Homework IV Hypothesis Testing, Problem Sheet

Abstract:
The problems are labeled by the keywords in characterizing what concepts we are dealing with along with the symbol ⋆ which measures the hardness of the problem, the more ⋆, the harder the problem. For the solutions in finding exact result and the remarks, the end of solution is labeled by "||", for those problems asking us in proving or disproving some statements, the end is labeled by the usual □.

Problem 1: ⋆ ⋆ ⋆
(**Keywords**: Exponential, LRT, Test Statistic)

Suppose that we have two independent random samples $X_1, \cdots, X_n$ are such that $X_i \sim \text{Exponential}(\theta)\,\forall i = 1,\cdots,n$ and $Y_1, \cdots, Y_m$ are such that $Y_j \sim \text{Exponential}(\mu)\ \forall j = 1,\cdots,m$.

(a) Find the LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

(b) Show that the test in (a) can be based on the statistic $T = \dfrac{\Sigma X_i}{\Sigma X_i + \Sigma Y_i}$.

(c) Find the distribution of T when H_0 is true.

Problem 2: ⋆ ⋆
(**Keywords**: Uniform, Power Function, Size)

Let X_1, X_2 be i.i.d. Uniform$(\theta, \theta + 1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing test:

$$\varphi_1(X_1) : \text{Reject } H_0 \text{ if } X_1 > 0.95,$$
$$\varphi_2(X_1, X_2) : \text{Reject } H_0 \text{ if } X_1 + X_2 > C.$$

(a) Find the value of C so that φ has the same size as φ_1.

(b) Calculate the power function of each test.

Problem 3: ⋆ ⋆
(**Keywords**: Bernoulli, CLT, Test Function)

For a random sample $X_1, \cdots, X_n$ of Bernoulli(p) variables, it is desired to test

$$H_0 : p = 0.49 \text{ versus } H_1 : p = 0.51.$$

Use the **Central Limit Theorem** to determine, approximately, the sample size needed so that the two probabilities of error are both about 0.01. Use a test function that rejects H_0 if $\sum_{i=1}^{n} X_i$ is large.

Problem 4: ⋆
(**Keywords**: Normal, Type I Error, UMP Test, Neyman-Pearson Lemma)

Show that for a random sample $X_1, \cdots, X_n$ from a $N(0,\sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\varphi\left(\sum X_i^2\right) = \begin{cases} 1 & , \text{ if } \sum X_i^2 > c \\ 0 & , \text{ if } \sum X_i^2 \leq c \end{cases}.$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

Problem 5: ★ ★

(**Keywords**: _Beta_, _LRT_, _Test Statistic_)

Suppose that $X_1, \cdots, X_n$ are i.i.d. with a Beta(μ,1) pdf and $Y_1, \cdots, Y_m$ are i.i.d. with a Beta(θ,1) pdf. Also assume that the Xs are independent of the Ys.

(a) Find an LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

(b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum \log X_i}{\sum \log X_i + \sum \log Y_i}.$$