Exercise 1: Determinants with Solution
Review on Linear Algebra
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Abstract:
The problems focus mainly on the determinants we covered in the first section. Some interesting application with concepts from other sections are also involved, however, background from other sections are not necessary, one can use both elementary and advanced techniques to solve the problems. The problems are labeled with difficulties by stars, ⋆ means simple, ⋆ ⋆ means hard, ⋆ ⋆ ⋆ means challenging, while ⋆ ⋆ ⋆ ⋆ takes amounts of time.

Probem 1: ⋆

Calculate the determinant $\begin{vmatrix} 0 & a & b & 0 \\ a & 0 & 0 & b \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{vmatrix}$.

Solution:

Observe that we can change this matrix into the diagonal one:

Interchange row 2 and row 3 yields $\begin{vmatrix} 0 & a & b & 0 \\ 0 & c & d & 0 \\ a & 0 & 0 & b \\ c & 0 & 0 & d \end{vmatrix}$. Then interchange column 1 and column 3 yields $\begin{vmatrix} b & a & 0 & 0 \\ d & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{vmatrix}$. (Since the interchange between rows (resp. columns) happens twice, the sign remains unchanged.)

Simple calculation yields

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)(bc - ad) = -(ad - bc)^2. \qquad ||$$

Problem 2: ⋆

Calculate the determinant $\begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 4 & 3 & 2 & \lambda+1 \end{vmatrix}$.

Solution:

By column-expansion with respect to column 1, one has

$$(-1)^{1+1}\lambda \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 3 & 2 & \lambda+1 \end{vmatrix} + (-1)^{4+1}4 \begin{vmatrix} -1 & 0 & 0 \\ \lambda & -1 & 0 \\ 0 & \lambda & -1 \end{vmatrix}. \qquad (1.1)$$

Note that the latter one is diagonal hence it is $-4(-1)=4$, it left us to deal with the first one, which we apply column-expansion with respect to the "new" column 1 again, hence

$$(1.1)\ = \lambda\left((-1)^{1+1}\lambda\begin{vmatrix}\lambda & -1\\ 2 & \lambda+1\end{vmatrix} + (-1)^{3+1}3\begin{vmatrix}-1 & 0\\ \lambda & -1\end{vmatrix}\right)+4$$
$$= \lambda\left(\lambda^3+\lambda^2+2\lambda+3\right)+4 = \lambda^4+\lambda^3+2\lambda^2+3\lambda+4. \qquad \|$$

Problem 3: ★ ★

Let $\alpha := (-1,0,1)^T$ and $A := \alpha\alpha^T$. Let $n \in \mathbb{N}^+$, calculate $|aI - A^n|$.

Solution 1:

By assumption, $A := \alpha\alpha^T = \begin{bmatrix}-1\\0\\1\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{bmatrix}$.

Observe that $A^2 = AA = (\alpha\alpha^T)(\alpha\alpha^T) = \alpha(\alpha^T\alpha)\alpha^T = 2\alpha\alpha^T = 2A$. One has, by simplification, $A^n = 2^nA = \begin{bmatrix}2^n & 0 & -2^n\\ 0 & 0 & 0\\ -2^n & 0 & 2^n\end{bmatrix} = \begin{bmatrix}2^n & 0 & -2^n\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$.

Now $aI - A^n = \begin{bmatrix}a-2^n & 0 & 2^n\\ 0 & a & 0\\ 0 & 0 & a\end{bmatrix}$. Hence $|aI - A^n| = (a-2^n)a^2$. ||

Solution 2:

Similarly we have $A = \begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 1\end{bmatrix}$ and $A^2 = 2A = \begin{bmatrix}2 & 0 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ hence the eigenvalue of A^2 is $\lambda = 2$, $\lambda = 0$, and $\lambda = 0$. It follows that for the matrix $aI - A^n$, one has the eigenvalues being $\lambda = a - 2^n$, $\lambda = a$, and $\lambda = a$. The determinant equals the product of all eigenvalues hence result follows. ||

Problem 4: ★

Calculate the determinant of $\begin{bmatrix}a & 0 & -1 & 1\\ 0 & a & 1 & -1\\ -1 & 1 & a & 0\\ 1 & -1 & 0 & a\end{bmatrix}$.

Solution:

Adding Row 3 to Row 4 yields $\begin{bmatrix}a & 0 & -1 & 1\\ 0 & a & 1 & -1\\ -1 & 1 & a & 0\\ 0 & 0 & a & a\end{bmatrix}$. Multiplying Row 3 by a and adding it to Row 1 yields $\begin{bmatrix}0 & a & a^2-1 & 1\\ 0 & a & 1 & -1\\ -1 & 1 & a & 0\\ 0 & 0 & a & a\end{bmatrix}$. Applying column-expansion with respect to Column 1 yields $(-1)^{3+1}(-1)\begin{vmatrix} a & a^2-a & 1 \\ a & 1 & -1 \\ 0 & a & a \end{vmatrix}$.

Multiplying Row 2 with -1 and adding it to Row 1 yields

$-\begin{vmatrix} 0 & a^2-2 & 2 \\ a & 1 & -1 \\ 0 & a & a \end{vmatrix}$. Applying Column-expansion with the new "Column 1"

again gives us $-(-1)^{2+1}a\begin{vmatrix} a^2-2 & 2 \\ a & a \end{vmatrix} = a(a^3-4a) = a^4-4a^2$. ||

Problem 5: ★ ★

Caculate the determinant $\begin{bmatrix} 1-a & a & 0 & 0 & 0 \\ -1 & 1-a & a & 0 & 0 \\ 0 & -1 & 1-a & a & 0 \\ 0 & 0 & -1 & 1-a & a \\ 0 & 0 & 0 & -1 & 1-a \end{bmatrix}$.

Solution:

Recall in **Algorithm 4** we offer a method to solve such a matrix. First we add Column 2, Column 3, Column 4, and Column 5 to Column 1, this gives us

$$\begin{vmatrix} 1 & a & 0 & 0 & 0 \\ 0 & 1-a & a & 0 & 0 \\ 0 & -1 & 1-a & a & 0 \\ 0 & 0 & -1 & 1-a & a \\ -a & 0 & 0 & -1 & 1-a \end{vmatrix}.$$

We now apply column-expansion with respect to Column 1, this gives us

$$(-1)^{1+1}\begin{vmatrix} 1-a & a & 0 & 0 \\ -1 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \\ 0 & 0 & -1 & 1-a \end{vmatrix} + (-1)^{5+1}(-a)\begin{vmatrix} a & 0 & 0 & 0 \\ 1-a & a & 0 & 0 \\ -1 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \end{vmatrix}. \tag{1.2}$$

It is not hard to recognize that the second part is diagonal, hence we have the second part being $-a^5$, it left us to calculate the first part. Similarly, we add each column to the first column and we shall have

$\begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \\ -a & 0 & -1 & 1-a \end{vmatrix}$. Applying column-expansion with respect to the first column yields

$$(-1)^{1+1}\begin{vmatrix}1-a & a & 0\\ -1 & 1-a & a\\ 0 & -1 & 1-a\end{vmatrix}+(-1)^{4+1}(-a)\begin{vmatrix}a & 0 & 0\\ 1-a & a & 0\\ -1 & 1-a & a\end{vmatrix}. \quad (1.3)$$

We have the latter part of (1.3) being a^4, it left to calculate the other part, we add each column to the first column again, this gives us $\begin{vmatrix}1 & a & 0\\ 0 & 1-a & a\\ -a & -1 & 1-a\end{vmatrix}$.

Do the column-expansion with respect to the first column one has

$$(-1)^{1+1}\begin{vmatrix}1-a & a\\ -1 & 1-a\end{vmatrix}+(-1)^{3+1}(-a)\begin{vmatrix}a & 0\\ 1-a & a\end{vmatrix}. \quad (1.4)$$

Simple calculation shows (1.4) $=(1-a)^2+a-a^3=1-a^3+a^2-a$.
Note that (1.2) = (1.3) $-a^5$ and (1.3) = (1.4) $+a^4=1+a^4-a^3+a^2-a$
hence (1.2) $=1-a^5+a^4-a^3+a^2-a$. ||

Remark:

We can generalize this abstractly, for a matrix with this form, we can add each columns to the first column and apply the column-expansion technique till it becomes the determinant of 2×2. Denote D_n the determinant for $n\times n$, we have

$$D_n=D_1+(-1)^{2+1}\lambda^2+(-1)^{3+1}\lambda^3+\cdots+(-1)^{n+1}\lambda^n,$$

where λ is the left-bottom entry when summing all the columns to the first column. Note that we can also apply the row-expansion by the first row, algorithm is analogous. ||

Problem 6: ★ ★

Calculate the determinant of $A=\begin{bmatrix}2a & 1 & & & & \\ a^2 & 2a & 1 & & & \\ & a^2 & 2a & 1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & a^2 & 2a & 1\\ & & & & a^2 & 2a\end{bmatrix}$.

Solution:

We wish our equation is in the diagonal form, to that end, observe that for each Row_{i+1} where $i=1,2,\cdots,n-1$, we can multiply $\dfrac{-ia}{i+1}$ to Row_i and add it to Row_{i+1}. It follows that, till the process terminates, we have

$$\det A = \begin{vmatrix} 2a & 1 & & & & \\ & \frac{2}{3}a & 1 & & & \\ & & \frac{3}{4}a & 1 & & \\ & & & \cdots & \cdots & \cdots \\ & & & & \frac{n}{n-1}a & 1 \\ & & & & & \frac{n+1}{n}a \end{vmatrix} = 2 \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{n+1}{n} a^n.$$

It follows that RHS $= (n+1)a^n$, result follows. ||

Problem 7: ★ ★ ★ ★

Suppose that $A = [a_{ij}]$ is a nonzero matrix with $1 \le i, j \le 3$. Suppose that A_{ij} is the (i,j)-entry of the adjoint A^* and $a_{ij} + A_{ij} = 0$. Find $|A|$.

Solution:

[Claim]: $a_{ij} + A_{ij} = 0 \Rightarrow A^T + A^* = 0$.

$$\text{We have } A + A^* = \begin{bmatrix} a_{11}+A_{11} & a_{12}+A_{12} & a_{13}+A_{13} \\ a_{21}+A_{21} & a_{22}+A_{22} & a_{23}+A_{23} \\ a_{31}+A_{31} & a_{32}+A_{32} & a_{33}+A_{33} \end{bmatrix} = 0$$

$$\text{Therefore } A^T + A^* = \begin{bmatrix} a_{11}+A_{11} & a_{21}+A_{21} & a_{31}+A_{31} \\ a_{12}+A_{12} & a_{22}+A_{22} & a_{32}+A_{32} \\ a_{13}+A_{13} & a_{23}+A_{23} & a_{33}+A_{33} \end{bmatrix} = 0.$$

According to **Theorem 1.2** (b), we have $|A| = |A^T|$ since $A = 3 \times 3$, according to **Corollary 2.7.1**, we have $|A^*| = |A|^{n-1} = |A|^2$, lastly according to our **[Claim]**, we have $A^T = -A^* \Rightarrow |A^T| = (-1)^3|A^*|$. Summarizing, we have

$$|A| = |A^T| = (-1)^3|A^*| = (-1)^3|A|^2$$

a direct consequence is that $|A| = -|A|^2$, the solution is that either $|A| = 0$ or $|A| = -1$.

According to **Theorem 1.6**, we have

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j},$$

according to **[Claim]**, we have

$$A^T = -A^* \Rightarrow a_{1j} = -A_{1j}, a_{2j} = -A_{2j}, \text{and } a_{3j} = -A_{3j}.$$

It follows that

$$|A| = -a_{1j}^2 - a_{2j}^2 - a_{3j}^2 = -\sum_{i=1}^{3} a_{ij}^2 < 0$$

since A is nonzero. Hence $|A| \ne 0$, it follows that $|A| = -1$. ||

Problem 8: ★ ★

Suppose that A and B are both 3×3 matrices and $|A| = 3, |B| = 2$, $|A^{-1} + B| = 2$. Find $|A + B^{-1}|$.

Solution:

We have

$$A + B^{-1} = (B^{-1}B)A + B^{-1}(A^{-1}A) = B^{-1}(B + A^{-1})A.$$

Suppose B is invertible, according to **Theorem 2.9**, (iv), one has $|B^{-1}| = |B|^{-1}$. Moreover, since all of them are 3×3, it follows from **Theorem 1.2** (b) that

$$\begin{aligned} |A + B^{-1}| &= |B^{-1}(B + A^{-1})A| \\ &= |B^{-1}||B + A^{-1}||A| \\ &= |B|^{-1}|B + A^{-1}||A| \\ &= \frac{1}{2} \cdot 2 \cdot 3 = 3. \qquad \text{(Since } B + A^{-1} = A^{-1} + B\text{)} \end{aligned}$$

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Problem 9: ★

Suppose a 3×3 matrix A has eigenvalues 1, 2, and 2. Find $|4A^{-1} - I_3|$.

Solution:

Note that:

(1) $\det A = \prod_{i=1}^{n} \lambda_i$ where each λ_i is an eigenvalue of A.

(2) A has eigenvalue $\lambda \Leftrightarrow A^{-1}$ has eigenvalue λ^{-1}.

(3) $\det(cA + I) = \prod_{i=1}^{n} (c\lambda_i + 1)$.

First we have the eigenvalues of A^{-1} being 1, $\frac{1}{2}$, and $\frac{1}{2}$ by (2).

Then we have, by (3)

$$|4A^{-1} - I_3| = \prod_{i=1}^{3} (4\lambda_i - 1) = (4 \cdot 1 - 1)(4 \cdot \frac{1}{2} - 1)(4 \cdot \frac{1}{2} - 1) = 3.$$ ||

Problem 10: ★

Suppose that $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and B is such that $BA = B + 2I_2$, find $|B|$.

Solution:

Since $BA = B + 2I_2$ we have $BA - B = 2I_2 = B(A - I_2)$.

Now we take determinant from both sides of $2I_2 = B(A - I_2)$, one has

$$\begin{aligned} & |2I_2| = |B(A - I_2)| \\ \Leftrightarrow\ & 2^n|I_2| = |B(A - I_2)| \qquad (\text{Since } |\alpha A| = \alpha^n |A|) \\ \Leftrightarrow\ & 2^2 = |B||A - I_2| \qquad (\text{by } \textbf{Theorem 1.2} \text{ (b)}) \\ \Leftrightarrow\ & 4 = |B| \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \\ \Leftrightarrow\ & |B| = 2. \end{aligned}$$

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Problem 11: ★

Suppose v_1, v_2 and v_3 are 3-dimensional real vectors. Suppose that the matrices $A := [v_1 \quad v_2 \quad v_3]$ and $B := \begin{bmatrix} v_1 + v_2 + v_3 & v_1 + 2v_2 + 4v_3 & v_1 + 3v_2 + 9v_3 \end{bmatrix}$. We know that $|A| = 1$, find $|B|$.

Solution:

We can rewrite B as

$$B = [v_1 \quad v_2 \quad v_3] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = AC. \text{ where } C := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$

It follows from **Theorem 1.2** (b) that $|B| = |AC| = |A||C| = |C|$. It left us to find $\det C$:

$$\det C = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} \sim \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 5 \end{vmatrix}.$$

Applying column-expansion with respect to Column 1 yields

$$\det C = (-1)^{1+1} \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2.$$

Hence $|B| = 2$. ||

Problem 12: ⋆

Assume $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and B is a matrix such that $ABA^* = 2BA^* + I_3$.

Find $|B|$.

Solution:

Since $ABA^* = 2BA^* + I_3$ we have $ABA^* - 2BA^* = I_3$, it follows that

$$(A - 2I_3)BA^* = I_3. \tag{1.5}$$

Take the determinant operation on both sides of (1.5) yields

$$|(A - 2I_3)BA^*| = |I_3|.$$

By **Theorem 1.2** (b) and the fact that $|I_n| = 1$ we have

$$|A - 2I_3||B||A^*| = 1.$$

By **Corollary 2.7.1** we have

$$|A - 2I_3||B||A|^{n-1} = 1 = |A - 2I_3||B||A|^2 = 1.$$

Since $A - 2I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ we have $|A - 2I_3| = (-1)^{3+3}(-1) \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1.$

We also have $|A| = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ hence $|A|^2 = 9$, it follows that

$|B| = \dfrac{1}{9}$. ||

Problem 13: ⋆ ⋆ ⋆

Suppose A is a 3×3 matrix, assume $v_1, v_2,$ and v_3 are three linearly independent vectors and $Av_1 = v_1 + v_2, Av_2 = v_2 + v_3, Av_3 = v_1 + v_3$, find

$|A|$.

Solution:

We have

$$A(v_1, v_2, v_3) = (Av_1, Av_2, Av_3) = (v_1 + v_2, v_2 + v_3, v_1 + v_3).$$

It follows that

$$A(v_1, v_2, v_3) = (v_1, v_2, v_3)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since $|(v_1, v_2, v_3)| \neq 0$ by linear independence and by **Theorem 1.2** (b),

$$|A||(v_1, v_2, v_3)| = |(v_1, v_2, v_3)|\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}. \tag{1.6}$$

where $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \sim \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = (-1)^{1+1}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$. Since in (1.6) $|(v_1, v_2, v_3)|$ is a nonzero constant we can divide it by both sides, it follows that $|A| = 2$. ||