Hahn-Banach 1st Geometric Form

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Abstract:
In order for convenience in checking definition along learning in Math, I found it necessary to gather definitions of the same kind so that distinguishing the differences among them will no longer be a tedious job. This work is very hard to do even many materials are available, if you want to use for commercial use, please contact me for permission. Furthermore, the work is still long from finished, if you want to contribute for more or better definitions, please contact me.

Lemma 1.2:

Let $C \subseteq E$ be an open set with $0 \in C$, $\forall x \in E$, set $p(x) := \inf\{\alpha > 0 \,|\, \alpha^{-1}x \in C\}$, then $p$ is called the gauge, or the Minkowski functional of C, and $p$ satisfies that:

(i) $p(\lambda x) = \lambda p(x) \forall \lambda > 0 \forall x \in E$ (positive homogeneous)
(ii) $p(x+y) \leq p(x) + p(y) \forall x, y \in E$ (subadditive)
(iii) $\forall x \in E \, \exists M$ such that $0 \leq p(x) \leq M\|x\|$ (bounded above)
(iv) $C := \{x \in E \,|\, p(x) < 1\}$

Proof:

(i) is trivial.

(iii) Let $r > 0$ such that $B_r(0) \subseteq C$, we have $p(x) \leq \frac{1}{r}\|x\| \forall x \in E, M := \frac{1}{r}$.

(iv) $C \subseteq \{x \in E \,|\, p(x) < 1\}$: Suppose $x \in C$, since C is open, $(1+\epsilon)x \in C$ for some $\epsilon > 0$, therefore $p(x) \leq \frac{1}{1+\epsilon} < 1$, then $x \in \{x \in E \,|\, p(x) < 1\}$.

$C \supseteq \{x \in E \,|\, p(x) < 1\}$: if $x \in \{x \in E \,|\, p(x) < 1\}$, then $\exists \alpha \in (0,1)$, such that $\alpha^{-1}x \in C(B_{\alpha^{-1}}(x) \subseteq C)$, thus $x = \alpha(\alpha^{-1}x) + (1-\alpha)0$ since C is convex, then $x \in C$.

(ii) Let $x, y \in E$ and $\epsilon > 0$, by (i) and (iv), we have $\frac{x}{p(x)+\epsilon} \in C$, and $\frac{y}{p(y)+\epsilon} \in C$, thus $\frac{x}{p(x)+\epsilon}t + \frac{y}{p(y)+\epsilon}(1-t) \in C \ \forall t \in [0,1]$, choose the value $t = \frac{p(x)+\epsilon}{p(x)+p(y)+2\epsilon}$, then there exists $\frac{x+y}{p(x)+p(y)+2\epsilon} \in C$, apply (i) and (iv) again, we have $p(x+y) < p(x) + p(y) + 2\epsilon \ \forall \epsilon > 0$, since $\epsilon$ is arbitrary, let $\epsilon \to 0$.

□

Lemma 1.3:

Let $C \subseteq E$ be a non-empty open convex set and let $x_0 \in E$ with $x_0 \notin C$, there exists $f \in E^*$ such that $f(x) < f(x_0) \forall x \in C$; in particular, the hyperplane $[f = f(x_0)]$ separates $\{x_0\}$ and C.

Proof:

If $0 \notin C$, we can use translation to make $\{0\}$ always be an element of C, therefore, we can apply Lemma 1.2, the gauge $p$ of C, $p(x) := \inf\{\alpha > 0 \,|\, \alpha^{-1}x \in C\}$, with the desired properties:

(i) $p(\lambda x) = \lambda p(x) \forall \lambda > 0 \forall x \in E$ (positive homogeneous)

(ii) $p(x+y) \leq p(x) + p(y) \forall x, y \in E$ (subadditive)

(iii) $\forall x \in E \, \exists M$ such that $0 \leq p(x) \leq M\|x\|$ (bounded above)

(iv) $C := \{x \in E \,|\, p(x) < 1\}$.

Consider the linear subspace $G := \mathbb{R}x_0$ and the linear functional $g : G \to \mathbb{R}$ defined by $g(tx_0) = t, t \in \mathbb{R}$. It is clear that $p(x) \geq g(x) \forall x \in G$. Now apply Helly, Hahn-Banach Theorem, then there exists a linear functional $f$ on $E$ which extends $g$ such that $f(x) \leq p(x) \forall x \in E$. In particular, we have $f(x_0) = 1$, and by (iii), $f$ is continuous, by (iv), $f(x) < 1 \forall x \in C$.

□

Theorem 1.3: Hahn-Banach, 1st geometric form

Let $A, B \subseteq E$ be two non-empty convex subsets such that $A \cap B = \emptyset$, if one of them is open, then there exists a closed hyperplane that separates A and B.

Proof:

***[Claim I]***: $C := A - B = \{x - y \,|\, x \in A, y \in B\}$ is convex.

Take $z_1, z_2 \in C, \theta \in [0,1]$ such that $z_1 = x_1 - y_1, z_2 = x_2 - y_2$, where $x_1, x_2 \in A, y_1, y_2 \in B$. Then consider the equation

$$\begin{aligned}\theta z_1 + (1-\theta)z_2 &= \theta(x_1 - y_1) + (1-\theta)(x_2 - y_2)\\ &= \theta x_1 + (1-\theta)x_2 - [\theta y_1 + (1-\theta)y_2] \in A - B = C.\end{aligned}$$

***[Claim II]***: C is open.

To show C is open is to show that take $x \in A, y \in B, x - y \in C^\circ$.
Assume that A is open, fix $z \in A, \exists \epsilon > 0$ such that $x - y + tz \in C \,\forall |t| \leq \epsilon$, since A is open, then $x + tz \in A$, consequently $x - y \in C^\circ$.

***[Claim III]***: $0 \notin C$.

If not, $\exists 0 \in C = A - B$, $\exists z \in A, y \in B$ such that $0 = z - y$, i.e., $z = y$, but $A \cap B = \emptyset$ by assumption, contradicts, hence $0 \notin C$.

By the Claims, all the requirements of Lemma 1.3 are met, now apply Lemma 1.3. There exists $f \in E^*$ such that $f(z) < 0 \forall z \in C$, i.e., $f(x) < f(y) \forall x \in A, y \in B$. Fix a constant $\alpha$ such that $\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y)$.

Then $:= [f = \alpha]$ separates A and B.

□