Weak topologies

David Lecomte

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1 Preliminaries from general topology

In this section, we are given a set X, a collection of topological spaces $(\mathrm{Y}_i)_{i\in\mathrm{I}}$ and a collection of maps $(f_i)_{i\in\mathrm{I}}$ such that each f_i maps X into Y_i.

We wish to define a topology on X that makes all the f_i's continuous. And we want to do this in the cheapest way, that is: there should be no more open sets in X than required for this purpose.

Obviously, all the $f_i^{-1}(\mathrm{O}_i)$, where O_i is an open set in Y_i should be open in X. Then finite intersections of those should also be open. And then any union of finite intersections should be open. By this process, we have created as few open sets as required. Yet it is not clear that the collection obtained is closed under finite intersections. It actually is, as a consequence of the following lemma:

Lemma 1
Let X be a set and let $\mathscr{O} \subset \mathscr{P}(\mathrm{X})$ be a collection of subsets of X, such that

- *$\emptyset$ and X are in $\mathscr{O}$;*
- *$\mathscr{O}$ is closed under finite intersections.*

Then $\mathscr{T} = \{\bigcup_{\mathrm{O}\in\mathcal{O}} \mathrm{O} \mid \mathcal{O} \subset \mathscr{O}\}$ is a topology on X.

Proof: By definition, $\mathscr{T}$ contains X and $\emptyset$ since those were already in $\mathscr{O}$. Furthermore, $\mathscr{T}$ is closed under unions, again by definition. So all that's left is to check that $\mathscr{T}$ is closed under finite intersections.

Let A_1 and A_2 be two elements of $\mathscr{T}$. Then there exist $\mathcal{O}_1$ and $\mathcal{O}_2$, subsets of $\mathscr{O}$, such that

$$\mathrm{A}_1 = \bigcup_{\mathrm{O}\in\mathcal{O}_1} \mathrm{O} \qquad \text{and} \qquad \mathrm{A}_2 = \bigcup_{\mathrm{O}\in\mathcal{O}_2} \mathrm{O}$$

It is then easy to check by double inclusion that

$$A_1 \cap A_2 = \bigcup_{\substack{O_1 \in \mathcal{O}_1 \\ O_2 \in \mathcal{O}_2}} O_1 \cap O_2$$

Letting $\mathcal{O}$ denote the collection $\{O_1 \cap O_2 \mid O_1 \in \mathcal{O}_1 \quad O_2 \in \mathcal{O}_2\}$, which is a subset of $\mathscr{O}$ since the latter is closed under finite intersections, we get

$$A_1 \cap A_2 = \bigcup_{O \in \mathcal{O}} O$$

This set belongs to $\mathscr{T}$. By induction, $\mathscr{T}$ is closed under finite intersections. □

Corollary 2
The collection of all unions of finite intersection of sets of the form $f_i^{-1}(O_i)$ where $i \in I$ and O_i is an open set in Y_i is a topology. It is called the weak topology on X generated by the $(f_i)_{i \in I}$'s and we denote it by $\sigma\big(X, (f_i)_{i \in I}\big)$.

By definition, the functions $(f_i)_{i \in I}$ are continuous for this topology.

Of course, a topology is useless if it is too complicated for us to deal with it. It turns out there is a nice characterization of converging sequences, and continuous function from a topological space into a weakly topologized space.

Theorem 3
Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. It converges in the topology $\sigma\big(X, (f_i)_{i \in I}\big)$ to some $x \in X$ if and only if

$$\forall i \in I \qquad \lim_{n \to \infty} f_i(x_n) = f_i(x)$$

Remember that in a topological space $(X, \mathscr{T})$,

- a sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ if and only if

  $$\forall O \in \mathscr{T} \text{ containing x} \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \qquad x_n \in O$$

  Notice also that if the topology $\mathscr{T}$ is not Hausdorff, there is no unicity for the limit of a sequence. For example, if x and y are distinct points that cannot be separated by open sets, the constant sequence equal to x will converge both to x and y.

- a function $f : X \longrightarrow Y$, where Y is a topological space, is continuous at a point x if and only if the preimage by f of any open set (in Y) containing $f(x)$ is in $\mathscr{T}$.

It is easy to show that in such a case, for every sequence $(x_n)_{n\in\mathbb{N}}$ converging to x, the sequence $\big(f(x_n)\big)_{n\in\mathbb{N}}$ converges to $f(x)$. But *this property does not characterize continuity at x* in general!!! We need more assumptions on X, one of them being for example that x has a countable basis of neighbourhoods.

So we see that there are bad habits, inherited from working with normed or metric spaces all the time, that we have to lose. Here are more of them:

- If X is compact for the topology $\mathscr{T}$, there is no reason to believe that any sequence has a converging subsequence.

  What can be shown is the following: if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X compact, there exists a point x such that any open set containing x contains infinitely many terms of the sequence $(x_n)_{n\in\mathbb{N}}$. However, this does not allow us to build a subsequence of $(x_n)_{n\in\mathbb{N}}$ converging to x because there could be too many open neighbourhoods of x.

  However, if x has a countable basis of neighbourhoods, we can build such a subsequence.

- If A is any subset of X, the closure of A cannot be characterized anymore as the set of limit points of A. It is true that limit points of A are in $\overline{\mathrm{A}}$, but $\overline{\mathrm{A}}$ could contain strictly the set of limit points of A.

  The only characterization of points $x \in \overline{\mathrm{A}}$ is that every open set containing x meets A.

Proof: Suppose first that the sequence converges in the weak topology to some $x \in \mathrm{X}$. Since for every $j \in \mathrm{I}$, the function f_j is continuous for $\sigma\big(\mathrm{X}, (f_i)_{i\in\mathrm{I}}\big)$, we have

$$\lim_{n\to\infty} f_j(x_n) = f_j(x)$$

Conversely, suppose that there exists x in X such that

$$\forall i \in \mathrm{I} \qquad \lim_{n\to\infty} f_i(x_n) \text{ exists and equals } f_i(x)$$

Let O be any open set containing x. By definition, there exist a finite subset J of I, and open sets $(\mathrm{O}_j)_{j\in\mathrm{J}}$ such that $\mathrm{O}_j \subset \mathrm{Y}_j$ for all $j \in \mathrm{J}$, such that

$$x \in \bigcap_{j=1}^{n} f_{i_j}^{-1}(\mathrm{O}_j)$$

which means that $$\forall j \in \mathrm{J} \qquad f_j(x) \in \mathrm{O}_j$$

Given $j \in \mathrm{J}$, we know that the sequence $\big(f_j(x_n)\big)_{n\in\mathbb{N}}$ converges to $f_j(x)$. Then, since O_j contains $f_j(x)$, there exists $\mathrm{N}_j \in \mathbb{N}$ such that

$$\forall n \geqslant \mathrm{N}_j \qquad f_j(x_n) \in \mathrm{O}_j$$

Letting $\mathrm{N} = \operatorname*{Max}_{j\in\mathrm{J}} \mathrm{N}_j$, we have

$$\forall n \geqslant \mathrm{N} \quad \forall j \in \mathrm{J} \qquad f_j(x_n) \in \mathrm{O}_j$$

In other words, $$\forall n \geqslant \mathrm{N} \qquad x_n \in \bigcap_{j\in\mathrm{J}} f_j^{-1}(\mathrm{O}_j) \subset \mathrm{O}$$

So $(x_n)_{n\in\mathbb{N}}$ converges to x for the topology $\sigma\big(\mathrm{X}, (f_i)_{i\in\mathrm{I}}\big)$. □

Theorem 4
Let $(\mathrm{Z}, \mathscr{T})$ be a topological space, and $\varphi : \mathrm{Z} \longrightarrow \mathrm{X}$ be map. Then φ is continuous for the topologies $\mathscr{T}$ and $\sigma\big(\mathrm{X}, (f_i)_{i\in\mathrm{I}}\big)$ if and only if for every $i \in \mathrm{I}$, $f_i \circ \varphi$ is continuous.

Proof: Easy, try it. □

An example of a weak topology, aside from the ones we will present in those note, is the topology of pointwise convergence. It is defined as follows: let A be any set and let X be the set of functions $\mathrm{A} \longrightarrow \mathbb{R}$. For every a in A, define the function $\varphi_a : \mathrm{X} \longmapsto \mathbb{R}$ by

$$\forall f \in \mathrm{X} \qquad \varphi_a(f) = f(a)$$

The topology of pointwise convergence is $\sigma\big(\mathrm{X}, (f_a)_{a\in\mathrm{A}}\big)$. In this topology, a sequence of functions converges if and only if it converges pointwise, in view of **Theorem 3**. One can show that this topology is not metrizable, this is the topic of a problem in the fall 2003 qualifying exam.

2 The topology $\sigma(\mathrm{X}, \mathrm{X}^\star)$

In this section, X is a normed space. Unless stated otherwise, we do not assume that it is complete.

Definition 5 The weak topology on X is the topology $\sigma\big(\mathrm{X}, (f)_{f\in\mathrm{X}^\star}\big)$. For convenience, it is simply denoted $\sigma(\mathrm{X}, \mathrm{X}^\star)$.

A first thing we want to check is that the weak topology on X is Hausdorff, which will guarantee us the unicity of limits.

Theorem 6
The topology $\sigma(\mathrm{X}, \mathrm{X}^\star)$ is Hausdorff.

Proof: Let x and y be two distinct points in X. Since $\|x-y\|>0$, there exists a positive ϵ such that $\mathcal{B}(x,\epsilon)$ does not contain y. Since $\mathcal{B}(x,\epsilon)$ is convex open, we know that it can be strictly separated from $\{y\}$ by a hyperplane by the geometric form of the Hahn-Banach theorem: there exist $f\in \mathrm{X}^\star$ and $\alpha\in\mathbb{R}$ such that

$$\forall u\in\mathcal{B}(x,\epsilon)\qquad (f,u)<\alpha<(f,y)$$

In particular, $$(f,x)<\alpha<(f,y)$$

Therefore, $$x\in f^{-1}\big((-\infty,\alpha)\big)\qquad\text{and}\qquad y\in f^{-1}\big((\alpha,+\infty)\big)$$

Those two sets are weakly open, since they are preimages of open subsets of $\mathbb{R}$ by a linear functional. And they are disjoint. Thus $\sigma(\mathrm{X},\mathrm{X}^\star)$ is Hausdorff. □

Then we prove a few easy facts comparing the weak topology and the norm (also called strong) topology on X.

Proposition 7

1. *The weak topology is weaker than the norm topology: every weakly open (resp. closed) set is strongly open (resp. closed).*
2. *A sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x if and only if*

$$\forall f\in\mathrm{X}^\star\qquad \lim_{n\to\infty}(f,x_n)=(f,x)$$

3. *A strongly converging sequence converges weakly.*
4. *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging weakly to x, then $(x_n)_{n\in\mathbb{N}}$ is bounded and*

$$\|x\|\leqslant\liminf_{n\to\infty}\|x_n\|$$

5. *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging weakly to x and $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{X}^\star$ converging strongly to f, then*

$$\lim_{n\to\infty}(f_n,x_n)\text{ exists and equals }(f,x)$$

Proof: The first point is clear: the norm topology already makes all linear functionals continuous. Since the weak topology is the weakest with this property, it is weaker than the strong topolgy. So every weakly open set is strongly open, and by taking complements, every weakly closed set is strongly closed.

The second point is just a restatement of **Theorem 3** in the particular case of the weak topology on X.

The third point is clear as well: if $(x_n)_{n\in\mathbb{N}}$ converges strongly to x, then

$$\forall f \in \mathrm{X}^\star \qquad \big|(f,x)-(f,x_n)\big| = \big|(f,x-x_n)\big| \leqslant \|f\|\,\|x-x_n\| \xrightarrow[n\to\infty]{} 0$$

and therefore $(x_n)_{n\in\mathbb{N}}$ converges weakly to x by 2.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging weakly to x. We have already seen at least once in quals that $(x_n)_{n\in\mathbb{N}}$ is bounded, as a consequence of the Banach-Steinhaus theorem (see for example the spring 2002 exam). Also, if f is a bounded linear functional on X,

$$\forall n \in \mathbb{N} \qquad \big|(f,x_n)\big| \leqslant \|f\|\,\|x_n\|$$

so by taking liminfs, we get

$$\big|(f,x)\big| \leqslant \|f\|\,\liminf_{n\to\infty}\|x_n\|$$

Since

$$\|x\| = \operatorname*{Sup}_{\substack{f\in\mathrm{X}^\star\\ f\neq 0}} \frac{\big|(f,x)\big|}{\|f\|}$$

it follows that

$$\|x\| \leqslant \liminf_{n\to\infty}\|x_n\|$$

Finally, let $(x_n)_{n\in\mathbb{N}}$ be sequence in X converging weakly to x and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{X}^\star$ converging strongly to f. We have for every integer n

$$\big|(f,x)-(f_n,x_n)\big| = \big|(f,x)-(f,x_n)+(f,x_n)-(f_n,x_n)\big| \leqslant \big|(f,x-x_n)\big| + \|f_n-f\|\,\|x_n\|$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded by 4, the righthandside tends to 0 and 5 is proved. □

The next step is to identify a basis of neighbourhoods for $\sigma(\mathrm{X},\mathrm{X}^\star)$.

Theorem 8
Let $x_0 \in \mathrm{X}$. A basis of neighbourhoods of x_0 for the weak topology is given by the collection of sets of the form

$$\mathrm{W}_{\epsilon,f_1,\ldots,f_n} = \big\{x\in\mathrm{X} \;\big|\; \forall i\in\{1,\ldots,n\} \quad \big|(f_i,x)-(f_i,x_0)\big| < \epsilon\big\}$$

$$n\in\mathbb{N} \quad \epsilon>0 \quad f_1,\ldots,f_n\in\mathrm{X}^\star$$

Proof: Remember that a collection $\mathscr{X}$ of open sets containing x is a basis of neighbourhoods of x_0 if and only if every open set containing x contains an element of $\mathscr{X}$.

That every set $\mathrm{W}_{\epsilon,f_1,\ldots,f_n}$ are weakly open is clear, since all $f_1,\ldots,f_n$ are weakly continuous. Furthermore, it contains x_0 since

$$\forall i\in\{1,\ldots,n\} \qquad \big|(f_i,x_0)-(f_i,x_0)\big| = 0 < \epsilon$$

Now, let O be any weakly open set containing x_0. By definition of the topology $\sigma(\mathrm{X},\mathrm{X}^\star)$, it is a union of finite intersections of preimages of open sets in $\mathbb{R}$ by bounded linear functionals. So there exist a finitely many bounded linear functionals $f_1, \ldots, f_n$ and open subsets $\mathrm{O}_1, \ldots, \mathrm{O}_n$ of $\mathbb{R}$, such that

$$x_0 \in \bigcap_{j=1}^{n} f_j^{-1}(\mathrm{O}_j) \subset \mathrm{O}$$

Then for every $j \in \{1, \ldots, n\}$, the real number (f_j, x_0) belongs to O_j. Since this set is open, there is a positive ϵ_j such that

$$\big((f_j, x_0) - \epsilon_j, (f_j, x_0) + \epsilon_j\big) \subset \mathrm{O}_j$$

Let $$\epsilon = \operatorname*{Min}_{1 \leqslant j \leqslant n} \epsilon_j > 0$$

so that $$\forall j \in \{1, \ldots, n\} \qquad \big((f_j, x_0) - \epsilon, (f_j, x_0) + \epsilon\big) \subset \mathrm{O}_j$$

Then, if x belongs to $\mathrm{W}_{\epsilon, f_1, \ldots, f_n}$, we have by definition of this set

$$\forall j \in \{1, \ldots, n\} \qquad (f_j, x) \in \big((f_j, x_0) - \epsilon, (f_j, x_0) + \epsilon\big) \subset \mathrm{O}_j$$

which means that $$x \in \bigcap_{j=1}^{n} f_j^{-1}(\mathrm{O}_j) \subset \mathrm{O}$$

Thus $$\mathrm{W}_{\epsilon, f_1, \ldots, f_n} \subset \mathrm{O}$$ □

Now that we have two topologies on X, we may wonder if by any chance they coincide. The answer is given by the following

Proposition 9
The weak topology and the strong topology on X *coincide if and only if* X *is finite dimensional.*

Proof: Suppose first that X is finite dimensional. We already know that the weak topology is included in the strong topology, through **Proposition 7**. All that is left to show is the converse.

Since X is finite dimensional, it has a basis $(e_1, \ldots, e_n)$. Any $x \in \mathrm{X}$ has a unique decomposition along this basis, which means that

$$\exists!(x_1, \ldots, x_n) \in \mathbb{R}^n \qquad x = \sum_{i=1}^{n} x_i e_i$$

Define then $$\|x\|_\infty = \operatorname*{Max}_{1 \leqslant i \leqslant n} |x_i|$$

It is easy to check that it is a norm on X.

Remember that, as a consequence of the finite dimensionality of X, all norms are equivalent and the strong topology on X is the topology defined by any norm. So if O is any strong open set, it is in particular open for $\| \|_\infty$. This means that for every $x \in \mathrm{O}$, there exists a positive ϵ_x such that

$$\mathcal{B}_\infty(x, \epsilon_x) \subset \mathrm{O}$$

Therefore
$$\mathrm{O} = \bigcup_{x \in \mathrm{O}} \mathcal{B}_\infty(x, \epsilon_x)$$

So if we show that any open ball is weakly open, we get that the strongly open set O is weakly open, as union of weakly open sets.

Let x be any point in X and ϵ be any positive real number. Then

$$\mathcal{B}_\infty(x, \epsilon) = \big\{y \in \mathrm{X} \;\big|\; \|y - x\|_\infty < \epsilon\big\} = \big\{y \in \mathrm{X} \;\big|\; \forall i \in \{1, \ldots, n\} \quad |y_i - x_i| < \epsilon\big\}$$

But the functionals $f_1, \ldots, f_n$, defined by

$$\forall x = \sum_{j=1}^{n} x_j e_j \in \mathrm{X} \qquad (f_i, x) = x_i$$

are clearly in $\mathrm{X}^\star$. And we can then write

$$\mathcal{B}_\infty(x, \epsilon) = \big\{y \in \mathrm{X} \;\big|\; \forall i \in \{1, \ldots, n\} \quad \big|(f_i, y) - (f_i, x)\big| < \epsilon\big\}$$

which proves that $\mathcal{B}_\infty(x, \epsilon)$ is weakly open, by **Theorem 8**.

So any strong open set is weakly open: the weak and strong topology on a finite dimensional space coincide.

Now let's suppose that X is infinite dimensional, and let's show that the weak and strong topologies do not coincide. Let S be the unit sphere in X:

$$\mathrm{S} = \big\{x \in \mathrm{X} \;\big|\; \|x\| = 1\big\}$$

Then S is strongly closed. But 0 belongs to the weak closure of S. Indeed, let O be any weak neighbourhood of 0. By **Theorem 8**, there exist $\epsilon > 0$ and $f_1, \ldots, f_n$ in $\mathrm{X}^\star$ such that

$$\mathrm{W} = \big\{x \in \mathrm{X} \;\big|\; \big|(f_i, x)\big| < \epsilon\big\} \subset \mathrm{O}$$

The map
$$\begin{aligned} \Phi :\ & \mathrm{X} \longrightarrow \mathbb{R}^n \\ & x \longmapsto \big((f_1, x), \ldots, (f_n, x)\big) \end{aligned}$$

is linear and

$$\operatorname{Ker} \Phi = \big\{x \in \mathrm{X} \;\big|\; (f_i, x) = 0 \quad \forall 1 \leqslant i \leqslant n\big\} = \bigcap_{i=1}^{n} \operatorname{Ker} f_i$$

By the rank-nullity theorem,

$$\mathrm{Dim}\,\mathrm{Ker}\,\Phi + \mathrm{Dim}\,\mathrm{Im}\,\Phi = \mathrm{Dim}\,\mathrm{X} = \infty$$

Since
$$\mathrm{Dim}\,\mathrm{Im}\,\Phi \leqslant n$$

it follows that $\mathrm{Ker}\,\Phi$ is infinite dimensional, and can certainly not be equal to $\{0\}$. So there exists $x \neq 0$ such that

$$\forall i \in \{1, \ldots, n\} \qquad (f_i, x) = 0$$

Then
$$\forall \lambda \in \mathbb{R} \quad \forall i \in \{1, \ldots, n\} \qquad \big|(f_i, \lambda x)\big| = 0 < \epsilon$$

which proves that
$$\forall \lambda \in \mathbb{R} \qquad \lambda x \in \mathrm{W} \subset \mathrm{O}$$

Now, taking $\lambda = \frac{1}{\|x\|}$ shows that O intersect S. So any weakly open neighbourhood of 0 intersects S: 0 is in the weak closure of S. Thus the weak and strong closure of S are different. □

> Notice how in the proof we showed that weakly open sets are very big: they at least contain lines when the dimension of X is infinite.
>
> Using exactly the same strategy, one can show that the weak closure of S contains the closed unit ball $\overline{\mathcal{B}}$ of X. And the next theorem will establish that $\overline{\mathcal{B}}$ is weakly closed. So we get that in an infinite dimensional normed space,
>
> $$\overline{\mathrm{S}}^{\sigma(\mathrm{X},\mathrm{X}^\star)} = \overline{\mathrm{B}}$$

The next theorem answers the important question: if the weak topology is strictly weaker than the strong topology (in infinite dimension), are there sets for which we can guarantee that strongly closed implies weakly closed?

Theorem 10
Let C *be a nonempty convex set in* X*. Then* C *is strongly closed if and only if it is weakly closed.*

Proof: Of course, by **Proposition 7**, if C is weakly closed, it is strongly closed. We are more interested in the converse.

Suppose that C is strongly closed. We have $\mathrm{C} \subset \overline{\mathrm{C}}^{\sigma(\mathrm{X},\mathrm{X}^\star)}$ and we want to show the converse inclusion. So let x be in the complement of C. By the geometric form of the Hahn-Banach theorem, C (convex strongly closed) and $\{x\}$ (convex strongly compact) can be strictly separated by a hyperplane: there exist $f \in \mathrm{X}^\star$ and $\alpha \in \mathbb{R}$ such that

$$\forall y \in \mathrm{C} \qquad (f, y) < \alpha < (f, x)$$

Thus the weakly open set $f^{-1}\big((\alpha, +\infty)\big)$ contains x and does not intersect C. So x is not in the weak closure of C. Which achieves the proof. □

Now we worry about bounded linear maps.

Theorem 11
Let X *and* Y *be two Banach spaces and let* T *be a linear map between* X *and* Y*. Then* T *is continuous strong-strong if and only if it is continuous weak-weak.*

Proof: The fact that continuous strong-stong implies continuous weak-weak is a consequence of **Theorem 4**. The converse (or something very close) has already been studied in the 2003 spring qualifying exam, and follows at once from the closed graph theorem.□

And to finish this section, we wonder about whether the weak topology is metrizable. We will first need a lemma from linear algebra:

Lemma 12
Let $f_1, \ldots, f_n, f$ *be linear functionals on a vector space* X*. Then* f *is a linear combination of* $f_1, \ldots, f_n$ *if and only if*

$$\bigcap_{i=1}^{n} Ker\, f_i \subset Ker\, f$$

Proof: If f is a linear combination of $f_1, \ldots, f_n$, we can write

$$f = \sum_{i=1}^{n} \alpha_i f_i \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{R}$$

and it clear from here that $\bigcap_{i=1}^{n} \operatorname{Ker} f_i \subset \operatorname{Ker} f$.

Conversely, suppose that this inclusion holds. Assume also, for now, that $f_1, \ldots, f_n$ are linearly independent. Define

$$\forall x \in \mathrm{X} \qquad \Phi(x) = \big((f_1, x), \ldots, (f_n, x), (f, x)\big)$$

Then $(0, \ldots, 0, 1)$ cannot be in the image of Φ, since if $(f_i, x) = 0$ for $1 \leqslant i \leqslant n$, then $(f, x) = 0$ as well. So $\operatorname{Im}\Phi$ is a strict subspace of $\mathbb{R}^{n+1}$ and there exists a nonzero linear functional on $\mathbb{R}^{n+1}$ that vanishes on $\operatorname{Im}\Phi$:

$$\exists (\alpha_1, \ldots, \alpha_n, \alpha) \in \mathbb{R}^{n+1} \setminus \{0\} \quad \forall (x_1, \ldots, x_n, x) \in \operatorname{Im}\Phi \qquad \alpha x + \sum_{i=1}^{n} \alpha_i x_i = 0$$

Then
$$\forall x \in \mathrm{X} \qquad \alpha(f, x) + \sum_{i=1}^{n} \alpha_i (f_i, x) = 0$$

and
$$\forall x \in \mathrm{X} \qquad \Big(\alpha f + \sum_{i=1}^{n} \alpha_i f_i, x\Big) = 0$$

which implies that
$$\alpha f + \sum_{i=1}^{n} \alpha_i f_i = 0$$

Now, α cannot be 0 since $f_1, \ldots, f_n$ are linearly independent. Thus we can divide by α and f is a linear combination of $f_1, \ldots, f_n$.

If we don't assume that $f_1, \ldots, f_n$ are linearly independent, up to renaming, we can suppose that $f_1, \ldots, f_p$ are linearly independent and that $f_{p+1}, \ldots, f_n$ are linear combinations of those. Then by the first part of the proposition,

$$\bigcap_{i=1}^{p} \operatorname{Ker} f_i = \bigcap_{i=1}^{n} \operatorname{Ker} f_i \subset \operatorname{Ker} f$$

From what we just proved, f is a linear combination of $f_1, \ldots, f_p$, which is fair enough.$\square$

Corollary 13

The weak topology on a normed space X *is metrizable if and only if* X *is finite dimensional.*

Proof: From **Proposition 9**, if X is finite dimensional, the weak topology is the norm topology and therefore comes from a metric.

Conversely, suppose that there is a metric d on X, such that the induced topolgy coincides with the weak topology. For every positive integer k, define

$$\mathrm{B}_k = \left\{ x \in \mathrm{X} \;\middle|\; \mathrm{d}(x, 0) < \frac{1}{k} \right\}$$

B_k is open for d and thus is a weak open neighbourhood of 0. By **Theorem 8**, there exist a finite collection F_k of bounded linear functionals and a positive ϵ_k such that

$$\mathrm{W}_k = \left\{ x \in \mathrm{X} \;\middle|\; \left|(f, x)\right| < \epsilon_k \quad \forall f \in \mathrm{F}_k \right\} \subset \mathrm{B}_k$$

Let
$$\mathrm{F} = \bigcup_{k \geqslant 1} \mathrm{F}_k$$

which is a countable subset of $\mathrm{X}^\star$. We show that F spans $\mathrm{X}^\star$. This implies (see the spring 2004 qual) that $\mathrm{X}^\star$ (complete) is finite dimensional. Thus $\mathrm{X}^{\star\star}$ is finite dimensional as well, and therefore X follows the herd since it injects in $\mathrm{X}^{\star\star}$.

So let g be any bounded linear functional on X and consider

$$\mathrm{W} = \left\{ x \in \mathrm{X} \;\middle|\; \left|(g, x)\right| < 1 \right\}$$

This is a weak neighbourhood of 0. So there exists a k big enough so that $\mathrm{B}_k \subset \mathrm{W}$ and as a consequence $\mathrm{W}_k \subset \mathrm{W}$. Let $x \in \bigcap_{f \in \mathrm{F}_k} \operatorname{Ker} f$. Then

$$\forall \lambda \in \mathbb{R} \quad \forall f \in \mathrm{F}_k \qquad \left|(f, \lambda x)\right| = 0 < \epsilon_k$$

Hence
$$\forall \lambda \in \mathbb{R} \qquad \lambda x \in \mathrm{W}_k \subset \mathrm{W}$$

which means that
$$\forall \lambda \in \mathbb{R} \qquad |\lambda| \left|(g, x)\right| < 1$$

Necessarily $\qquad (g, x) = 0 \qquad$ that is $\qquad x \in \operatorname{Ker} g$

According to **Lemma 12**, $g \in \operatorname{Span} \mathrm{F}_k$. Which achieves showing that

$$\mathrm{X}^\star = \operatorname{Span} \mathrm{F} \qquad \square$$

3 The topology $\sigma(\mathrm{X}^\star, \mathrm{X})$

Remember that any element of X can be seen as a bounded linear function on $\mathrm{X}^\star$, through evaluation at x:

$$\forall f \in \mathrm{X}^\star \qquad (x, f) = (f, x)$$

Definition 14 The weak$\star$ topology on $\mathrm{X}^\star$ is the topology $\sigma(\mathrm{X}^\star, (x)_{x\in\mathrm{X}})$. For convenience, it is simply noted $\sigma(\mathrm{X}^\star, \mathrm{X})$.

We rapidly check the usual properties of this new topology:

Theorem 15
The topology $\sigma(\mathrm{X}^\star, \mathrm{X})$ is Hausdorff.

Proof: This is easier than **Theorem 6**, since it does not even involve the Hahn-Banach theorem. We let f and g be distinct elements of $\mathrm{X}^\star$. Thus there exists $x \in \mathrm{X}$ such that

$$(f, x) \neq (g, x)$$

Assuming, for example, that $(f, x) < (g, x)$, we can find a real number α such that

$$(f, x) < \alpha < (g, x)$$

so that $\qquad f \in x^{-1}\big((-\infty, \alpha)\big) \qquad$ and $\qquad g \in x^{-1}\big((\alpha, +\infty)\big)$

Those are two disjoint weak$\star$ open sets that separate f and g. $\square$

Proposition 16

1. *The weak$\star$ topology on $\mathrm{X}^\star$ is weaker than the weak topology $\sigma(\mathrm{X}^\star, \mathrm{X}^{\star\star})$, itself weaker than the norm topology.*
2. *A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{X}^\star$ is weak$\star$ convergent to f if and only if*
$$\forall x \in \mathrm{X} \qquad \lim_{n\to\infty} (f_n, x) = (f, x)$$
3. *A strongly converging sequence in $\mathrm{X}^\star$ is weak$\star$ convergent.*
4. *If $(f_n)_{n\in\mathbb{N}}$ is weak$\star$ convergent to f, then $(f_n)_{n\in\mathbb{N}}$ is bounded and*
$$\|f\| \leqslant \liminf_{n\to\infty} \|f_n\|$$

5. *If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence in* $\mathrm{X}^\star$*, weak*$\star$ *convergent to* f*, and if* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *converging strongly to* x*, then*

$$\lim_{n\to\infty} (f_n, x_n) \text{ exists and equals } (f, x)$$

Proof: Same as **Proposition 7**. □

Theorem 17
Let $f_0 \in \mathrm{X}^\star$*. A basis of neighbourhoods of* f_0 *for the weak*$\star$ *topology is given by the collection of sets of the form*

$$\mathrm{W}_{\epsilon,x_1,\dots,x_n} = \left\{ f \in \mathrm{X}^\star \;\middle|\; \forall i \in \{1,\dots,n\} \quad \left|(f,x_i) - (f_0,x_i)\right| < \epsilon \right\}$$

$$n \in \mathbb{N} \quad \epsilon > 0 \quad x_1,\dots,x_n \in \mathrm{X}$$

Proof: Same as **Theorem 8**. □

> Now, at this point, we might wonder why in the world someone would be that obsessed with weakening topologies.
>
> The basic answer is that, if there are less open sets, it is easier to extract finite subcovers from open covers. So we are hoping to get more compact sets.
>
> And indeed, the Banach-Alaoglu-Bourbaki theorem will show that the closed unit ball in $\mathrm{X}^\star$ actually is compact for the weaker $\sigma(\mathrm{X}^\star, \mathrm{X})$. This is, for example, the starting point of the (really neat) Gelfand theory of Banach algebras. See Professor Katznelson's book for that.
>
> Aside from that very specific example, compactness is generally a good thing because it allows us to show that things exist: minimums or maximums of continuous functions, fixed points, and other stuff such as converging subsequences (even though compactness alone is not enough for that).

Proposition 18
Let $\varphi \in \mathrm{X}^{\star\star}$ *and suppose that* φ *is weak*$\star$ *continuous. Then*

$$\exists x \in \mathrm{X} \quad \forall f \in \mathrm{X}^\star \qquad (\varphi, f) = (f, x)$$

Proof: Since φ is weak$\star$ continuous, the set

$$\mathrm{V} = \left\{ f \in \mathrm{X}^\star \;\middle|\; \left|(\varphi, f)\right| < 1 \right\}$$

is weak$\star$ open and contains 0. According to **Theorem 17**, there exist $x_1,\dots,x_n$ in X and a positive ϵ such that

$$\mathrm{W} = \left\{ f \in \mathrm{X}^\star \;\middle|\; \left|(f, x_i)\right| < \epsilon \quad \forall 1 \leqslant i \leqslant n \right\} \subset \mathrm{V}$$

If $f \in \bigcap_{i=1}^{n} \operatorname{Ker} x_i$, then

$$\forall \lambda \in \mathbb{R} \quad \forall 1 \leqslant i \leqslant n \qquad (\lambda f, x_i) = 0 < \epsilon$$

so that
$$\forall \lambda \in \mathbb{R} \qquad \lambda f \in \mathrm{W} \subset \mathrm{V}$$

Thus
$$\forall \lambda \in \mathbb{R} \qquad |\lambda| \, \big|(\varphi, f)\big| < 1$$

Necessarily,
$$f \in \operatorname{Ker} \varphi$$

According to **Lemma 12**, φ is a linear combination of $x_1, \dots, x_n$. □

Proposition 19
Let H *be a hyperplane in* $\mathrm{X}^\star$ *and suppose that* H *is closed for the weak*$\star$ *topology. Then there exist* $x \in \mathrm{X}$ *and* $\alpha \in \mathbb{R}$ *such that*

$$\mathrm{H} = \big\{ f \in \mathrm{X}^\star \mid (f, x) = \alpha \big\}$$

Proof: H is a hyperplane, that is there exist $\xi \in \mathrm{X}^{\star\star}$ and $\alpha \in \mathbb{R}$ such that

$$\mathrm{H} = \big\{ f \in \mathrm{X}^\star \mid (\xi, f) = \alpha \big\}$$

By assumption, H is weak$\star$ closed. So if $f_0 \in \mathrm{X}^\star \setminus \mathrm{H}$, there exists a weak$\star$ open set containing f_0 that does not intersect H. So there are $x_1, \dots, x_n$ in X and $\epsilon > 0$ such that

$$\mathrm{W} \cap \mathrm{H} = \emptyset \qquad \text{where} \qquad \mathrm{W} = \big\{ f \in \mathrm{X}^\star \mid \big|(f, x_i) - (f_0, x_i)\big| < \epsilon \text{ for all } i \big\}$$

We also have $(\xi, f_0) \neq \alpha$ since $f_0 \notin \mathrm{H}$ so either $(\xi, f_0) < \alpha$ or $(\xi, f_0) > \alpha$. Assume we are in the first case.

Suppose that for some f in W, we have $(\xi, f) > \alpha$. Then

$$\varphi : t \longmapsto \big(\xi, tf + (1-t) f_0\big)$$

is continuous on $[0, 1]$, such that

$$\varphi(0) = (\xi, f_0) < \alpha \qquad \text{and} \qquad \varphi(1) = (\xi, f) > \alpha$$

So by the intermediate value theorem, there is f' in $[f_0, f]$ which belongs to H. But that segment is contained in W (this set is convex, it is easy to check) so W intersects H: this is absurd. Therefore,

$$\forall f \in \mathrm{W} \qquad (\xi, f) < \alpha$$

Now, consider
$$\begin{aligned} \mathrm{W} - f_0 &= \{ f - f_0 \mid f \in \mathrm{W} \} \\ &= \big\{ f - f_0 \mid f \in \mathrm{X}^\star \quad \big|(f, x_i) - (f_0, x_i)\big| < \epsilon \text{ for all } i \big\} \\ \mathrm{W} - f_0 &= \big\{ f \in \mathrm{X}^\star \mid \big|(f, x_i)\big| < \epsilon \text{ for all } i \big\} \end{aligned}$$

This is a weak⋆ neighbourhood of 0. Also,

$$\forall f \in \mathrm{W} - f_0 \qquad (\xi, f) = (\xi, \underbrace{f + f_0}_{\in \mathrm{W}}) - (\xi, f_0) < \alpha - (\xi, f_0)$$

Since $-f$ belongs as well to $\mathrm{W} - f_0$, we have

$$\forall f \in \mathrm{W} - f_0 \qquad \big|(\xi, f)\big| < \alpha - (\xi, f_0)$$

If we had supposed that $(\xi, f_0) > \alpha$, then we would have obtained that

$$\forall f \in \mathrm{W} - f_0 \qquad \big|(\xi, f)\big| < (\xi, f_0) - \alpha$$

So we showed that, for any choice of f_0 in $\mathrm{X}^\star \setminus \mathrm{H}$, there exists a weak⋆ neighbourhood of 0 contained in the pre-image under ξ of the open interval $\big(-|(\xi, f_0) - \alpha|, |(\xi, f_0) - \alpha|\big)$. It follows that ξ is weak⋆ continuous at 0 and by linearity, it is weak⋆ continuous. **Proposition 18** then tells us that ξ is actually in X. Which achieves the proof. □

Those two propositions are satisfying, in the sense that the weak⋆ topology allows us to separate X from $\mathrm{X}^{\star\star}$: the only linear functionals that are weak⋆ continuous are elements of X. Similarly, the only weak⋆ closed hyperplanes are the ones induced by elements of X.

Thus, we see that if the space X is not reflexive, then $\sigma(\mathrm{X}^\star, \mathrm{X})$ is *strictly* included in $\sigma(\mathrm{X}^\star, \mathrm{X}^{\star\star})$: there are hyperplanes closed for the latter that are not closed for the former, namely the ones induced by linear functionals on $\mathrm{X}^\star$ that are not in X.

Finally, here comes the Banach-Alaoglu-Bourbaki theorem:

Theorem 20
The closed unit ball in $\mathrm{X}^\star$ *is weak⋆ compact.*

Proof: We first check that $\overline{\mathcal{B}}_{\mathrm{X}^\star}$ is closed for the weak⋆ topology. Let f_0 be in the weak⋆ closure of this set. Let ϵ be any positive real number. By definition of $\|f_0\|$, there exists $x \in \mathrm{X}$, with norm 1, such that

$$(f_0, x) > \|f_0\| - \epsilon$$

The weak⋆ open neighbourhood

$$\mathrm{W} = \big\{ f \in \mathrm{X}^\star \mid \big|(f, x) - (f_0, x)\big| < \epsilon \big\}$$

has to intersect $\overline{\mathcal{B}}_{\mathrm{X}^\star}$ non-trivially: there exists $f \in \mathrm{X}^\star$ such that

$$\|f\| \leqslant 1 \qquad \text{and} \qquad \big|(f, x) - (f_0, x)\big| < \epsilon$$

In particular $\qquad \|f_0\| - \epsilon < (f_0, x) < (f, x) + \epsilon \leqslant \|f\|\,\|x\| + \epsilon \leqslant 1 + \epsilon$

and it follows that $\qquad \|f_0\| \leqslant 1 + 2\epsilon$

This holds for every positive ϵ, so $\|f_0\| \leqslant 1$ which proves our claim.

Let Y be the space $\mathbb{R}^{\mathrm{X}}$ of all real-valued functions on X, together with the product topology. That is, the weakest topology on Y that makes all the evaluations

$$\begin{aligned} \mathrm{e}_x : \ \mathrm{Y} &\longrightarrow \mathbb{R} & \qquad\qquad x \in \mathrm{X} \\ \omega &\longmapsto \mathrm{e}_x(\omega) = \omega(x) \end{aligned}$$

continuous. Since every element of $\mathrm{X}^\star$ is a function on X, we have an injection

$$\begin{aligned} \mathrm{J} : \ \mathrm{X}^\star &\longrightarrow \mathrm{Y} \\ f &\longmapsto \mathrm{J}(f) \end{aligned}$$

where

$$\forall x \in \mathrm{X} \qquad \mathrm{J}(f)(x) = (f, x)$$

We first check that J is continuous when $\mathrm{X}^\star$ has the weak$\star$ topology. According to **Theorem 4**, this is the case by definition of the weak$\star$ topology, since

$$\forall x \in \mathrm{X} \quad \forall f \in \mathrm{X}^\star \qquad (\mathrm{e}_x \circ \mathrm{J})(f) = \mathrm{J}(f)(x) = (f, x)$$

We also make sure that J^{-1} is continuous on $\mathrm{J}(\mathrm{X}^\star)$. This is also a consequence of **Theorem 4** and the definition of the product topology:

$$\forall x \in \mathrm{X} \quad \forall f \in \mathrm{J}(\mathrm{X}^\star) \qquad \big(\mathrm{J}^{-1}(f), x\big) = (f, x) = \mathrm{e}_x(f)$$

So J is a homeomorphism onto its image. Notice that

$$\forall x \in \mathrm{X} \quad \forall f \in \overline{\mathcal{B}}_{\mathrm{X}^\star} \qquad \big|\mathrm{J}(f)(x)\big| = \big|(f, x)\big| \leqslant \|f\| \, \|x\| \leqslant \|x\|$$

so

$$\mathrm{J}\big(\overline{\mathcal{B}}_{\mathrm{X}^\star}\big) \subset \prod_{x \in \mathrm{X}} \big[-\|x\|, \|x\| \big]$$

The righthandside is compact by Tychonoff's theorem; the lefthandside is closed, since J^{-1} is continuous and $\overline{\mathcal{B}}_{\mathrm{X}^\star}$ is weak$\star$ closed. Therefore, $\mathrm{J}\big(\overline{\mathcal{B}}_{\mathrm{X}^\star}\big)$ is compact. Again, J^{-1} is continuous and the continuous image of a compact set is compact. So $\overline{\mathcal{B}}_{\mathrm{X}^\star}$ is compact for the weak$\star$ topology. □

And now, we finally achieved something: the closed unit ball in $\mathrm{X}^\star$, which is certainly not compact when X is infinite dimensional, has a topology that makes it compact. There is also a neat immediate consequence of this theorem:

Corollary 21
If X *is reflexive, the closed unit ball of* X *is weakly compact.*

Proof: We identify X and $\mathrm{X}^{\star\star}$, since X is reflexive. Then $\sigma(\mathrm{X}, \mathrm{X}^\star)$ and $\sigma(\mathrm{X}^{\star\star}, \mathrm{X}^\star)$ are the same topology on X. But in the latter, the closed unit ball of X is compact, by **Theorem 20**. Simple as that. □

This is the "trivial" direction of Kakutani's theorem, which actually asserts that the converse is true: if the closed unit ball of X is weakly compact, then X is reflexive. This will be proved in the next section.

4 Weak topologies, reflexivity and uniform convexity

4.1 Kakutani and consequences

Lemma 22
$\mathcal{B}_{\mathrm{X}}$ is weak$\star$ dense in $\overline{\mathcal{B}}_{\mathrm{X}^{\star\star}}$.

Proof: We let ξ_0 be in the unit ball in $\mathrm{X}^{\star\star}$ and we suppose that there is a weak$\star$ neighbourhood of ξ_0 that does not intersect $\mathcal{B}_{\mathrm{X}}$: there are $f_1, \ldots, f_n$ in $\mathrm{X}^\star$ and a positive ϵ such that

$$\mathrm{W} \cap \mathcal{B}_{\mathrm{X}} = \emptyset \qquad \text{where} \qquad \mathrm{W} = \left\{\xi \in \mathrm{X}^{\star\star} \mid \left|(\xi, f_i) - (\xi_0, f_i)\right| < \epsilon \text{ for all } i\right\}$$

In other words, $\quad \forall x \in \mathcal{B}_{\mathrm{X}} \quad \exists i \in \{1, \ldots, n\} \qquad \left|(f_i, x) - (\xi_0, f_i)\right| \geqslant \epsilon \qquad$ (**1**)

Define
$$\forall x \in \mathrm{X} \qquad \Phi(x) = \begin{bmatrix} (f_1, x) \\ \vdots \\ (f_n, x) \end{bmatrix}$$

and let's use $\| \ \|_\infty$ on $\mathbb{R}^n$. Then (**1**) tells us precisely that

$$\forall x \in \mathcal{B}_{\mathrm{X}} \qquad \left\|\Phi(x) - \boldsymbol{\alpha}\right\|_\infty \geqslant \epsilon \qquad \text{where} \qquad \boldsymbol{\alpha} = \begin{bmatrix} (\xi_0, f_1) \\ \vdots \\ (\xi_0, f_n) \end{bmatrix}$$

In other words, $\boldsymbol{\alpha}$ is not in the closure of the convex set $\Phi(\mathcal{B}_{\mathrm{X}})$ so those can be separated by a hyperplane in $\mathbb{R}^n$: there exist real numbers $\beta_1, \ldots, \beta_n, \omega$ such that

$$\forall x \in \mathcal{B}_{\mathrm{X}} \qquad \sum_{i=1}^n \beta_i (f_i, x) < \omega < \sum_{i=i}^n \beta_i (\xi, f_i)$$

This implies that
$$\left\|\sum_{i=1}^n \beta_i f_i\right\| < \omega < \left(\xi, \sum_{i=1}^n \beta_i f_i\right) \leqslant \left\|\sum_{i=1}^n \beta_i f_i\right\|$$

and we have a contradiction. Therefore, W intersect $\mathcal{B}_{\mathrm{X}}$ non-trivially. □

Theorem 23 (Kakutani)
A Banach space X *is reflexive if and only if* $\overline{\mathcal{B}}_{\mathrm{X}}$ *is weakly compact.*

Proof: One direction was shown in **Corollary 21**. So now suppose that $\overline{\mathcal{B}}_{\mathrm{X}}$ is weakly compact. Then it is compact in $\mathrm{X}^{\star\star}$ for the $\sigma(\mathrm{X}^{\star\star}, \mathrm{X}^\star)$ topology. In particular, it is weak$\star$ closed. By **Lemma 22**, its weak$\star$ closure is $\overline{\mathcal{B}}_{\mathrm{X}^{\star\star}}$. Thus

$$\overline{\mathcal{B}}_{\mathrm{X}} = \overline{\mathcal{B}}_{\mathrm{X}^{\star\star}}$$

and X is reflexive. □

This theorem has very important consequences.

Corollary 24
Let X *be a reflexive Banach space. Every closed subspace is reflexive.*

Proof: Let Y be a closed subspace of X. There are, on Y, two topologies: the restriction of the weak topology on X and the weak topology $\sigma(\mathrm{Y}, \mathrm{Y}^\star)$. We check that those two coincide.

Let $y_0 \in \mathrm{Y}$ and consider an elementary $\sigma(\mathrm{Y}, \mathrm{Y}^\star)$-neighbourhood of y_0:

$$\mathrm{W} = \big\{ y \in \mathrm{Y} \mid \big|(f_i, y) - (f_i, y_0)\big| < \epsilon \text{ for all } i \big\} \qquad n \in \mathbb{N} \quad f_1, \ldots, f_n \in \mathrm{Y}^\star \quad \epsilon > 0$$

By Hahn-Banach, $f_1, \ldots, f_n$ can be extended to bounded linear functionals $g_1, \ldots, g_n$ on X. Thus, since the g_i's coincide with the f_i's on Y, we have:

$$\mathrm{W} = \big\{ y \in \mathrm{Y} \mid \big|(g_i, y) - (g_i, y_0)\big| < \epsilon \text{ for all } i \big\} = \mathrm{Y} \cap \big\{ x \in \mathrm{X} \mid \big|(g_i, x) - (g_i, y_0)\big| < \epsilon \text{ for all } i \big\}$$

so W is an open subset of Y for the restriction of the $\sigma(\mathrm{X}, \mathrm{X}^\star)$ topology.

Conversely, let W be an elementary neighbourhood of y_0 for the trace of the $\sigma(\mathrm{X}, \mathrm{X}^\star)$ topology. There exist $\epsilon > 0$ and $g_1, \ldots, g_n$ in $\mathrm{X}^\star$ such that

$$\mathrm{W} = \mathrm{Y} \cap \big\{ x \in \mathrm{X} \mid \big|(g_i, x) - (g_i, y_0)\big| < \epsilon \text{ for all } i \big\} = \big\{ y \in \mathrm{Y} \mid \big|(g_i, y) - (g_i, y_0)\big| < \epsilon \text{ for all } i \big\}$$

Since $g_1, \ldots, g_n$ are bounded on X, their restrictions $f_1, \ldots, f_n$ to Y are also bounded. Thus

$$\mathrm{W} = \big\{ y \in \mathrm{Y} \mid \big|(f_i, y) - (f_i, y_0)\big| < \epsilon \text{ for all } i \big\}$$

is open for the weak topology on Y.

The closed unit ball $\overline{\mathcal{B}}_{\mathrm{Y}}$ is convex and strongly closed in X. Therefore, it is a weakly closed subset of $\overline{\mathcal{B}}_{\mathrm{X}}$, which is compact by Kakutani's theorem. Therefore, $\overline{\mathcal{B}}_{\mathrm{Y}}$ is compact for $\sigma(\mathrm{X}, \mathrm{X}^\star)$. Since that topology coincides on Y with $\sigma(\mathrm{Y}, \mathrm{Y}^\star)$, it follows that $\overline{\mathcal{B}}_{\mathrm{Y}}$ is weakly compact in Y. By Kakutani, Y is reflexive. □

Corollary 25
A Banach space X *is reflexive if and only if* $\mathrm{X}^\star$ *is reflexive.*

Proof: Suppose that X is reflexive. Then $\mathrm{X} = \mathrm{X}^{\star\star}$ and bounded linear functionals on $\mathrm{X}^{\star\star}$ are the same as bounded linear functionals on X. In other words, $\mathrm{X}^{\star\star\star} = \mathrm{X}^\star$.

Conversely, let's assume that $\mathrm{X}^\star$ is reflexive. The weak$\star$ topology and the weak topology on $\mathrm{X}^{\star\star}$ then coincide. By Banach-Alaoglu-Bourbaki, $\mathrm{X}^{\star\star}$ is reflexive. But X is a closed subspace of $\mathrm{X}^{\star\star}$ so by **Corollary 24**, X is reflexive. □

Corollary 26
Let X *be a reflexive Banach space. Any closed convex bounded set is weakly compact.*

Proof: If C is closed and convex, it is weakly closed. Since it is bounded, it is included in $\overline{\mathcal{B}}(0, \mathrm{R})$ for some R and this set is weakly compact by Kakutani. Closed subsets of compact sets are compact, so C is weakly compact. ☐

This is all for the moment in general. More will come later, once we study the connection between weak topologies and separability.

4.2 Uniformly convex Banach spaces

Let's prove the Milmann-Pettis theorem about uniformly convex Banach spaces. Although it is not a consequence of Kakutani, it uses Banach-Alaoglu-Bourbaki as well as **Lemma 22** so it derives from all the work done so far.

Definition 27 A Banach space X is called uniformly convex if and only if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in \mathrm{X} \quad \|x\| = \|y\| = 1 \qquad \left\|\frac{x+y}{2}\right\| > 1 - \delta \implies \|x - y\| < \epsilon$$

We first show that this definition is in fact equivalent to an apparently stronger statement:

Lemma 28
Let X *be a uniformly convex Banach space. Then*

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in \overline{\mathcal{B}}_{\mathrm{X}} \qquad \left\|\frac{x+y}{2}\right\| > 1 - \delta \implies \|x - y\| < \epsilon$$

> Be careful, this is ugly and does not result from a simple scaling argument. I have not been able to find anything simpler...

Proof: Let ϵ be a positive real number. By uniform convexity of X, we are given a positive α such that

$$\forall x, y \in \mathrm{X} \quad \|x\| = \|y\| = 1 \qquad \left\|\frac{x+y}{2}\right\| > 1 - \alpha \implies \|x - y\| \leqslant \frac{\epsilon}{2}$$

Let
$$\delta = \mathrm{Min}\left(\frac{\alpha}{2}, \frac{\epsilon}{4}\right)$$

so that
$$2\delta \leqslant \alpha \qquad \text{and} \qquad 2\delta \leqslant \frac{\epsilon}{2}$$

Take x, y non-zero in X, both with norm less than 1 and such that

$$\left\|\frac{x+y}{2}\right\| > 1-\delta$$

Suppose also, for example, that $\|y\| \leqslant \|x\|$. It will be useful to get also a lower bound for $\|y\|$. This is done by noticing that

$$y = 2 \times \frac{x+y}{2} - x$$

so that
$$\|y\| \geqslant 2\left\|\frac{x+y}{2}\right\| - \|x\| \geqslant 2 - 2\delta - \|x\|$$

and
$$\|y\| - \|x\| \geqslant 2 - 2\delta - 2\|x\| \geqslant -2\delta$$

We scale x and y to bring them on the unit sphere: let

$$x_0 = \frac{x}{\|x\|} \qquad \text{and} \qquad y_0 = \frac{y}{\|y\|}$$

Then
$$\begin{aligned}
\|x_0 + y_0\| &= \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| = \left\|\frac{x}{\|x\|} + \frac{y}{\|x\|} - \frac{y}{\|x\|} + \frac{y}{\|y\|}\right\| \\
&\geqslant \left\|\frac{x+y}{\|x\|}\right\| - \left\|\frac{y}{\|x\|} - \frac{y}{\|y\|}\right\| \geqslant \frac{2-2\delta}{\|x\|} - \left\|\frac{(\|y\| - \|x\|)y}{\|x\|\,\|y\|}\right\| \\
&\geqslant \frac{2-2\delta}{\|x\|} - \frac{\|x\| - \|y\|}{\|x\|} = \frac{2-2\delta+\|y\|-\|x\|}{\|x\|} \\
\|x_0 + y_0\| &\geqslant \frac{2-4\delta}{\|x\|} \geqslant 2 - 4\delta \geqslant 2 - 2\alpha
\end{aligned}$$

Thus
$$\left\|\frac{x_0+y_0}{2}\right\| \geqslant 1 - \alpha$$

and from uniform convexity,
$$\|x_0 - y_0\| \leqslant \frac{\epsilon}{2}$$

Now we have to relate this to $\|x - y\|$. It is the same kind of mess as what we just did:

$$\begin{aligned}
\|x_0 - y_0\| &= \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| = \left\|\frac{x}{\|x\|} - \frac{y}{\|x\|} + \frac{y}{\|x\|} - \frac{y}{\|y\|}\right\| \\
&\geqslant \left\|\frac{x-y}{\|x\|}\right\| - \frac{\|x\| - \|y\|}{\|x\|} \geqslant \frac{\|x-y\|}{\|x\|} - \frac{2\delta}{\|x\|} \\
\|x_0 - y_0\| &\geqslant \|x - y\| - 2\delta
\end{aligned}$$

Therefore
$$\|x - y\| \leqslant \|x_0 - y_0\| + 2\delta \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
□

Theorem 29 (Milmann-Pettis)
A uniformly convex Banach space is reflexive.

Proof: Let ϵ be a positive number, which provides us with a positive δ such that

$$\forall x, y \in \overline{\mathcal{B}}_{\mathrm{X}} \qquad \left\|\frac{x+y}{2}\right\| > 1-\delta \implies \|x-y\| \leqslant \epsilon$$

Let $\xi_0 \in \mathrm{X}^{\star\star}$ with $\|\xi_0\| = 1$ and α be any positive real number. There exists $f \in \mathrm{X}^\star$ with $\|f\| = 1$ such that

$$(\xi_0, f) > 1-\delta$$

Define $$\mathrm{W} = \left\{\xi \in \mathrm{X}^{\star\star} \mid (\xi, f) > 1-\delta\right\}$$

This is a weak$\star$ neighbourhood of ξ_0 and by **Lemma 22**, it intersects $\overline{\mathcal{B}}_{\mathrm{X}}$ non-trivially: there exists $x_0 \in \mathrm{X}$, with $\|x_0\| \leqslant 1$, such that

$$(f, x_0) > 1-\delta$$

Now, assume that $\|\xi_0 - x_0\| > \epsilon$. This means that ξ_0 is not in the ball $\overline{\mathcal{B}}_{\mathrm{X}^{\star\star}}(x_0, \epsilon)$. This set is weak$\star$ closed, as we proved in **Theorem 20** so its complement is open. Thus,

$$\overline{\mathcal{B}}_{\mathrm{X}^{\star\star}}(x_0, \epsilon)^c \cap \mathrm{W}$$

is a weak$\star$ neighbourhood of ξ_0. By **Lemma 22**, it contains a $y_0 \in \mathrm{X}$ with $\|y_0\| \leqslant 1$ and y_0 satisfies

$$\|x_0 - y_0\| > \epsilon \qquad \text{and} \qquad (f, y_0) > 1-\delta$$

Therefore $$\left(f, \frac{x_0+y_0}{2}\right) > 1-\delta$$

which implies $$\left\|\frac{x_0+y_0}{2}\right\| > 1-\delta$$

From uniform convexity, $$\|x_0 - y_0\| \leqslant \epsilon$$

and this is a contradiction. Thus $\|x_0 - \xi_0\| \leqslant \epsilon$, or in other words

$$\forall \epsilon > 0 \qquad \mathcal{B}(\xi_0, \epsilon) \cap \overline{\mathcal{B}}_{\mathrm{X}} \neq \emptyset$$

So ξ_0 is in the (strong) closure of $\overline{\mathcal{B}}_{\mathrm{X}}$, which is already closed: ξ_0 is in fact in X. □

5 Weak topologies and separability

We finally investigate how the property of separability influences the weak topologies. Let's remember that

Definition 30 A normed space X is called separable if and only if it contains a countable dense subset.

Theorem 31
Let X *be a Banach space such that* $\mathrm{X}^\star$ *is separable. Then* X *is separable.*

Proof: Let $(f_n)_{n\in\mathbb{N}}$ be a countable dense subset of $\mathrm{X}^\star$. For every n, there exists x_n in X with norm 1, such that

$$(f_n, x_n) > \frac{\|f_n\|}{2}$$

Let L_0 be the $\mathbb{Q}$-vector subspace of X spanned by $(x_n)_{n\in\mathbb{N}}$, and L be the $\mathbb{R}$-vector subspace of X spanned by $(x_n)_{n\in\mathbb{N}}$. Then L_0 is countable, and dense in L. So if we show that L is dense in X, we're done.

Let $f \in \mathrm{X}^\star$, that vanishes on L. Since $(f_n)_{n\in\mathbb{N}}$ is dense in $\mathrm{X}^\star$, given a positive ϵ, there exists $n \in \mathbb{N}$ such that $\|f_n - f\| < \epsilon$. Then

$$\frac{\|f_n\|}{2} < (f_n, x_n) = (f_n - f, x_n) < \epsilon$$

and

$$\|f\| \leqslant \|f_n - f\| + \|f_n\| \leqslant 3\epsilon$$

This is true for all positive ϵ, so $f = 0$ and L is dense. □

> Note that there is no converse to this theorem, that is, if X is separable, $\mathrm{X}^\star$ has no reason to be separable as well.
>
> Think, for example, of $\mathrm{L}^1(\mathbb{R})$ which is separable while its dual, $\mathrm{L}^\infty(\mathbb{R})$ is not.

Corollary 32
A Banach space X *is reflexive and separable if and only if* $\mathrm{X}^\star$ *is reflexive and separable.*

Proof: Assume that X is reflexive and separable. Then $\mathrm{X}^{\star\star}$ is reflexive and separable since it is identified with X. By **Corollary 25** and **Theorem 31**, $\mathrm{X}^\star$ is reflexive and separable.

Conversely, if $\mathrm{X}^\star$ is reflexive and separable, **Corollary 25** and **Theorem 31** imply that X is reflexive and separable. □

Theorem 33
A normed space X *is separable if and only if the weak⋆ topology on* $\overline{\mathcal{B}}_{X^\star}$ *is metrizable.*

> Notice that there is no contradiction at all with the fact, seen earlier, that a weak topology is never metrizable on an infinite dimensional space. Indeed, if we remember how that was proved, we used the fact that weak open sets contained entire lines. Obviously, if we restrict the topology to a bounded set such as the unit ball, we cannot use that fact anymore.

Proof: Let X be a separable normed space, which means there is a countable dense subset A in X. We define D to be $A \cap \mathcal{B}_X(0,1)$; because D is countable, we can enumerate its elements:

$$D = \{x_n \mid n \in \mathbb{N}\}$$

Finally, we define :

$$\forall (f,g) \in S^2 \qquad d(f,g) = \sum_{n\in\mathbb{N}} \frac{|(f-g,x_n)|}{2^n}$$

where S is the closed unit ball in $X^\star$. We claim that d is a metric on S and that the toplogy $\mathscr{T}$ it induces on S coincides with the weak topology $\sigma(X^\star, X)$. In order to do this, there are a few things to check.

<u>1</u> : D is dense in $\overline{\mathcal{B}}_X$

Let x be in the unit ball of X and let O be an open set containing x. Then there is some ϵ such that $\mathcal{B}_X(x,\epsilon)$ is included in O. And because $\mathcal{B}_X$ is open, we can take ϵ smaller so that $\mathcal{B}_X(x,\epsilon)$ is included in $\mathcal{B}_X$. Since A is dense, $\mathcal{B}_X(x,\epsilon)$ intersects A at some y. And y is then in D. Which shows that D is dense in $\mathcal{B}_X$.

And we conclude using the fact that $\mathcal{B}_X$ is dense in $\overline{\mathcal{B}}_X$.

<u>2</u> : d is a metric on S

We first check that d is properly defined, i.e. that it does not take the value ∞. This is easy :

$$\forall (f,g) \in S^2 \qquad \sum_{n\in\mathbb{N}} \frac{|(f-g,x_n)|}{2^n} \leqslant \sum_{n\in\mathbb{N}} \frac{\|f-g\|\,\|x_n\|}{2^n} \leqslant 2\sum_{n\in\mathbb{N}} \frac{1}{2^n}$$

which is finite.

Triangular inequality and symmetry for d are trivial to check. The only thing remaining is to see what happens if $d(f,g) = 0$. In this case, we get

$$\forall x \in D \qquad (f-g,x) = 0$$

Now take any y in X with norm less than 1. From **1.1**, there is some sequence $(y_n)_{n\in\mathbb{N}}$ in D converging to y. So that :

$$(f-g,y) = (f-g, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} \underbrace{(f-g,y_n)}_{=0}$$

since $f-g$ is continuous on X. Which shows that

$$\forall y \in \mathrm{X} \qquad \big(\|y\| \leqslant 1 \Longrightarrow (f-g,y)=0\big)$$

Hence
$$\|f-g\| = 0$$

and
$$f = g$$

d is a metric on S.

3 : $\mathscr{T} \subset \boldsymbol{\sigma}(\mathrm{X}^\star, \mathrm{X})$

Let f_0 be in S and take an open set $\mathcal{O}$ in $\mathscr{T}$ containing f_0. Then we can find a positive real number r such that :

$$\mathcal{B}_{\mathrm{d}}(f_0, r) = \{f \in \mathrm{S} \mid \mathrm{d}(f_0,f) < r\} \subset \mathcal{O}$$

Now take an integer k such that $\dfrac{1}{2^k} < \dfrac{r}{2}$ and $\epsilon < \dfrac{r}{2}$. Define :

$$\mathrm{V} = \big\{f \in \mathrm{S} \mid |(f-f_0,x_j)| < \epsilon \text{ for } j=0,\dots,k\big\}$$

so that V is a weak⋆ open set in S containing f_0. Then

$$\forall f \in \mathrm{V} \qquad \mathrm{d}(f_0,f) = \sum_{n=0}^{k} \frac{|(f-f_0,x_n)|}{2^n} + \sum_{n\geqslant k+1} \underbrace{\frac{|(f-f_0,x_n)|}{2^n}}_{\leqslant \frac{1}{2^{n-1}}} \leqslant \epsilon + \frac{1}{2^k} < r$$

which shows
$$\mathrm{V} \subset \mathcal{B}_{\mathrm{d}}(f_0,r) \subset \mathcal{O}$$

Thus for every f_0 in S, every $\mathscr{T}$-open neighbourhood of f_0 contains a weak⋆ neighbourhood of f_0.

4 : $\boldsymbol{\sigma}(\mathrm{X}^\star, \mathrm{X}) \subset \mathscr{T}$

Let f_0 be in S and take $\mathcal{O}$ a weak⋆ open set in S containing f_0. Then $\mathcal{O}$ is a union of elementary weak⋆ open sets containing f_0 so that there exists $\epsilon > 0$ and points $y_1,\dots,y_k$ in X such that :

$$\mathrm{V} = \big\{f \in \mathrm{S} \mid |(f-f_0,y_j)| < \epsilon \text{ for } j=1,\dots,k\big\} \subset \mathcal{O}$$

Now, take $m = \mathrm{Max}\,\big\{\|y_j\| \mid j=1,\dots,k\big\}$ so that

$$\forall j \in \{1,\dots,k\} \qquad \frac{y_j}{m} \in \mathrm{S}$$

Given a positive η, and $j \in \{1, \dots, k\}$, there is some n_j in $\mathbb{N}$ such that $\|x_{n_j} - y_j/m\| < \eta$. And we take a positive r such that

$$\forall j \in \{1, \dots, k\} \qquad m2^{n_j} r < \frac{\epsilon}{2}$$

Then for all f in $\mathcal{B}_{\mathrm{d}}(f_0, r)$, we have :

$$\sum_{n \in \mathbb{N}} \frac{|(f - f_0, x_n)|}{2^n} < r$$

so that

$$\forall j \in \{1, \dots, k\} \qquad |(f - f_0, x_{n_j})| \leqslant 2^{n_j} r$$

Finally, $\forall j \in \{1, \dots, k\}$

$$\begin{aligned} |(f - f_0, y_j)| &= m \left| \left(f - f_0, \frac{y_j}{m} \right) \right| \\ &\leqslant m \underbrace{|(f - f_0, x_{n_j})|}_{\leqslant 2^{n_j} r} + m \underbrace{\left| \left(f - f_0, x_{n_j} - \frac{y_j}{m} \right) \right|}_{\leqslant 2\eta} \\ &\leqslant \frac{\epsilon}{2} + 2\eta m \end{aligned}$$

Recall that η was arbitrary ; so just take it small enough so that $2\eta m < \dfrac{\epsilon}{2}$ and we get $f \in \mathrm{V}$. Hence

$$\mathcal{B}_{\mathrm{d}}(f_0, r) \subset \mathrm{V} \subset \mathcal{O}$$

So for every f_0 in S and every weak$\star$ open neighbourhood $\mathcal{O}$ of f_0 in S, there is a $\mathscr{T}$-neighbourhood of f_0 contained in $\mathcal{O}$.

5 : Conclusion

Both topologies $\mathscr{T}$ and $\star - \sigma(\mathrm{X}^\star, \mathrm{X})$ coincide.

Now, suppose that S together with the weak$\star$ topology is metrizable: there exists a distance d on S such that the induced topology coincides with $\sigma(\mathrm{X}^\star, \mathrm{X})$. Define

$$\forall n \in \mathbb{N}^\star \qquad \mathrm{B}_n = \mathcal{B}_{\mathrm{d}}\Big(0, \frac{1}{n}\Big) = \Big\{ f \in \mathrm{S} \;\Big|\; \mathrm{d}(0, f) < \frac{1}{n} \Big\}$$

Each B_n is an weak$\star$ open neighbourhood of 0, so there exist a positive ϵ_n and $\mathrm{F}_n \subset \mathrm{X}$, finite, such that

$$\mathrm{W}_n = \big\{ f \in \mathrm{X}^\star \mid \big|(f, x)\big| < \epsilon_n \text{ for all } x \in \mathrm{F}_n \big\} \subset \mathrm{B}_n$$

Let

$$\mathrm{F} = \bigcup_{n \in \mathbb{N}^\star} \mathrm{F}_n$$

This is a countable subset of X. Let's show that it spans a dense subspace of X. Let f be a bounded linear functional that takes the value 0 at each $x \in \mathrm{F}$. Then

$$f \in \bigcap_{n \in \mathbb{N}^\star} \mathrm{W}_n \subset \bigcap_{n \in \mathbb{N}^\star} \mathrm{B}_n = \{0\}$$

So f is 0 and as a consequence of the Hahn-Banach theorem, Span F is dense. This is a separable set (take finite linear combinations of elements of F with rational coefficients). Thus X is separable. □

This theorem has a dual equivalent, though more difficult to prove:

Theorem 34
Let X *be a Banach space. Its dual is separable if and only if* $\overline{\mathrm{B}}_{\mathrm{X}}$ *is metrizable for the weak topology* $\sigma(\mathrm{X}, \mathrm{X}^\star)$*.*

We end our work with the two important consequences of everything done so far:

Theorem 35
Let X *be a separable Banach space. Any bounded sequence in* $\mathrm{X}^\star$ *has a weak⋆ converging subsequence.*

Proof: Let $(f_n)_{n\in\mathbb{N}}$ be any bounded sequence in $\mathrm{X}^\star$. There exists a positive real number M such that

$$\forall n \in \mathbb{N} \qquad g_n = \frac{f_n}{\mathrm{M}} \in \overline{\mathcal{B}}_{\mathrm{X}^\star}$$

The sequence $(g_n)_{n\in\mathbb{N}}$ takes its values in the unit ball of $\mathrm{X}^\star$, which is weak⋆ compact by Banach-Alaoglu-Bourbaki. And the weak⋆ topology on it is metrizable by **Theorem 33**. Therefore, $(g_n)_{n\in\mathbb{N}}$ has a weak⋆ converging subsequence, as well as $(f_n)_{n\in\mathbb{N}}$.

Theorem 36
Let X *be a reflexive Banach space. Any bounded sequence has a weakly converging subsequence.*

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X and let M be the closure in X of Span $(x_n)_{n\in\mathbb{N}}$. Then M is a closed subspace of X. By **Corollary 24**, M is reflexive. By construction, M is also separable since the $\mathbb{Q}$-vector subspace spanned by the $(x_n)_{n\in\mathbb{N}}$ is dense.

By **Theorem 34**, there is a $\sigma(\mathrm{M}^{\star\star}, \mathrm{M}^\star)$ converging subsequence of $(x_n)_{n\in\mathbb{N}}$, with limit x. But since M is reflexive, the topologies $\sigma(\mathrm{M}^{\star\star}, \mathrm{M}^\star)$ and $\sigma(\mathrm{M}, \mathrm{M}^\star)$ coincide. So $(x_n)_{n\in\mathbb{N}}$ converges to x for the topology $\sigma(\mathrm{M}, \mathrm{M}^\star)$.

Now, if f is any bounded linear functional on X, its restriction to M is of course bounded on M. Therefore,

$$\lim_{n\to\infty} (f, x_n) = (f, x)$$

So $(x_n)_{n\in\mathbb{N}}$ converges to x for $\sigma(\mathrm{X}, \mathrm{X}^\star)$. □