Homework III Interval Estimators, Problem Sheet

Abstract:
The problems are labeled by the keywords in characterizing what concepts we are dealing with along with the symbol ⋆ which measures the hardness of the problem, the more ⋆, the harder the problem. For the solutions in finding exact result and the remarks, the end of solution is labeled by "||", for those problems asking us in proving or disproving some statements, the end is labeled by the usual □.

Recall that in using the pivotal quantity we require two conditions (i) it is a function of the sample measurements and the unknown parameter θ, where θ is the only unknown parameter, and (ii) its probability distribution does not depend on the parameter θ.

Problem 1: ⋆
(**Keywords**: *Pivotal Quantity*, *Exponential*, *Confidence Interval*, *Confidence Coefficient*)

Suppose that we are to obtain a single observation Y from an exponential distribution with mean θ. Use Y to form a confidence interval for θ with confidence coefficient 0.90.

Problem 2: ⋆
(**Keywords**: *Pivotal Quantity*, *Uniform*, *Confidence Interval*, *Confidence Coefficient*)

Suppose that we take a sample of size $n = 1$ from a uniform distribution defined on the interval $[0,\theta]$, where θ is unknown. Find a 95% lower confidence bound for θ.

Remark: Location-Scale Pivots

Form of PDF	**Type of PDF**	**Pivot**
$f(x-\mu)$	Location	$\overline{X}-\mu$
$\frac{1}{\sigma}f(\frac{x}{\sigma})$	Scale	$\frac{\overline{X}}{\sigma}$
$\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$	Location-Scale	$\frac{\overline{X}-\mu}{\sigma}$

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Problem 3: ⋆
(**Keywords**: *Pivotal Quantity*, *Normal*, *Confidence Interval*, *Confidence Coefficient*)

Let $\hat{\theta}$ be a statistic that is normally distributed with mean θ and standard error $\sigma_{\hat{\theta}}$. Find a confidence interval for θ that possesses a confidence coefficient equal to $(1-\alpha)$.

Problem 4: ⋆ ⋆
(**Keywords**: *Interval Estimation*, *Confidence Coefficient*)

If $\hat{\theta}_L(x)$ and $\hat{\theta}_U(x)$ satisfy $\mathbb{P}(\hat{\theta}_L(x) \le \theta) = 1-\alpha_1$ and $\mathbb{P}(\hat{\theta}_U(x) \ge \theta) = 1-\alpha_2$ and we know $\hat{\theta}_L(x) \le \hat{\theta}_U(x)$ for all x. Show that
$\mathbb{P}\big(\hat{\theta}_L(X) \le \theta \le \hat{\theta}_U(X)\big) = 1-\alpha_1-\alpha_2$.

Problem 5: ⋆

(**Keywords**: *Confidence Interval*)

Find a $1-\alpha$ confidence interval for θ, given $X_1, \cdots, X_n$ i.i.d. with pdf

(i) $f(x|\theta) = 1, \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$.

(ii) $f(x|\theta) = 2x/\theta^2, 0 < x < \theta$.

Problem 6: ★ ★

(**Keywords**: *Location Family*, *Confidence Set*, *Coverage Probability*)

If $X_1, \cdots, X_n$ are i.i.d. from a location pdf $f(x-\theta)$, show that the confidence set $C(x_1, \cdots, x_n) = \{\theta | \bar{x} - k_1 \leq \theta \leq \bar{x} + k_2\}$, where k_1 and k_2 are constants, has constant coverage probability.

Problem 7: ★ ★ ★

(**Keywords**: *Normal*, *Confidence Bound*, *Unbiased*)

Let $X_1, \cdots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ population, where σ^2 is known. Show that the usual one-sided $(1-\alpha)$ 100% upper confidence bound $\{\theta | \theta \leq \bar{x} + z_\alpha \sigma/\sqrt{n}\}$ is unbiased, and so is the corresponding lower confidence bound.