

Exercise 1: Determinants Question Sheet
Review on Linear Algebra
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Abstract:

The problems focus mainly on the determinants we covered in the first section. Some interesting application with concepts from other sections are also involved, however, background from other sections are not necessary, one can use both elementary and advanced techniques to solve the problems. The problems are labeled with difficulties by stars, ★ means simple, ★★ means hard, ★★★ means challenging, while ★★★★★ takes amounts of time.

Problem 1: ★

Calculate the determinant $\begin{vmatrix} 0 & a & b & 0 \\ a & 0 & 0 & b \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{vmatrix}$.

Problem 2: ★

Calculate the determinant $\begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 4 & 3 & 2 & \lambda + 1 \end{vmatrix}$.

Problem 3: ★★

Let $\alpha := (-1, 0, 1)^T$ and $A := \alpha\alpha^T$. Let $n \in \mathbb{N}^+$, calculate $|aI - A^n|$.

Problem 4: ★

Calculate the determinant of $\begin{bmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{bmatrix}$.

Problem 5: ★★

Calculate the determinant $\begin{bmatrix} 1-a & a & 0 & 0 & 0 \\ -1 & 1-a & a & 0 & 0 \\ 0 & -1 & 1-a & a & 0 \\ 0 & 0 & -1 & 1-a & a \\ 0 & 0 & 0 & -1 & 1-a \end{bmatrix}$.

Problem 6: ★★

Calculate the determinant of $A = \begin{bmatrix} 2a & 1 & & & & \\ a^2 & 2a & 1 & & & \\ & a^2 & 2a & 1 & & \\ & & \dots & \dots & \dots & \\ & & & a^2 & 2a & 1 \\ & & & & a^2 & 2a \end{bmatrix}$.

Problem 7: ★ ★ ★ ★

Suppose that $A = [a_{ij}]$ is a nonzero matrix with $1 \leq i, j \leq 3$. Suppose that A_{ij} is the (i,j) -entry of the adjoint A^* and $a_{ij} + A_{ij} = 0$. Find $|A|$.

Problem 8: ★ ★

Suppose that A and B are both 3×3 matrices and $|A| = 3, |B| = 2, |A^{-1} + B| = 2$. Find $|A + B^{-1}|$.

Problem 9: ★

Suppose a 3×3 matrix A has eigenvalues 1, 2, and 2. Find $|4A^{-1} - I_3|$.

Problem 10: ★

Suppose that $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and B is such that $BA = B + 2I_2$, find $|B|$.

Problem 11: ★

Suppose v_1, v_2 and v_3 are 3-dimensional real vectors. Suppose that the matrices $A := [v_1 \ v_2 \ v_3]$ and $B := [v_1 + v_2 + v_3 \ v_1 + 2v_2 + 4v_3 \ v_1 + 3v_2 + 9v_3]$. We know that $|A| = 1$, find $|B|$.

Problem 12: ★

Assume $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and B is a matrix such that $ABA^* = 2BA^* + I_3$.

Find $|B|$.

Problem 13: ★ ★ ★

Suppose A is a 3×3 matrix, assume $v_1, v_2,$ and v_3 are three linearly independent vectors and $Av_1 = v_1 + v_2, Av_2 = v_2 + v_3, Av_3 = v_1 + v_3$, find $|A|$.