was applied, i.e. $J = \Delta p$.

We can also discuss the momentum and force. One has $F_{ave} = \frac{\Delta p}{\Delta t}$. Then we can derive the following result.

Theorem 5: Newton's Second Law of Motion in Terms of Momentum

The next external force on a system is equal to the rate of change of the momentum of the system caused by the force $F = \frac{dp}{dt}$.

Proof:

By definition we have $F = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma$.

□

Theorem 6: Law of Conservation Momentum

If the value of a physical quantity is constant in time, we say that the quantity is conserved. That is, the total momentum of a closed system is conserved: $\sum_{j=1}^{N} p_j =$ constant.